I. First-Order Ordinary Differential Equations
5. Graphical Methods

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Contents

5. Graphical Methods
- 5.1. Phase-Line Portraits for Autonomous Equations
- 5.2. Plots of Explicit Solutions
- 5.3. Contour Plots of Implicit Solutions
  - 5.3.1. Rendered by Computer
  - 5.3.2. Critical Points and Values (not covered)
- 5.4. Direction Fields
  - 5.4.1. Rendered by Computer
  - 5.4.2. Sketched by Hand (not covered)

Exercises on Graphical Methods

Navigation to other Chapters

Return to Main Webpage

5. Graphical Methods

Sometimes the best way to understand the solution of a differential equation is by graphical methods. Of course sometimes these methods can be applied when analytic methods fail to yield explicit solutions. But even when analytic methods can yield explicit solutions, it is often better to gain some understanding of the solutions through a graphical method. Often we can find out everything we need to know graphically, thereby saving ourselves from a complicated analytic calculation.

5.1. Phase-Line Portraits for Autonomous Equations.

This method can be applied to autonomous equations of the form

$$\frac{\mathrm{d}y}{\mathrm{d}t} = g(y)\,. \tag{5.1}$$

It has the virtue that it can often be carried out quickly without the aid of a calculator or computer. It requires that g be continuous wherever it is defined over an interval (y_L, y_R) and that g be differentiable at each of its zeros in that interval. Then Theorem 4.1 asserts that every point of (y_L, y_R) has a unique solution of (5.1) passing through it.

Any solution $y = Y(t)$ of (5.1) can be viewed as giving the position of a point moving along the interval (y_L, y_R) as a function of time. We can determine the direction that this point moves as time increases from the sign of $g(y)$:

- where $g(y) = 0$ the point does not move because $Y'(t) = \frac{\mathrm{d}y}{\mathrm{d}t} = 0$,
- where $g(y) > 0$ the point moves to the right because $Y'(t) = \frac{\mathrm{d}y}{\mathrm{d}t} > 0$,
- where $g(y) < 0$ the point moves to the left because $Y'(t) = \frac{\mathrm{d}y}{\mathrm{d}t} < 0$.

We can present the sign analysis of $g(y)$ on a graph of the interval (y_L, y_R) as follows.

1. Identify all points where $g(y)$ is undefined. Theses points are often isolated. Check that $g(y)$ is continuous elsewhere. Plot these isolated points on the interval (y_L, y_R) with a $\circ$.
2. Find all the zeros of $g(y)$. These are the stationary points of the equation. They are usually isolated. Check that $g(y)$ is differentiable at each of them. Plot these isolated points on the interval (y_L, y_R) with a $\bullet$.
3. These undefined and stationary points partition (y_L, y_R) into subintervals. Determine the sign of $g(y)$ on each of these subintervals. Plot right arrows on each subinterval where $g(y)$ is positive and left arrows on each subinterval where $g(y)$ is negative. These arrows indicate the direction that solutions of the equation will move along (y_L, y_R) as time increases.

The resulting graph is called the *phase portrait* for equation (5.1) along the interval (y_L, y_R) of the *phase-line*. It gives a rather complete picture of how all solutions of (5.1) behave when they take values in (y_L, y_R).

Remark. Often $g(y)$ is differentiable everywhere it is defined. In that case it is continuous everywhere it is defined and is differentiable at each of its zeros.

Example. Describe the behavior of all solutions of the equation

$$\frac{\mathrm{d}y}{\mathrm{d}t} = 4y - y^3 \,.$$

Solution. This equation is autonomous and is defined and differentiable everywhere. Because $4y - y^3 = y(2+y)(2-y)$, the stationary points of this equation are $y = -2$, $y = 0$, and $y = 2$. To get a complete picture we should sketch a phase-line portrait over an interval that includes all of these points, say $(-4, 4)$. Clearly $g(y)$ is positive over $(-4, -2)$, negative over $(-2, 0)$, positive over $(0, 2)$, and negative over $(2, 4)$. Therefore the phase-line portrait is

$$\begin{array}{ccccccccc} & + & & - & & + & & - & \\ & \rightarrow\rightarrow\rightarrow\rightarrow & \bullet & \leftarrow\leftarrow\leftarrow\leftarrow & \bullet & \rightarrow\rightarrow\rightarrow\rightarrow & \bullet & \leftarrow\leftarrow\leftarrow\leftarrow & y \\ & & -2 & & 0 & & 2 & & \end{array}$$

It is clear from this phase-line portrait that:

- every solution $y(t)$ that initially lies within $(-\infty, -2)$ will move towards -2 as t increases with $y(t) \to -2$ as $t \to \infty$;
- every solution $y(t)$ that initially lies within $(-2, 0)$ will move towards -2 as t increases with $y(t) \to -2$ as $t \to \infty$;
- every solution $y(t)$ that initially lies within $(0, 2)$ will move towards 2 as t increases with $y(t) \to 2$ as $t \to \infty$;
- every solution $y(t)$ that initially lies within $(2, \infty)$ will move towards 2 as t increases with $y(t) \to 2$ as $t \to \infty$.

