I. First-Order Ordinary Differential Equations
3. Separable Equations

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3. Separable Equations

The next simplest class of first-order equations to treat after linear ones is that of so-called separable equations. These have the form

$$\frac{dy}{dt} = f(t)\, g(y)\,. \tag{3.1}$$

Here the derivative of y with respect to t is given as a function of t times a function of y. Equation (3.1) becomes explicit when $g(y)$ is a constant, so we will exclude that case.

3.1. Recipe for Autonomous Equations.

Equation (3.1) is said to be *autonomous* when $f(t)$ is a constant. In that case $f(t)$ can be absorbed into $g(y)$ and (3.1) becomes

$$\frac{dy}{dt} = g(y)\,. \tag{3.2}$$

The word "autonomous" has Greek roots and means "self governing." Equation (3.2) is called autonomous because it depends only on y. Autonomous equations arise naturally in applications when the laws governing dynamics are the same at every time.

If $g(y_o) = 0$ at some point y_o then it is clear that $y(t) = y_o$ is a solution of (3.2) that is defined for every t. Because this solution does not depend on t it is called a *stationary solution*. Every zero of g is also called a *stationary point* because it yields such a stationary solution. These points are also sometimes called either *equilibrium points*, *critical points*, or *fixed points*.

Example. Consider the autonomous equation

$$\frac{dy}{dt} = 4y - y^3\,.$$

Because $4y - y^3 = y(2-y)(2+y)$, we see that $y = 0$, $y = 2$, and $y = -2$ are stationary points of this equation.

Now let us consider how to find the nonstationary solutions of (3.2). Just as we did for the linear case, we will reduce the autonomous case to the explicit case. The trick to doing this is to consider t to be a function of y. This trick works over intervals over which the solution $y(t)$ is a strictly monotonic function of t. This will be the case over intervals where $g(y(t))$ is never zero — i.e. over intervals where the solution $y(t)$ does not hit a stationary point. In that case, by the chain rule we have

$$\frac{dt}{dy} = \frac{1}{\dfrac{dy}{dt}} = \frac{1}{g(y)}\,.$$

This is an explicit equation for the derivative of t with respect to y. It can be integrated to obtain

$$t = \int \frac{dy}{g(y)} = G(y) + c\,, \qquad \text{where } G'(y) = \frac{1}{g(y)} \text{ and } c \text{ is any constant}\,. \tag{3.3}$$

Equation (3.3) is called an *implicit solution* of equation (3.2) because if we solve equation (3.3) for y as a differentiable function of t then the result will be a solution of (3.2) wherever $g(y) \neq 0$. Indeed, suppose that $Y(t)$ is differentiable and satisfies

$$t = G\big(Y(t)\big) + c\,.$$

Upon differentiating both sides of this equation with respect to t we see that

$$1 = G'\big(Y(t)\big)\,\frac{\mathrm{d}Y(t)}{\mathrm{d}t} = \frac{1}{g\big(Y(t)\big)}\,\frac{\mathrm{d}Y(t)}{\mathrm{d}t}\,, \qquad \text{wherever } g\big(Y(t)\big) \neq 0\,.$$

It follows that $y = Y(t)$ satifies (3.2).

Being able to solve (3.3) for y means finding an inverse function of G — namely, a function G^{-1} with property that

$$G^{-1}\big(G(y)\big) = y \quad \text{for every } y \text{ in some interval within the domain of } G\,.$$

For every such an inverse function, a family of explicit solutions to (3.2) is then given by

$$y = G^{-1}\big(t - c\big)\,. \tag{3.4}$$

As we will see in examples below, such a solution may not exist for every value of $t - c$, or there may be more than one solution.

Remark. This recipe will fail to yield a family of explicit solutions to (3.2) if either we are unable to find an expression for the primitive $G(y)$ in order to obtain (3.3), or we are unable find an explicit inverse function of $G(y)$ in order to obtain (3.4) from (3.3).

Example. Find all solutions of

$$\frac{\mathrm{d}y}{\mathrm{d}t} = y^2\,.$$

Solution. This equation is autonomous. Its right-hand side is defined for every y. It has the stationary point $y = 0$. Its nonstationary solutions are given implicitly by

$$t = \int \frac{\mathrm{d}y}{y^2} = -\frac{1}{y} + c\,.$$

We can solve for y explicitly to find the family of solutions

$$y = \frac{1}{c - t}\,.$$

Notice that this solution is not defined at $t = c$. Therefore it is really two solutions — one for $t < c$ with interval of definition $(-\infty, c)$ and one for $t > c$ with interval of definition (c, ∞). Because

$$\lim_{t \to c^-} \frac{1}{c - t} = +\infty\,, \qquad \lim_{t \to c^+} \frac{1}{c - t} = -\infty\,,$$

we say that the solution for $t < c$ "blows-up" to ∞ as $t \to c^-$ while the solution for $t > c$ "blows-up" to $-\infty$ as $t \to c^+$. (Here $t \to c^-$ means that t approaches c from below while $t \to c^+$ means that t approaches c from above.) Also notice that none of these solutions ever hits the stationary point $y = 0$.

Remark. The above example already shows two important differences between nonlinear and linear equations. For one, it shows that solutions of nonlinear equations can "blow-up" even when the right-hand side of the equation is continuous everywhere. This is in marked contrast with linear equations where the solution will not blow-up if the coefficient and forcing are continuous everywhere. Moreover, it shows that for solutions of nonlinear equations the interval of definition cannot be read off from the equation. This also is in marked contrast with linear equations where we can read off the interval of definition from the coefficient and the forcing.

Example. Find all solutions of

$$\frac{\mathrm{d}y}{\mathrm{d}t} = \frac{1}{2y}\,.$$

Solution. This equation is autonomous. Because the right-hand side of this equation is undefined when $y = 0$, no solution can take the value 0. Because its right-hand side never vanishes, this equation has no stationary points. Its nonstationary solutions are given implicitly by

$$t = \int 2y\,\mathrm{d}y = y^2 + c\,.$$

We can solve for y explicitly to find two families of solutions

$$y = \pm\sqrt{t - c}\,.$$

Notice that these functions are not defined for $t < c$ and are not differentiable at $t = c$. Therefore they are solutions only when $t > c$. One family gives positive solutions while the other gives negative solutions. Their interval of definition is (c, ∞). The derivatives of these solutions are given by

$$\frac{\mathrm{d}y}{\mathrm{d}t} = \frac{1}{2y} = \pm\frac{1}{2\sqrt{t - c}}\,.$$

Notice that these derivatives "blow-up" as $t \to c^+$ while the solutions themselves approach the forbidden value 0.

Remark. The above example shows another important difference between nonlinear and linear equations. Specifically, it shows that sometimes we need more than one family of nonstationary solutions to give a general solution. This is in marked contrast with linear equations where a general solution is always given by the single family 2.8. It also shows that the derivative of a solution can "blow-up" at an endpoint of its interval of definition even though the solution does not. This phenomenon does not happen for linear equations so long as the coefficient and forcing are continuous.

The next example shows that finding explicit nonstationary solutions by recipe (3.4) can get complicated even when $g(y)$ is a fairly simple polynomial. In fact, it will show that even the simpler task of finding the implicit relation (3.3) satisfied by nonstationary solutions sometimes requires us to recall basic techniques of integration.

