I. First-Order Ordinary Differential Equations
4. General Theory

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4. General Theory

So far we have used analytical methods to construct solutions of first-order differential equations and solve initial-value problems. These methods have required the equation to be either linear or separable. More generally, they can be applied to any equation that can be transformed into either of those forms. They also have required finding certain primitives. Later we will learn other analytic methods, but these also will require finding certain primitives. However, most first-order differential equations will lead to a primitive that cannot be found explicitly. Moreover, even when analytic methods do work, they can produce an expression that makes the behavior of the solution hard to see. Therefore it is helpful to know nonanalytic methods that can be applied to a broad class of first-order equations.

In subsequent chapters we will develop graphical and numerical methods that can be broadly applied. These methods require a little theoretical groundwork in order to explain when they can be applied.

4.1. **Well-Posed Problems.** The notion of a well-posed problem is central to science and engineering. It motivated by the idea that mathematical problems in science and engineering are used to predict or explain something. A problem is called *well-posed* if

(i) the problem has a solution,
(ii) the solution is unique,
(iii) the solution depends continuously upon the problem.

The motivations for the first two points are fairly clear: a problem with no solution will not give a prediction, and a problem with many solutions gives too many. The third point is crucial. It recognizes that mathematical problems are models of reality. Nearby models are equally valid. To have predictive value, the solutions of nearby models should lie close to each other. For example, if our model is an initial-value problem associated with a differential equation then its solution should not change much if the initial value is a bit different or if a coefficient in the differential equation is altered slightly. This is what is meant by saying the solution depends continuously upon the problem.

The solution of a well-posed problem can be approximated accurately by a wealth of techniques. The solution of a problem that is not well-posed is very difficult, if not impossible, to approximate accurately. This is why scientists and engineers want to know which problems are well-posed and which are not.

In this chapter we consider initial-value problems of the form

$$\frac{\mathrm{d}y}{\mathrm{d}t} = f(t, y)\,, \qquad y(t_I) = y_I\,. \tag{4.1}$$

We will give conditions on $f(t, y)$ that insure this problem has a unique solution. Later we will use this theory to develop methods by which we can study solutions of these problems when analytical methods either do not apply or become complicated.

4.2. **Existence and Uniqueness.** Here we will address only the existence and uniqueness of solutions. We begin with a definition that a picture should help clarify.

Definition 4.1. *Let S be a set in the ty-plane. A point (t_o, y_o) is said to be in the* interior *of S if there exists a rectangle $(t_L, t_R) \times (y_L, y_R)$ that contains the point (t_o, y_o) and also lies within the set S.*

Our basic existence and uniqueness theorem is the following.

Theorem 4.1. *Let $f(t, y)$ be a function defined over a set S in the ty-plane such that*

- *f is continuous over S,*
- *f is differentiable with respect to y over S,*
- *$\partial_y f$ is continuous over S.*

Then for every inital time t_I and every initial value y_I such that (t_I, y_I) is in the interior of S there exists a unique solution $y = Y(t)$ to initial-value problem (4.1) that is defined over the largest time interval (a, b) such that

- *t_I is in (a, b),*
- *$\{(t, Y(t)) : t \in (a, b)\}$ lies within the interior of S.*

Moreover, $Y'(t)$ is continuous over that interval.

Remark. This is not the most general theorem we could state, but it is one that applies to most equations you will face in this course, and is easy to apply. It asserts that $Y(t)$ will exist until $(t, Y(t))$ leaves S.

As the following examples show, applying this theorem is often a matter of simply checking that $f(t, y)$ and $\partial_y f(t, y)$ are continuous over the set S where they are defined. When this is the case, Theorem 4.1 ensures that for every initial data (t_I, y_I) in the interior of S the initial-value problem has a unique solution $y = Y(t)$ that is defined over some time interval (a, b) that contains t_I.