Of course, all of this information can be read off from the analytic general solution we had worked out earlier. But if this is all we wanted to know, sketching the phase portrait is certainly a faster way to get it.

Remark. The relationship between the above phase-line portrait and the sign of $g(y) = 4y - y^3$ can be seen from the graph of $g(y)$ over the interval $[-4, 4]$.

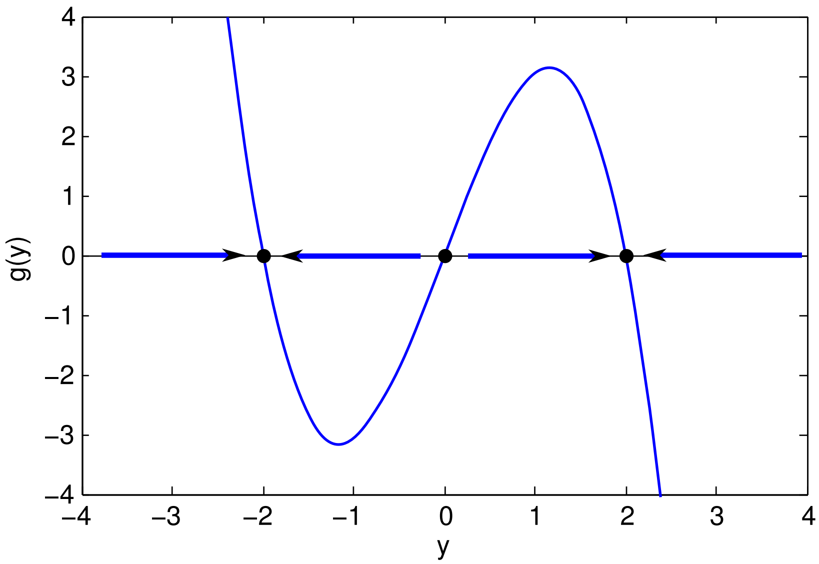


FIGURE 5.1. Plot of $g(y)$ vs y for $g(y) = 4y - y^3$, showing its relationship to the phase-line portrait.

However, as the above example illustrates, there is no need to sketch a graph $g(y)$ in order to sketch a phase-line portrait. The only information needed to sketch a phase-line portrait is the sign of $g(y)$, which is much less information than is needed to sketch a graph of $g(y)$.

Remark. The dynamics of $y(t)$ that we inferred from the above phase-line portrait can be visualized by plotting the graph of $y(t)$ for various initial data.

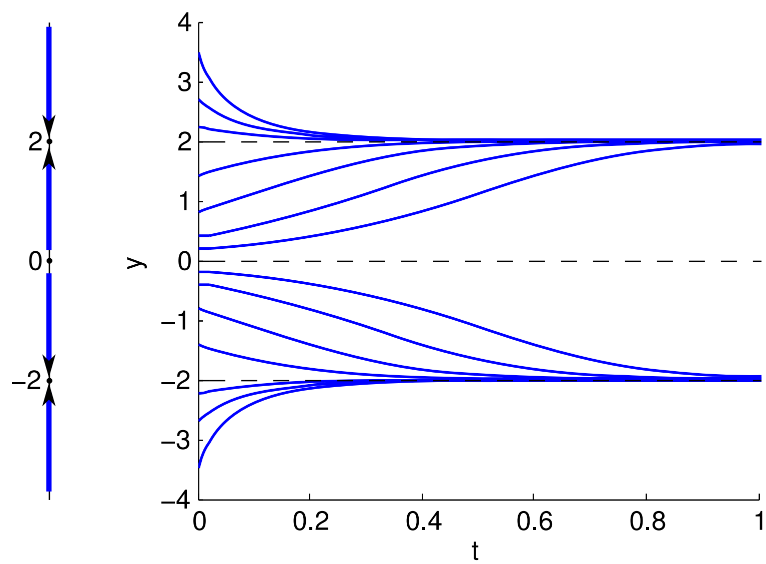


FIGURE 5.2. Several solutions of $\frac{dy}{dt} = 4y - y^3$ plotted versus time. The phase-line portrait corresponds to the vertical axis of this graph.

The phase-line portrait in the foregoing example shows that all solutions near -2 and 2 move towards them while all solutions near 0 move away from it. We say that the stationary points -2 and 2 are stable or attracting, while the stationary point 0 is unstable or repelling. A stationary point that has some solutions that move towards it and others that move away from it is called semistable. Phase portraits allow us to quickly classify the stability of every stationary point.

Example. Classify the stability of all the stationary points of

$$\frac{dx}{dt} = x(2-x)(4-x)^2 .$$

Describe the behavior of all solutions.