Example. Find all solutions of

$$\frac{\mathrm{d}y}{\mathrm{d}t} = 4y - y^3\,.$$

Solution. This equation is autonomous. Its right-hand side is defined for every y. Because $4y - y^3 = y(2-y)(2+y)$, we see that $y = 0$, $y = 2$, and $y = -2$ are stationary points for this equation. Its nonstationary solutions are given implicitly by

$$t = \int \frac{\mathrm{d}y}{4y - y^3} \,.$$

Because the factors of $4y - y^3$ are simple, the numerators that appear in the partial fraction identity of $1/(4y - y^3)$ above the factors y, $2 - y$, and $2 + y$ respectively can be read off as

$$\frac{1}{(2-y)(2+y)}\bigg|_{y=0} = \tfrac{1}{4} \,, \qquad \frac{1}{y(2+y)}\bigg|_{y=2} = \tfrac{1}{8} \,, \qquad \frac{1}{(2-y)y}\bigg|_{y=-2} = -\tfrac{1}{8} \,,$$

thereby yielding the partial fraction identity

$$\frac{1}{4y - y^3} = \frac{1}{y(2-y)(2+y)} = \frac{\frac{1}{4}}{y} + \frac{\frac{1}{8}}{2-y} - \frac{\frac{1}{8}}{2+y} \,.$$

(You should be able to write down such partial fraction identities directly.) Therefore nonstationary solutions are given implicitly by

$$\begin{aligned} t &= \int \frac{\frac{1}{4}}{y} + \frac{\frac{1}{8}}{2-y} - \frac{\frac{1}{8}}{2+y} \,\mathrm{d}y = \tfrac{1}{4}\log(|y|) - \tfrac{1}{8}\log(|2-y|) - \tfrac{1}{8}\log(|2+y|) + c \\ &= \tfrac{1}{8}\log(y^2) - \tfrac{1}{8}\log(|2-y|) - \tfrac{1}{8}\log(|2+y|) + c = \tfrac{1}{8}\log\left(\frac{y^2}{|4-y^2|}\right) + c \end{aligned}$$

where c is an arbitrary constant. This equation can be written as

$$8(t - c) = \log\left(\frac{y^2}{|4 - y^2|}\right),$$

which leads to

$$\frac{y^2}{|4 - y^2|} = e^{8(t-c)} \,.$$

This can then be broken down into two cases.

First, if $y^2 < 4$ then

$$\frac{y^2}{4 - y^2} = e^{8(t-c)} \,,$$

which implies that

$$y^2 = \frac{4}{1 + e^{-8(t-c)}} \,,$$

and finally that

$$y = \pm\sqrt{\frac{4}{1 + e^{-8(t-c)}}} \,.$$

These solutions exist for every time t. They vanish as $t \to -\infty$ and approach ± 2 as $t \to \infty$.

On the other hand, if $y^2 > 4$ then

$$\frac{y^2}{y^2 - 4} = e^{8(t-c)} \,,$$

which implies that

$$y^2 = \frac{4}{1 - e^{-8(t-c)}}\,.$$

So long as the denominator is positive, we find the solutions

$$y = \pm\sqrt{\frac{4}{1 - e^{-8(t-c)}}}\,.$$

The denominator is positive if and only if $t > c$. Therefore these solutions exist for every time $t > c$. They diverge (blow-up) to $\pm\infty$ as $t \to c^+$ and approach ± 2 as $t \to \infty$.

Therefore we have found the three stationary points $y = 0$, $y = 2$, and $y = -2$, and the four families of nonstationary solutions

$$
\begin{aligned}
y &= -\sqrt{\frac{4}{1 - e^{-8(t-c)}}} && \text{when } -\infty < y < -2 \text{ and } t > c\,,\\
y &= -\sqrt{\frac{4}{1 + e^{-8(t-c)}}} && \text{when } -2 < y < 0\,,\\
y &= +\sqrt{\frac{4}{1 + e^{-8(t-c)}}} && \text{when } 0 < y < 2\,,\\
y &= +\sqrt{\frac{4}{1 - e^{-8(t-c)}}} && \text{when } 2 < y < \infty \text{ and } t > c\,,
\end{aligned}
\tag{3.5}
$$

where c is an arbitrary constant. Notice that none of these nonstationary solutions ever hits one of the stationary points. This list includes every solution of the equation. Members of these families are shown in the graph below. □

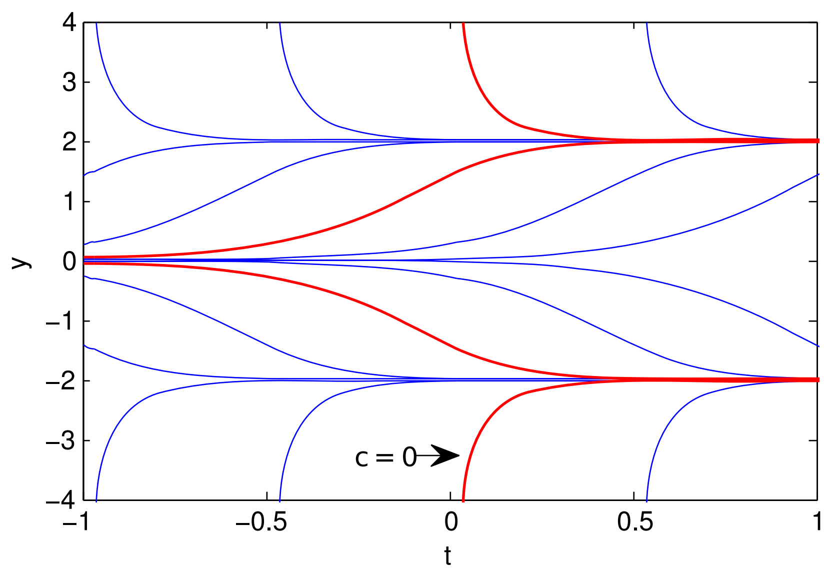

FIGURE 3.1. Nonstationary solutions to $y' = 4y - y^3$ are shown for various values of c. The solution for $c = 0$ is shown in red. Notice that every nonstationary solution approaches one of the stationary points $y = \pm 2$ as $t \to \infty$.

3.2. **Recipe for Separable Equations.** Now let us consider the general separable equation (3.1), which has the form

$$\frac{\mathrm{d}y}{\mathrm{d}t} = f(t)\, g(y)\,. \tag{3.6}$$

The method for solving separable equations is a slight modification of the one for solving autonomous equations. Indeed, the difficulties that arise are the same ones that arose in the autonomous case. The difference will be that the recipe for implicitly solving separable equations will require us to find two primitives, while recipe (3.3) for implicitly solving autonomous equations requires us to find only one primitive.

If $g(y_o) = 0$ at some point y_o then it is clear that $y(t) = y_o$ is a solution of (3.6) that is defined for every t. As before, this solution is called a *stationary solution*. Every zero of g is called a *stationary point* because it yields such a stationary solution.

To find the nonstationary solutions of equation (3.6) we first put the equation into its so-called *separated differential form*

$$\frac{1}{g(y)}\, \mathrm{d}y = f(t)\, \mathrm{d}t\,.$$

Then integrate both sides of this equation to obtain

$$\int \frac{1}{g(y)}\, \mathrm{d}y = \int f(t)\, \mathrm{d}t\,.$$

This is equivalent to

$$F(t) = G(y) + c\,, \tag{3.7}$$

where F and G satisfy

$$F'(t) = f(t)\,, \quad G'(y) = \frac{1}{g(y)}\,, \quad \text{and } c \text{ is some constant}\,.$$

Equation (3.7) is called an *implicit* solution of (3.6).

Remark. The separated differential form gives a good recipe by which to construct equation (3.7), but looks artificial. Fortunately, it is not needed to show that every solution of the differential equation (3.6) also satisfies equation (3.7) for some c. Indeed, if $Y(t)$ solves (3.6) then whenever $g(Y(t)) \neq 0$ we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\big(F(t) - G(Y(t))\big) = F'(t) - G'(Y(t))Y'(t) = f(t) - \frac{1}{g(Y(t))}\, Y'(t) = 0\,.$$

This implies that $F(t) - G(Y(t)) = c$ for some constant c. Therefore $Y(t)$ also satisfies equation (3.7) with this c.