Example. Determine (t_I, y_I) for which a unique solution exists to the initial-value problem

$$\frac{\mathrm{d}y}{\mathrm{d}t} = \frac{\log(|y|)}{1 + t^2 - y^2}, \qquad y(t_I) = y_I .$$

Solution. Because $f(t, y) = \frac{\log(|y|)}{1+t^2-y^2}$ is defined everywhere except where $1 + t^2 - y^2 = 0$ or $y = 0$, we try taking S to be all points in the ty-plane except those points on the hyperbola $1 + t^2 - y^2 = 0$ (where $y = \pm\sqrt{1 + t^2}$) and those points on the t-axis (where $y = 0$). Clearly, f is continuous over S, f is differentiable with respect to y over S with

$$\partial_y f(t, y) = \frac{1}{1 + t^2 - y^2} \frac{1}{y} + \frac{\log(|y|)}{(1 + t^2 - y^2)^2} 2y ,$$

and $\partial_y f$ is continuous over S. Every point in S is also in the interior of S. Therefore Theorem 4.1 insures that for every (t_I, y_I) in S the initial-value problem has a unique solution $y = Y(t)$ that is defined over some time interval (a, b) that contains t_I. The solution extends to the largest interval (a, b) for which $(t, Y(t))$ remains within S. $\square$

Example. Determine (t_I, y_I) for which a unique solution exists to the initial-value problem

$$\frac{\mathrm{d}y}{\mathrm{d}t} = \sqrt{1 + t^2 + y^2}\,, \qquad y(t_I) = y_I\,.$$

Solution. Because $f(t, y) = \sqrt{1 + t^2 + y^2}$ is defined over $(-\infty, \infty) \times (-\infty, \infty)$, we try taking S to be the entire ty-plane. Clearly, f is continuous over S, f is differentiable with respect to y over S with

$$\partial_y f(t, y) = \frac{y}{\sqrt{1 + t^2 + y^2}}\,,$$

and $\partial_y f$ is continuous over S. Every point in S is also in the interior of S. Therefore Theorem 4.1 insures that for every (t_I, y_I) in the ty-plane the initial-value problem has a unique solution $y = Y(t)$ that is defined over some time interval (a, b) that contains t_I. Either the solution extends to all t or $Y(t)$ blows up in finite time because those are the only ways for $(t, Y(t))$ to leave S. (In fact, it extends to all t.) □

Remark. The initial-value problems in the above examples cannot be solved by analytic methods. However, Theorem 4.1 insures that their solutions exist and are unique. In subsequent chapters we will see how to visualize and how to approximate them.

Now let us apply this general theorem to the special cases that were treated in the previous two chapters — namely, linear and separable equations.

Example. Let (t_L, t_R) be an interval. Let $a(t)$ and $f(t)$ be continuous over (t_L, t_R). Let t_I be in (t_L, t_R) and y_I be any real number. Apply Theorem 4.1 to the linear initial-value problem

$$\frac{\mathrm{d}y}{\mathrm{d}t} = f(t) - a(t)y\,, \qquad y(t_I) = y_I\,.$$

Solution. Consider the set S in the ty-plane given by $S = (t_L, t_R) \times (-\infty, \infty)$. The right-hand side of the differential equation satisfies

- $f(t) - a(t)y$ is continuous over S,
- $f(t) - a(t)y$ is differentiable with respect to y over S,
- $\partial_y\big(f(t) - a(t)y\big) = -a(t)$ is continuous over S.

The point (t_I, y_I) is in the interior of S because the set S itself is a rectangle that contains the point (t_I, y_I). Therefore Theorem 4.1 implies that there exists a unique solution $y = Y(t)$ to linear initial-value problem that is defined over the largest time interval (a, b) such that

- t_I is in (a, b),
- $\{(t, Y(t)) \,:\, t \in (a, b)\}$ lies within S.