We call (3.7) an implicit solution of (3.6) because if we solve equation (3.7) for y as a differentiable function of t then the result will be a solution of (3.6) whenever $g(y) \neq 0$. Indeed, suppose that $Y(t)$ is differentiable and satisfies

$$F(t) = G\big(Y(t)\big) + c\,.$$

Upon differentiating both sides of this equation with respect to t we see that

$$f(t) = F'(t) = G'\big(Y(t)\big)\,\frac{\mathrm{d}Y(t)}{\mathrm{d}t} = \frac{1}{g\big(Y(t)\big)}\,\frac{\mathrm{d}Y(t)}{\mathrm{d}t}\,, \qquad \text{wherever } g\big(Y(t)\big) \neq 0\,.$$

It follows that $y = Y(t)$ satifies the differential equation (3.6).

Being able to solve (3.7) for y means finding an inverse function of G — namely, a function G^{-1} with property that

$$G^{-1}\big(G(y)\big) = y \quad \text{for every } y \text{ in some interval within the domain of } G\,.$$

For every such an inverse function, a family of explicit solutions to (3.6) is then given by

$$y = G^{-1}\big(F(t) - c\big)\,. \tag{3.8}$$

As we will see in examples below, such a solution may not exist for every value of $F(t) - c$, or there may be more than one solution.

Remark. This recipe will fail to yield a family of explicit solutions to (3.6) if either we are unable to find expressions for the primitives $F(t)$ and $G(y)$ in order to obtain (3.7), or we are unable find an explicit inverse function of $G(y)$ in order obtain (3.8) from (3.7).

Example. Find all solutions of

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{x}{y}\,.$$

Solution. This equation is separable. Because its right-hand side is undefined when $y = 0$, no solution can take the value 0. Because there is no value of y that makes its right-hand side vanish, it has no stationary points. Its separated differential form is

$$2y\,\mathrm{d}y = -2x\,\mathrm{d}x\,.$$

(The factors of 2 make our primitives nicer.) Its solutions are given implicitly by

$$y^2 = -x^2 + c\,, \qquad \text{for any constant } c\,.$$

This can be solved explicitly to obain the families

$$y = \pm\sqrt{c - x^2}\,, \qquad \text{for any constant } c\,.$$

These are solutions provided $c - x^2 > 0$, which implies that $c > 0$. For every $c > 0$ each of these families gives a solution whose interval of definition is $(-\sqrt{c}, \sqrt{c})$.

Example. Find all solutions of

$$\frac{\mathrm{d}z}{\mathrm{d}x} = \frac{3x + xz^2}{z + x^2 z}\,.$$

Solution. This equation is separable. Because its right-hand side is undefined when $z = 0$, no solution can take the value 0. Because there is no value of z that makes its right-hand side vanish, it has no stationary points. Its separated differential form is

$$\frac{2z}{3 + z^2}\,\mathrm{d}z = \frac{2x}{1 + x^2}\,\mathrm{d}x\,.$$

(The factors of 2 make our primitives nicer.) Then because

$$F(x) = \int \frac{2x}{1+x^2}\,\mathrm{d}x = \log(1+x^2) + c_1\,, \quad G(z) = \int \frac{2z}{3+z^2}\,\mathrm{d}z = \log(3+z^2) + c_2\,,$$

its solutions are given implicitly by

$$\log(3+z^2) = \log(1+x^2) - c\,, \qquad \text{for any constant } c\,.$$

By exponentiating both sides of this equation we obtain

$$3 + z^2 = (1+x^2)e^{-c}\,,$$

which yields the families

$$z = \pm\sqrt{(1+x^2)e^{-c} - 3}\,, \qquad \text{for any constant } c\,.$$

These are solutions provided $(1+x^2)e^{-c} - 3 > 0$, or equivalently, provided

$$x^2 > 3e^c - 1\,.$$

For $c < -\log(3)$ each of the above families gives a single solution whose interval of definition is $(-\infty, \infty)$. For $c \geq -\log(3)$ each of the families gives two solutions, one whose interval of definition is $(-\infty, -\sqrt{3e^c - 1})$, and one whose interval of definition is $(\sqrt{3e^c - 1}, \infty)$.

3.3. **Separable Initial-Value Problems.** In order to pick a unique solution from among all the solutions we impose an additional condition. As for linear equations, we do this by imposing an *initial condition* of the form $y(t_I) = y_I$, where t_I is called the *initial time* while y_I is called the *initial value*. Differential equation (3.6) combined with this initial condition is the *initial-value problem*

$$\frac{\mathrm{d}y}{\mathrm{d}t} = f(t)\,g(y)\,, \qquad y(t_I) = y_I\,. \tag{3.9}$$

There are two possibilities: either $g(y_I) = 0$ or $g(y_I) \neq 0$.

- If $g(y_I) = 0$ then it is clear that a solution of (3.9) is the stationary solution $y(t) = y_I$, which is defined for every t.
- If $g(y_I) \neq 0$ then we can try to use recipe (3.6) to obtain an implicit relation

$$F(t) = G(y) + c\,, \qquad \text{where} \quad F'(t) = f(t) \quad \text{and} \quad G'(y) = \frac{1}{g(y)}\,.$$

In the second case the initial condition implies that

$$F(t_I) = G(y_I) + c\,,$$

whereby $c = F(t_I) - G(y_I)$. The solution of the initial-value problem (3.9) thereby satisfies

$$F(t) = G(y) + F(t_I) - G(y_I)\,.$$

To find the explicit solution, we must solve this equation for y as a function of t. There may be more than one solution of this equation. If so, we must be sure to take the one that satisfies the initial condition. This means we have to find the inverse function of G that recovers y_I — namely a function G^{-1} with the property that

$$G^{-1}\big(G(y)\big) = y \quad \text{for every } y \text{ in an interval within the domain of } G \text{ that contains } y_I\,.$$

In particular, G^{-1} must satisfy $G^{-1}\big(G(y_I)\big) = y_I$. There is a unique inverse function with this property because

$$G'(y_I) = \frac{1}{g(y_I)} \neq 0\,.$$

The solution of the initial-value problem (3.9) is then given by

$$y = G^{-1}\big(G(y_I) + F(t) - F(t_I)\big)\,.$$

This will be valid for all times t in some open interval that contains the initial time t_I. The largest such interval is the interval of definition for the solution.

Example. Find the solution to the initial-value problem

$$\frac{\mathrm{d}y}{\mathrm{d}t} = y^2\,, \qquad y(0) = y_I\,.$$

Identify its interval of definition.

Solution. We see that $y = 0$ is the only stationary point of this equation. Hence, if $y_I = 0$ then $y(t) = 0$ is a stationary solution whose interval of definition is $(-\infty, \infty)$. So let us suppose that $y_I \neq 0$. The solution is given implicitly by

$$t = \int \frac{\mathrm{d}y}{y^2} = -\frac{1}{y} + c\,.$$

The initial condition then implies that

$$0 = -\frac{1}{y_I} + c\,,$$

whereby $c = 1/y_I$. Therefore the solution is given implicitly by

$$t = -\frac{1}{y} + \frac{1}{y_I}\,.$$

This may be solved for y explicitly by first noticing that

$$\frac{1}{y} = \frac{1}{y_I} - t = \frac{1 - y_I t}{y_I}\,,$$

and then taking the reciporical of each side to obtain

$$y = \frac{y_I}{1 - y_I t}\,. \tag{3.10}$$

This is graphed in Figure 3.2 below for $y_I = -2$ and for $y_I = 2$. It is clear from this formula that the solution ceases to exist when $t = 1/y_I$. Therefore if $y_I > 0$ then the interval of definition of the solution is $(-\infty, 1/y_I)$ while if $y_I < 0$ then the interval of definition of the solution is $(1/y_I, \infty)$. Notice that both of these intervals contain the initial time $t = 0$. Finally, notice that our explicit solution recovers the stationary solution when $y_I = 0$ even though it was derived assuming that $y_I \neq 0$.

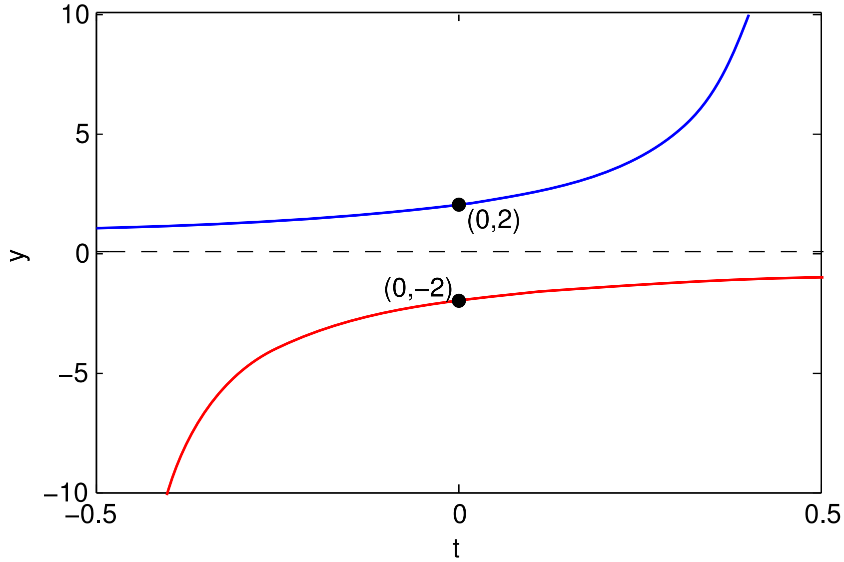

FIGURE 3.2. Solution of the initial-value problem $y' = y^2$, $y(0) = y_I$ for $y_I = -2$ and for y_I =2.

We might think that the blow-up seen in the last example had something to do with the fact that the stationary point $y = 0$ was bad for our recipe. However, the next example shows that blow-up happens even when there are no stationary points.

Example. Find the solution to the initial-value problem

$$\frac{\mathrm{d}y}{\mathrm{d}t} = 1 + y^2 \,, \qquad y(0) = y_I \,.$$

Give its interval of definition.

Solution. Because $1 + y^2 > 0$, we see there are no stationary points for this equation. Solutions are given implicitly by

$$t = \int \frac{\mathrm{d}y}{1 + y^2} = \tan^{-1}(y) + c \,.$$

The initial condition then implies that $0 = \tan^{-1}(y_I) + c$, whereby

$$c = -\tan^{-1}(y_I) \,.$$

Here we adopt the usual convention that $-\frac{\pi}{2} < \tan^{-1}(y_I) < \frac{\pi}{2}$. Therefore the solution is given implicitly by

$$t = \tan^{-1}(y) - \tan^{-1}(y_I) \,.$$

This may be solved for y explicitly by first noticing that

$$\tan^{-1}(y) = t + \tan^{-1}(y_I) \,,$$

and then taking the tangent of both sides to obtain

$$y = \tan\bigl(t + \tan^{-1}(y_I)\bigr) \,.$$

This solution is graphed for $y_I = 1$ in Figure 3.3 below. Because tan becomes undefined at $-\frac{\pi}{2}$ and $\frac{\pi}{2}$, the interval of definition for this solution is $(-\frac{\pi}{2} - \tan^{-1}(y_I), \frac{\pi}{2} - \tan^{-1}(y_I))$. Notice that this interval contains the initial time $t = 0$ because $-\frac{\pi}{2} < \tan^{-1}(y_I) < \frac{\pi}{2}$.

Remark. We can use the "tangent of a sum" identity to express this solution as

$$y = \frac{y_I + \tan(t)}{1 - y_I \tan(t)}, \qquad \text{for } -\tfrac{\pi}{2} - \tan^{-1}(y_I) < t < \tfrac{\pi}{2} - \tan^{-1}(y_I).$$

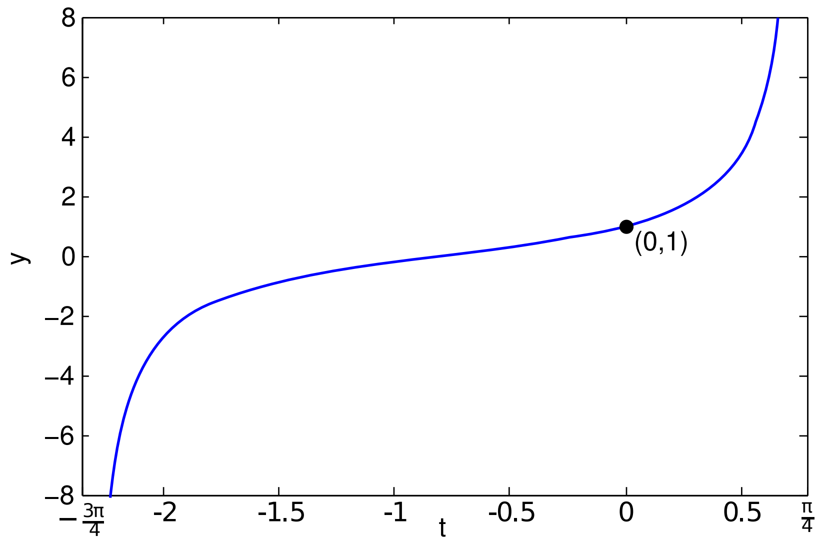

FIGURE 3.3. Solution to the initial-value problem $y' = 1 + y^2$, $y(0) = 1$.

The next example shows another way solutions of nonlinear equations can break down.

Example. Find the solution to the initial-value problem

$$\frac{\mathrm{d}y}{\mathrm{d}t} = \frac{1}{2y}, \qquad y(0) = y_I \neq 0.$$

Give its interval of definition.

Solution. This differential equation is autonomous. Because its right-hand side does is undefined when $y = 0$, no solution may take that value. In particuluar, we require that $y_I \neq 0$. Because $1/(2y) \neq 0$, we see there are no stationary points for this equation. Therefore solutions are given implicitly by

$$t = \int 2y \, \mathrm{d}y = y^2 + c.$$

The initial condition then implies that

$$0 = y_I^2 + c,$$

whereby $c = -y_I^2$. Therefore the solution is given implicitly by

$$t = y^2 - y_I^2.$$

This may be solved for y explicitly by first noticing that

$$y^2 = t + y_I^2 \,,$$

and then taking the square root of both sides to obtain

$$y = \pm\sqrt{t + y_I^2} \,.$$

We must then choose the sign of the square root so that the solution agrees with the initial data — i.e. positive when $y_I > 0$ and negative when $y_I < 0$. In either case we obtain

$$y = \operatorname{sign}(y_I)\sqrt{t + y_I^2} \,,$$

where we define

$$\operatorname{sign}(x) = \begin{cases} 1 & \text{if } x > 0 \,, \\ 0 & \text{if } x = 0 \,, \\ -1 & \text{if } x < 0 \,. \end{cases}$$

This solution is graphed for various values of y_I in Figure 3.4 below. Finally, notice that to keep the argument of the square root positive we must require that $t > -y_I^2$. Therefore the interval of definition for this solution is $(-y_I^2, \infty)$.

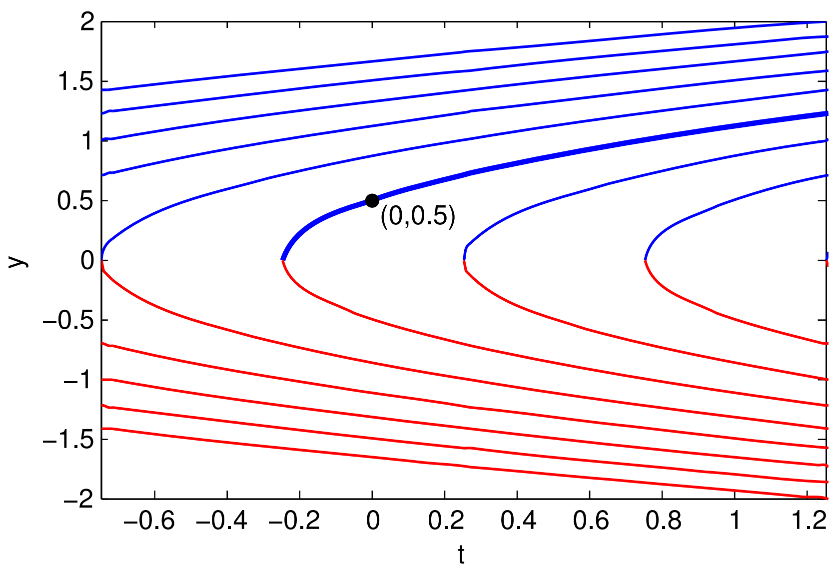

Figure 3.4. Solutions to the inital-value problem $\frac{dy}{dt} = \frac{1}{2y}$, $y(0) = y_I$ are shown for various values of y_I. The positive families are shown in blue and the negative shown in red. The solution with initial value $y_I = 0.5$ is highlighted. Its interval of definition is $(-.25, \infty)$.