□

Remark. Notice that Theorem 4.1 concludes less about the interval of definition of the solution than Theorem 2.1 did. Theorem 2.1, which specialized to linear equations, concludes that the interval of definition of the solution is at least (t_L, t_R), whereas Theorem 4.1 concludes only that it exists. Moreover, Theorem 2.1 concludes that the solution is given by the explicit solution for linear equations, whereas Theorem 4.1 concludes only that the solution exists.

Example. Let (t_L, t_R) and (y_L, y_R) be an intervals. Let $f(t)$ be continuous over (t_L, t_R). Let $g(y)$ be differentiable over (y_L, y_R) and $g'(y)$ be continuous over (y_L, y_R). Let t_I be in (t_L, t_R) and y_I be in (y_L, y_R). Apply Theorem 4.1 to the separable initial-value problem

$$\frac{\mathrm{d}y}{\mathrm{d}t} = f(t)g(y)\,, \qquad y(t_I) = y_I\,.$$

Solution. Consider the set S in the ty-plane given by $S = (t_L, t_R) \times (y_L, y_R)$. The right-hand side of the differential equation satisfies

- $f(t)g(y)$ is continuous over S,
- $f(t)g(y)$ is differentiable with respect to y over S,
- $\partial_y\big(f(t)g(y)\big) = f(t)g'(y)$ is continuous over S.

The point (t_I, y_I) is in the interior of S because the set S itself is a rectangle that contains the point (t_I, y_I). Therefore Theorem 4.1 implies that there exists a unique solution $y = Y(t)$ to separable initial-value problem that is defined over the largest time interval (a, b) such that

- t_I is in (a, b),
- $\{(t, Y(t)) \,:\, t \in (a, b)\}$ lies within S.

$\square$

Remark. Notice that Theorem 4.1 puts more conditions on g than Theorem 3.1 did. It also concludes less. Theorem 3.1, which specialized to separable equations, did not assume that $g(y)$ is differentiable over (y_L, y_R). Rather, it assumed only that g is differentiable at its zeros. Moreover, Theorem 3.1 concludes that the solution satisfies the implicit solution for separable equations, whereas Theorem 4.1 concludes only that the solution exists.

Exercises on General Theory

In the first group of problems determine a set S satisfying the hypotheses of the Existence and Uniqueness Theorem posed in this section.

(1) $y' = \frac{1}{1-t^2}$; $y(0) = 1$
Solution

(2) $\dot{y} = \frac{1}{y-t^2}$; $y(0) = 1$
Solution

(3) $y' = \sqrt{yt}$; $y(1) = 1$
Solution

(4) $y' = \log(1 - t^2 - y^2)$; $y(0) = 0$
Solution

(5) $t\dot{y} = \sqrt{e^y - 1}$; $y(1) = 1$
Solution

(6) $yy' = \log(t)$; $y(1) = 1$
Solution

(7) $y' = \frac{y+t}{y^2-1}$; $y(0) = 0$
Solution

(8) $\frac{1}{t^2} + \frac{w'}{\sqrt{w}} = 0$; $w(-1) = 1$.
Solution

(9) $\dot{x} = \frac{\tan(t)}{x^2+2tx+t^2}$; $x(0) = 1$
Solution

(10) $\theta' + \log(t)\theta = \sqrt{4-t^2}$; $\theta(1) = 2$
Solution

(11) $\dot{y} = \frac{\sqrt{x^2-y^2}}{\log(y)}$; $y(-1) = 1/2$
Solution

(12) $x' = \frac{(\log(t))^2}{x^2-2x-3}$; $x(1) = 0$
Solution

For the next few problems be sure to justify your responses.

(13) A linear first order equation in normal form is given by $y' + p(t)y = q(t)$ with initial condition $y(t_I) = y_I$. Use the general existence and uniqueness theorem to justify why we can guarantee a unique solution when both p and q are continuous function on an open interval containing t_I.
Solution

(14) Suppose we have a separable equation $y' = \frac{g(t)}{h(y)}$ with initial condition $y(t_I) = y_I$. What conditions must we place on g, h, t_I and y_I for the solution to be unique?
Solution

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