Remark. In the above example the solution does not blow-up as t approaches $-y_I^2$. Indeed, the solution $y(t)$ approaches 0 as t approaches $-y_I^2$. However, the derivative of the solution is given by

$$\frac{\mathrm{d}y}{\mathrm{d}t} = \frac{1}{2y} = \frac{\mathrm{sign}(y_I)}{2\sqrt{t + y_I^2}}\,,$$

which does blow-up as t approaches $-y_I^2$. This happens because the solution approaches the value 0 where the right-hand side of the equation is not defined.

Example. Find the solution to the initial-value problem

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{x}{y}\,, \qquad y(3) = -4\,.$$

Give its interval of definition.

Solution. This equation is separable. Its right-hand side is undefined when $y = 0$. It has no stationary points. Earlier we showed that its solutions are given implicitly by

$$y^2 = -x^2 - c\,, \qquad \text{for some constant } c\,.$$

Members of this family of curves corresponding to various values of c are shown in Figure 3.4 below. The initial condition then implies that $(-4)^2 = -3^2 - c$, from which we solve to find that $c = -(-4)^2 - 3^2 = -16 - 9 = -25$. It follows that

$$y = -\sqrt{25 - x^2}\,,$$

where the negative square root is needed to satisfy the initial condition. This is a solution when $25 - x^2 > 0$, so its interval of definition is $(-5, 5)$. It is highlighed in Figure 3.5.

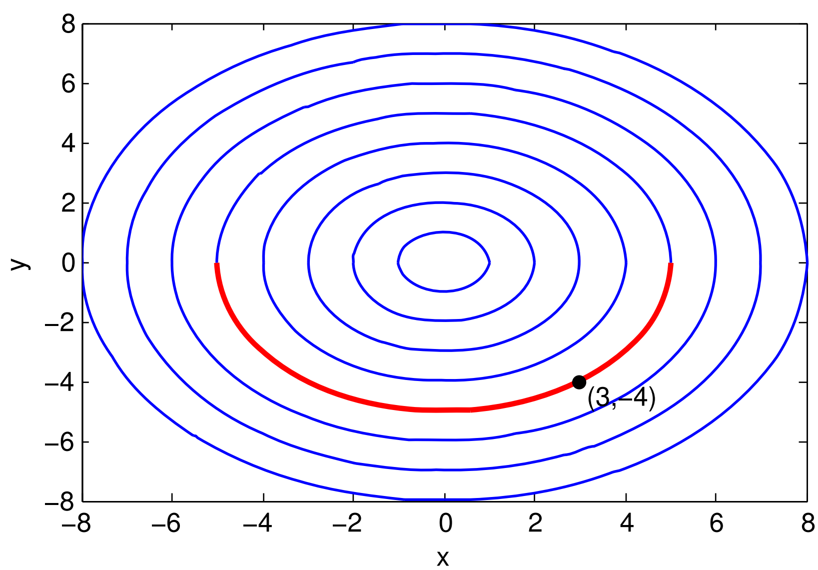

FIGURE 3.5. Solutions to $\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{x}{y}$ are shown as a family of implicit curves. The solution satisfying the initial condition $y(3) = -4$ is highlighted. Its interval of definition is seen to be $(-5, 5)$.

Example. Find the solution to the initial-value problem

$$\frac{\mathrm{d}z}{\mathrm{d}x} = \frac{3x + xz^2}{z + x^2 z}\,, \qquad z(1) = -3\,.$$

Give its interval of definition.

Solution. This equation is separable. Its right-hand side is undefined when $z = 0$. It has no stationary points. Earlier we showed that its solution is given implicitly by

$$\log(3 + z^2) = \log(1 + x^2) - c\,, \qquad \text{for some constant } c\,.$$

Members of this family of curves corresponding to various values of c are shown in Figure 3.6 below. The initial condition then implies that

$$\log\big(3 + (-3)^2\big) = \log\big(1 + 1^2\big) - c\,,$$

which gives $c = \log(2) - \log(12) = -\log(6)$. It follows that

$$\log(3 + z^2) = \log(1 + x^2) + \log(6) = \log\big((1 + x^2)\,6\big) = \log(6 + 6x^2)\,,$$

whereby $3 + z^2 = 6 + 6x^2$. Upon solving this for z we obtain

$$z = -\sqrt{3 + 6x^2}\,,$$

where the negative square root is needed to satisfy the initial condition. This is a solution when $3 + 6x^2 > 0$, so its interval of definition is $(-\infty, \infty)$. It is highlighed in Figure 3.6.

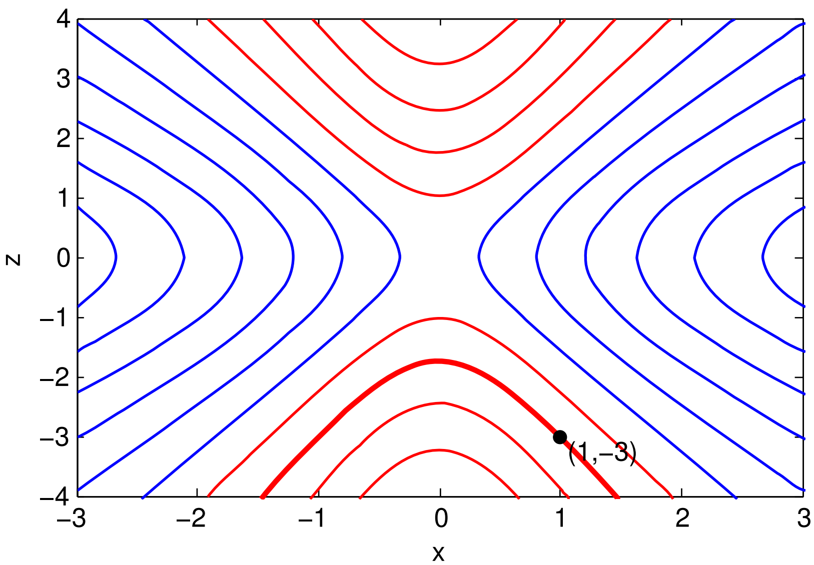

FIGURE 3.6. Solutions to $\frac{\mathrm{d}z}{\mathrm{d}x} = \frac{3x+xz^2}{z+x^2z}$ are shown for various c. Here $c < -\log(3)$ is shown in red and $c \geq -\log(3)$ is shown in blue. The solution satisfying the initial condition $z(1) = -3$ is highlighted.

3.4. **Theory for Separable Equations.** Up until now we have mentioned that we must be careful to check that the nonstationary solutions obtained from recipe (3.8) do not hit any of the stationary points, but we have not said why this leads to trouble. The next example illustrates the difficulty that arises.

Example. Find all solutions to the initial-value problem

$$\frac{\mathrm{d}y}{\mathrm{d}t} = 3y^{\frac{2}{3}}, \qquad y(0) = 0.$$

Solution. We see that $y = 0$ is a stationary point of this equation. Therefore $y(t) = 0$ is a stationary solution whose interval of definition is $(-\infty, \infty)$. However, let us carry out our recipe for nonstationary solutions to see where it leads. These solutions are given implicitly by

$$t = \int \frac{\mathrm{d}y}{3y^{\frac{2}{3}}} = \int \tfrac{1}{3} y^{-\frac{2}{3}} \,\mathrm{d}y = y^{\frac{1}{3}} + c.$$

Upon solving this for y we find $y = (t-c)^3$ where c is an arbitrary constant. Notice that each of these solutions hits the stationary point when $t = c$. The initial condition then implies that $0 = (0-c)^3 = -c^3$, whereby $c = 0$. We thereby have found two solutions of the initial-value problem: $y(t) = 0$ and $y(t) = t^3$.

In fact, as we will now show, there are many more solutions of the initial-value problem. Let a and b be any two numbers such that $a \leq 0 \leq b$ and define $y(t)$ by

$$y(t) = \begin{cases} (t-a)^3 & \text{for } t < a, \\ 0 & \text{for } a \leq t \leq b, \\ (t-b)^3 & \text{for } b < t. \end{cases}$$

We can understand such functions better by looking at their graph in Figure 3.7 below. It is clearly a differentiable function with

$$\frac{\mathrm{d}y}{\mathrm{d}t}(t) = \begin{cases} 3(t-a)^2 & \text{for } t < a, \\ 0 & \text{for } a \leq t \leq b, \\ 3(t-b)^2 & \text{for } b < t, \end{cases}$$

whereby it clearly satisfies the initial-value problem. Its interval of definition is $(-\infty, \infty)$. When $a = b = 0$ this reduces to $y(t) = t^3$.

Similarly, for every $a \leq 0$ we can construct the solution

$$y(t) = \begin{cases} (t-a)^3 & \text{for } t < a, \\ 0 & \text{for } a \leq t, \end{cases}$$

while for every $b \geq 0$ we can construct the solution

$$y(t) = \begin{cases} 0 & \text{for } t \leq b, \\ (t-b)^3 & \text{for } b < t. \end{cases}$$

The interval of definition for each of these solutions is also $(-\infty, \infty)$.

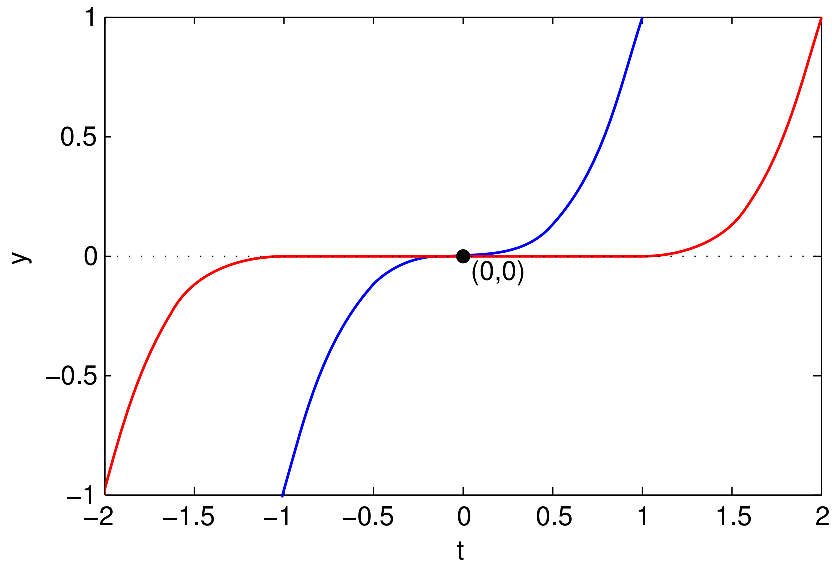

FIGURE 3.7. Illustration of nonuniqueness of solutions for the initial-value problem $y' = 3y^{2/3}$, $y(0) = 0$. The solution $y = t^3$ is shown in blue while the solution given above with $a = -1$ and $b = 1$ is shown in red.

Remark. The above example shows a very important difference between nonlinear and linear equations. Specifically, it shows that for nonlinear equations an initial-value problem may not have a unique solution.

The nonuniqueness seen in the previous example arises because $g(y) = 3y^{\frac{2}{3}}$ does not behave nicely at the stationary point $y = 0$. It is clear that g is continuous at 0, but because $g'(y) = 2y^{-\frac{1}{3}}$ we see that g is not differentiable at 0. The following fact states the differentiablity of g is enough to ensure that the solution of the initial-value problem exists and is unique.

Theorem 3.1. *Let $f(t)$ and $g(y)$ be functions defined over the open intervals (t_L, t_R) and (y_L, y_R) respecively such that*

- *f is continuous over (t_L, t_R),*
- *g is continuous over (y_L, y_R),*
- *g is differentiable at each of its zeros in (y_L, y_R).*

Then for every inital time t_I in (t_L, t_R), and every initial value y_I in (y_L, y_R) there exists a unique solution $y = Y(t)$ to the initial-value problem

$$\frac{\mathrm{d}y}{\mathrm{d}t} = f(t)\, g(y)\,, \qquad y(t_I) = y_I\,, \tag{3.11}$$

that is defined over the largest time interval (a, b) such that

- *t_I is in (a, b),*
- *$\{(t, Y(t)) : t \in (a, b)\}$ lies within the rectangle $(t_L, t_R) \times (y_L, y_R)$.*

Moreover, this solution is continuously differentiable and is determined by our recipe. This means that either $g(y_I) = 0$ and $Y(t) = y_I$ is a stationary solution, or $g(y_I) \neq 0$

and $Y(t)$ is a nonstationary solution that satisfies

$$G\big(Y(t)\big) = F(t)\,, \qquad Y(t_I) = y_I\,,$$

where $F(t)$ is defined for every t in (t_L, t_R) by the definite integral

$$F(t) = \int_{t_I}^{t} f(s)\,\mathrm{d}s\,,$$

while $G(y)$ is defined by the definite integral

$$G(y) = \int_{y_I}^{y} \frac{1}{g(x)}\,\mathrm{d}x\,,$$

whenever the point y is in (y_L, y_R) and neither y nor any point between y and y_I is a stationary point of g.

In particular, if f is continuous over $(-\infty, \infty)$ while g is differentiable over $(-\infty, \infty)$ then the initial-value problem (3.11) has a unique solution that either exists for all time or "blows up" in a finite time. Moreover, this solution is continuously differentiable and is determined by our recipe. We saw this "blow up" behavior in the examples above with $g(y) = y^2$ and $g(y) = 1 + y^2$. Indeed, it can be seen whenever $g(y)$ is a polynomial of degree two or more.

Remark. The above theorem implies that if the initial point y_I lies between two stationary points within (y_L, y_R) then the solution $Y(t)$ exists for all t in (t_L, t_R). This is because the uniqueness assertion implies $Y(t)$ cannot cross any stationary point, and therefore is trapped within (y_L, y_R). In particular, if g is differentiable over $(-\infty, \infty)$ then the only solutions that might "blow up" in a finite time are those that are not trapped above and below by stationary points.

Example. For $y' = y^2$ the only stationary point is $y = 0$. Because $g(y) = y^2 > 0$ when $y \neq 0$ we see that every nonstationary solution $Y(t)$ will be an increasing function of t. This fact is verified by formula (3.10) that we derived for the initial condition $y(0) = y_I$,

$$Y(t) = \frac{y_I}{1 - y_I t}\,.$$

When $y_I > 0$ the interval of definition is $(-\infty, 1/y_I)$ and we see that $Y(t) \to +\infty$ as $t \to 1/y_I$ while $Y(t) \to 0$ as $t \to -\infty$. In this case the solution is trapped below as $t \to -\infty$ by the stationary point $y = 0$. Similarly, when $y_I < 0$ the interval of definition is $(1/y_I, \infty)$ and we see that $Y(t) \to -\infty$ as $t \to 1/y_I$ while $Y(t) \to 0$ as $t \to \infty$. In this case the solution is trapped above as $t \to \infty$ by the stationary point $y = 0$. Figure 3.2 shows these solutions for the initial values $y_I = -2$ and $y_I = 2$.

Example. For $y' = 4y - y^3$ the stationary points are $y = -2$, $y = 0$, and $y = 2$. Explicit formulas for the nonstationary solutions in terms of an arbitary constant c are given by equation (3.5). For the initial condition $y(0) = y_I$ we obtain

$$c = -\frac{1}{8}\log\left(\frac{y_I^2}{|4 - y_I^2|}\right)\,.$$

Typical nonstationary solution are plotted in Figure 3.1. That figure shows that each nonstationary solution $Y(t)$ is trapped within one of the intervals $(-\infty, -2)$, $(-2, 0)$,

$(0,2)$, or $(2,\infty)$. Notice that for the solutions trapped within either $(-\infty,-2)$ or $(2,\infty)$ we have $y_I^2 > 4$, whereby $c < 0$.

- If $Y(t)$ lies within $(-\infty,-2)$ then its interval of definition is (c,∞). Moreover, $Y(t)$ is increasing with
$$\lim_{t\to c} Y(t) = -\infty\,, \qquad \lim_{t\to\infty} Y(t) = -2\,.$$
- If $Y(t)$ lies within $(-2,0)$ then its interval of definition is $(-\infty,\infty)$. Moreover, $Y(t)$ is decreasing with
$$\lim_{t\to-\infty} Y(t) = 0\,, \qquad \lim_{t\to\infty} Y(t) = -2\,.$$
- If $Y(t)$ lies within $(0,2)$ then its interval of definition is $(-\infty,\infty)$. Moreover, $Y(t)$ is increasing with
$$\lim_{t\to-\infty} Y(t) = 0\,, \qquad \lim_{t\to\infty} Y(t) = 2\,.$$
- If $Y(t)$ lies within $(2,\infty,)$ then its interval of definition is (c,∞). Moreover, $Y(t)$ is decreasing with
$$\lim_{t\to c} Y(t) = \infty\,, \qquad \lim_{t\to\infty} Y(t) = 2\,.$$

Remark. Even in such cases where we cannot find an explicit inverse function of $G(y)$ we often can determine the interval of definition of the solution directly from the equation

$$G(y) = F(t)\,, \quad \text{where} \quad F(t) = \int_{t_I}^{t} f(s)\,\mathrm{d}s\,, \quad G(y) = \int_{y_I}^{y} \frac{1}{g(x)}\,\mathrm{d}x\,.$$

For example, if $g(y) > 0$ over $[y_I,\infty)$ then $G(y)$ will be increasing over $[y_I,\infty)$ and the solution $Y(t)$ will be defined over the largest interval (a,b) such that t_I is in (a,b), (a,b) is contained within (t_L,t_R), and

$$F(t) < \lim_{y\to+\infty} G(y)\,.$$

If the above limit is finite and equal to $F(b)$ then the solution "blows up" as $t\to b^-$.

Similarly, if $g(y) > 0$ over $(-\infty,y_I]$ then $G(y)$ will be increasing over $(-\infty,y_I]$ and the solution $Y(t)$ will be defined over the largest interval (a,b) such that t_I is in (a,b), (a,b) is contained within (t_L,t_R), and

$$\lim_{y\to-\infty} G(y) < F(t)\,.$$

If the above limit is finite and equal to $F(b)$ then the solution "blows down" as $t\to b^-$.

The same kind of analysis can be carried out to determine the times at which a solution approaches a forbidden value. Two such examples are given in the next section.

3.5. **Analysis of Implicit Solutions.** Even when we cannot obtain explicit solutions by this recipe, often we can use the implicit solution to describe how all solutions behave and to determine their intervals of definition. We begin with an autonomous example.

Example. Solve the general initial-value problem

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{e^x}{1+x}, \qquad x(0) = x_I \neq -1.$$

Solution. The differential equation is autonomous. Its right-hand side is undefined when $x = -1$. It has no stationary point. Its separated differential form is

$$(1+x)e^{-x}\,\mathrm{d}x = \mathrm{d}t.$$

One integration-by-parts yields

$$\int (1+x)e^{-x}\,\mathrm{d}x = -(1+x)e^{-x} + \int e^{-x}\,\mathrm{d}x = -(2+x)e^{-x} + c,$$

so that the solutions of the differential equation satisfy

$$t = -(2+x)e^{-x} + c.$$

The initial condition implies $0 = -(2+x_I)e^{-x_I} + c$, whereby

$$c = (2+x_I)e^{-x_I}.$$

Therefore the solution of the initial-value problem satisfies

$$t - (2+x_I)e^{-x_I} = -(2+x)e^{-x}.$$

This equation cannot be solved for x to obtain explicit solutions.

Let $G(x) = -(2+x)e^{-x}$, so that the solution of the initial-value problem satisfies

$$t - c = G(x), \qquad \text{where } c = -G(x_I).$$

Because $G'(x) = (1+x)e^{-x}$, it can be seen that $G'(x) > 0$ for $x > -1$ while $G'(x) < 0$ for $x < -1$. This sign analysis shows that $G(x)$ has a minimum value of $G(-1) = -e$ while

$$\lim_{x\to-\infty} G(x) = +\infty, \qquad \lim_{x\to+\infty} G(x) = 0.$$

Because $x = -1$ is forbidden, there will be no solution unless $t - c > G(-1) = -e$.

If we sketch a graph of $G(x) = -(2+x)e^{-x}$ then we can see the following.

- The equation $G(x) = t-c$ has one solution with $x > -1$ provided $-e < t-c < 0$. This gives the solution of the initial-value problem when $x_I > -1$. It will be an increasing function $X(t)$ with interval of definition $(c-e, c)$ such that

$$\lim_{t\to(c-e)^+} X(t) = -1, \qquad \lim_{t\to c^-} X(t) = \infty.$$

- The equation $G(x) = t-c$ has one solution with $x < -1$ provided $-e < t-c < \infty$. This gives the solution of the initial-value problem when $x_I < -1$. It will be a decreasing function $X(t)$ with interval of definition $(c-e, \infty)$ such that

$$\lim_{t\to(c-e)^+} x(t) = -1, \qquad \lim_{t\to\infty} x(t) = -\infty.$$

All of this can be seen from the inverted plot of $G(x) = -(2+x)e^{-x}$ shown below.

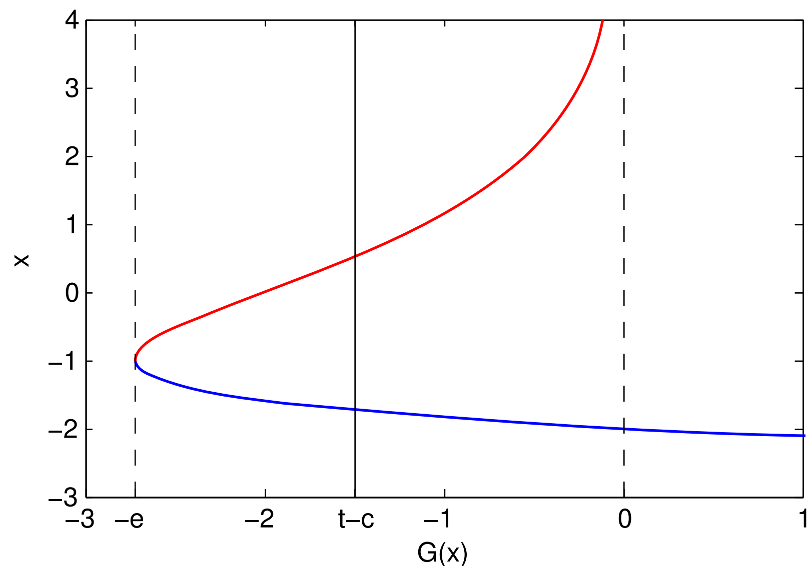

FIGURE 3.8. Inverted plot of function $G(x) = -(2+x)e^{-x}$. The part of the curve with $x > -1$ is shown in red, while $x < -1$ is shown in blue.

A similar analysis can be carried out for separable equations, but more care is required to determine the interval of definition.

Example. Solve the general initial-value problem

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{e^x \cos(t)}{1+x}\,, \qquad x(t_I) = x_I \neq -1\,.$$

Solution. The differential equation is separable. Its right-hand side is undefined when $x = -1$. It has no stationary point. Its separated differential form is

$$(1+x)e^{-x}\,\mathrm{d}x = \cos(t)\,\mathrm{d}t\,.$$

Because

$$\int (1+x)e^{-x}\,\mathrm{d}x = -(2+x)e^{-x} + c_1\,, \qquad \int \cos(t)\,\mathrm{d}t = \sin(t) + c_2\,,$$

the solutions of the differential equation satisfy

$$\sin(t) = -(2+x)e^{-x} + c\,.$$

The initial condition implies $\sin(t_I) = -(2+x_I)e^{-x_I} + c$, whereby

$$c = \sin(t_I) + (2+x_I)e^{-x_I}\,.$$

Therefore the solution of the initial-value problem satisfies

$$\sin(t) - \sin(t_I) - (2+x_I)e^{-x_I} = -(2+x)e^{-x}\,.$$

This equation cannot be solved for x to obtain explicit solutions.

Let $G(x) = -(2+x)e^{-x}$, so that the solution of the initial-value problem satisfies

$$\sin(t) - c = G(x)\,, \qquad \text{where } c = \sin(t_I) - G(x_I)\,.$$

This is the same $G(x)$ we analyzed in the previous example. From the graph of $G(x)$ produced by that analysis we conclude the following.

- The equation $G(x) = \sin(t) - c$ has one solution with $x > -1$ when $-e < \sin(t) - c < 0$. This solves the initial-value problem when $x_I > -1$. There are three possibilities.
  - If $-1 < c \leq 1$ then the solution $X(t) > -1$ exists for so long as $\sin(t) < c$. Its interval of definition is $\big(-\pi - \sin^{-1}(c), \sin^{-1}(c)\big)$, where $\sin^{-1}(c)$ is chosen so the interval contains t_I. We have $X(t) \to \infty$ as t approaches either endpoint.
  - If $1 < c < e - 1$ then the solution $X(t) > -1$ exists for all time and is oscillatory because $-e < \sin(t) - c < 0$ for every t. Its interval of definition is $(-\infty, \infty)$.
  - If $e - 1 \leq c < e + 1$ then the solution $X(t) > -1$ exists for so long as $c - e < \sin(t)$. Its interval of definition is $\big(\sin^{-1}(c-e), \pi - \sin^{-1}(c-e)\big)$, where $\sin^{-1}(c-e)$ is chosen so the interval contains t_I. We have $X(t) \to -1$ as t approaches either endpoint.
- The equation $G(x) = \sin(t) - c$ has one solution with $x < -1$ when $-e < \sin(t) - c$. This solves the initial-value problem when $x_I < -1$. There are two possibilites.
  - If $c < e-1$ then the solution $X(t) < -1$ exists for all time and is oscillatory because $-e < \sin(t) - c$ for every t. Its interval of definition is $(-\infty, \infty)$.
  - If $e - 1 \leq c < e + 1$ then the solution $X(t) < -1$ exists for so long as $c - e < \sin(t)$. Its interval of definition is $\big(\sin^{-1}(c-e), \pi - \sin^{-1}(c-e)\big)$, where $\sin^{-1}(c-e)$ is chosen so this interval contains t_I. We have $X(t) \to -1$ as t approaches either endpoint.

Exercises on Separable Equations

For #1 - 12 determine if the equation is separable, linear, or neither. If the function is separable or linear find a general solution. If you can make this solution explicit, then do so.

(1) $y' = yt$
ue
Short Answer
Solution

(2) $ty' = y + 1$
ue
Short Answer
Solution

(3) $t\dot{y} - y^2 \sin(t) = e^t$
ue
Short Answer
Solution

(4) $y' = \frac{2t}{y^4+4y^3}$
ue
Short Answer
Solution

(5) $y\dot{y} + \frac{\log(t)}{t} = 1.$
ue
Short Answer
Solution

(6) $\dot{\theta} = \frac{\theta+t}{\theta+1}$
ue
Short Answer
Solution

(7) $x' + \frac{x^2}{\tan(t)} = 0$
ue
Short Answer
Solution

(8) $t\dot{z} = e^z \log(t)$
ue
Short Answer
Solution

(9) $w'wt + \sin(tw) = 0$
ue
Short Answer
Solution

(10) $x^2y' + xy = x^3$
ue
Short Answer
Solution

(11) $\frac{dy}{dx} = \frac{x(e^{x^2}+1)}{3y^2}$
ue
Short Answer
Solution

(12)

$$y' - 3y = ty$$

Solve the above differential equation using two different methods.
ue

Short Answer
Solution

For #13- 22 find solutions to the given initial value problem. If you can find an explicit solution then do so.

(13) $y' + y^2 \cos(t) = 0$ where $y(0) = \frac{1}{2}$.
ue
Short Answer
Solution

(14) $\frac{1}{t^2} + \dfrac{y'}{\sqrt{y}} = 0;\ \ y(-1) = 1.$
ue
Short Answer
Solution

(15) $x' = e^{t+x}$ where $x(0) = 0$
ue
Short Answer
Solution

(16) $y' = \frac{\cos(t)}{5y^4+2y}$ where $y(0) = 2$
ue
Short Answer
Solution

(17) $u' = \frac{u^2+1}{t^2-1}$ where $u(0) = 1$
ue
Short Answer
Solution

(18) $e^t y' = \frac{y}{e^{-t}-1}$ where $y(1) = 1$
ue
Short Answer
Solution

(19) $ty' = y^3$ where $y(1) = y_0$. Your solution will depend on the value y_0, be sure you indicate how!
ue
Short Answer
Solution

(20) $y' = \frac{t}{y(1-t^2)};\quad y(0) = y_0$ Your solution will depend on the value y_0, be sure you indicate how!
ue
Short Answer
Solution

(21) $y' = \frac{2t}{\log(y)}$ where $y(0) = y_0$. Your solution will depend on the value y_0, be sure you indicate how!
ue
Short Answer
Solution

(22) $t\theta\theta' = 1$ where $\theta(1) = \theta_0$ Your solution will depend on the value θ_0, be sure you indicate how!
ue
Short Answer
Solution

In the following justify your responses.

(23) $v' = \frac{(v-1)^{\frac{1}{3}}}{1+t^2}$ where $v(0) = v_0$ Your solution will depend on the value y_0, be sure you indicate how! Then indicate which value of v_0 makes the solution not unique.

ue

Short Answer
Solution

(24) The goal of this problem is to analyze separable problems in a similar way to how we analyzed explicit equations. Recall from chapter 2 that an equation is explicit if it is of the form $y' = f(t)$. We solved such equations by applying the Fundamental Theorem of Calculus to integrate both sides and get an explicit solution for y.

Now consider the separable equation $\frac{dy}{dt} = \frac{f(t)}{g(y)}$ so that the equation separates as $g(y)dy = f(t)dt$.

(a) Suppose $G(y) + C_1 = \int g(y)dy$. Use implicit differentiation to find the derivative of $G(y)$ with respect to t.

(b) Suppose $F(t)+C_2 = \int f(t)dt$. What does the FTC tell you the derivative of $F(t)$ with respect to t is?

(c) Use (a), (b) and the FTC to justify why $G(y) = F(t) + C$ is an implicit general solution to the given separable equation.

ue

Solution

(25) Suppose we deposit \$1000 into a savings account with 5% interest compounded continuously. After 5 years we start taking \$200 out each year. How long will the money last?

ue

Short Answer
Solution

(26) What parameter values will guarantee that $x(t)$ will go to infinity as $t \to \infty$?

$$\frac{dx}{dt} = \alpha x(t - \beta)$$

ue

Short Answer
Solution

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