Solution. The stationary points of this equation are $x = 0$, $x = 2$, and $x = 4$. A sign analysis of $x(2-x)(4-x)^2$ shows that the phase portrait for this equation is

$$\begin{array}{ccccccc} - & & + & & - & & - \\ \leftarrow\leftarrow\leftarrow\leftarrow & \bullet & \rightarrow\rightarrow\rightarrow\rightarrow & \bullet & \leftarrow\leftarrow\leftarrow\leftarrow & \bullet & \leftarrow\leftarrow\leftarrow\leftarrow \; x \\ & 0 & & 2 & & 4 & \\ & \text{unstable} & & \text{stable} & & \text{semistable} & \end{array}$$

We thereby classify the stability of the stationary as indicated above. Moreover:

- every solution $x(t)$ that initially lies within $(-\infty, 0)$ will move towards $-\infty$ as t increases with $x(t) \to -\infty$ as $t \to t_*$, where t_* is some finite "blow-up" time;
- every solution $x(t)$ that initially lies within $(0, 2)$ will move towards 2 as t increases with $x(t) \to 2$ as $t \to \infty$;
- every solution $x(t)$ that initially lies within $(2, 4)$ will move towards 2 as t increases with $x(t) \to 2$ as $t \to \infty$;
- every solution $x(t)$ that initially lies within $(4, \infty)$ will move towards 4 as t increases with $x(t) \to 4$ as $t \to \infty$.

Perhaps the only one of these statements that might be a bit surprising is the last one. By just looking at the arrows on the phase-line you might have thought that such solutions would move past 4 and continue down to 2. But we have to remember that Theorem 4.1 tells us that nonstationary solutions never hit the stationary solutions. They merely approach them as $t \to \infty$.

Example. Sketch the phase-line portrait of the differential equation

$$\frac{dv}{dt} = \frac{(v^2-1)(v-3)^2}{(v^2+1)(v+3)^2}.$$

Classify each stationary point as being either stable, unstable, or semistable.

Solution. This equation is autonomous. Its right-hand side is undefined at $v = -3$ and is differentiable elsewhere. Its stationary points are found by setting

$$\frac{(v^2-1)(v-3)^2}{(v^2+1)(v+3)^2} = 0 .$$

Because $v^2 - 1 = (v+1)(v-1)$, the stationary points are $v = -1$, $v = 1$, and $v = 3$. (Notice that $v^2 + 1 > 0$.) Because the right-hand side is differentiable at each of these stationary points, no other solutions will touch them. (Uniqueness!)

A sign analysis of the right-hand side shows that the phase-line portrait is

$$\begin{array}{ccccccccccc} & + & & + & & - & & + & & + & \\ \rightarrow\rightarrow & + & \rightarrow\rightarrow\ \circ & \rightarrow\rightarrow\rightarrow\rightarrow & \bullet & \leftarrow\leftarrow\leftarrow\leftarrow & \bullet & \rightarrow\rightarrow\rightarrow\rightarrow & \bullet & \rightarrow\rightarrow\ + & \rightarrow\rightarrow \quad v \\ & -4 & -3 & & -1 & & 1 & & 3 & 4 & \\ & & \text{undefined} & & \text{stable} & & \text{unstable} & & \text{semistable} & & \end{array}$$

Remark. The terms stable, unstable, and semistable are applied only to solutions. The point $v = -3$ is not a solution, so these terms should not be applied to it.

The above phase-line portrait shows that as t increases the solutions $v(t)$ will move in *the direction of the arrows*. Moreover, uniqueness implies that a nonstationary solution will not touch any stationary one. Much information about the solutions can be read off from the phase-line portrait. For example, for each stationary point we can identify the set of initial-values $v(0)$ such that the solution $v(t)$ converges to that stationary point as $t \to \infty$.

- We see that for the stable stationary point -1 we have $v(t) \to -1$ as $t \to \infty$ if and only if $v(0)$ is in the interval $(-3, 1)$.
- We see that for the unstable stationary point 1 we have $v(t) \to 1$ as $t \to \infty$ if and only if $v(0) = 1$.
- We see that for the semistable stationary point 3 we have $v(t) \to 3$ as $t \to \infty$ if and only if $v(0)$ is in the interval $(1, 3]$.

5.2. **Plots of Explicit Solutions.** This method can be applied whenever we can obtain an explicit formula for a solution or a family of solutions. For example, if we want to see what a solution $y = Y(t)$ looks like over the time interval $[t_L, t_R]$ and we already have an explicit formula for $Y(t)$ then the simplest thing to do is to use the MATLAB command **fplot**. We can plot $Y(t)$ over a time interval $[t_L, t_R]$ by

>> fplot(@(t) Y(t), [t_L t_R])
>> xlabel 't', ylabel 'y'
>> title 'Plot of y = Y(t)'

Remark. You should get into the habit of labeling each axis and titling each graph.

Example. Consider the linear initial-value problem

$$\frac{\mathrm{d}y}{\mathrm{d}t} + y = 5\sin(2t)\,, \qquad y(0) = 1\,.$$

We have found that its solution is

$$y = 3e^{-t} - 2\cos(2t) + \sin(2t)\,.$$

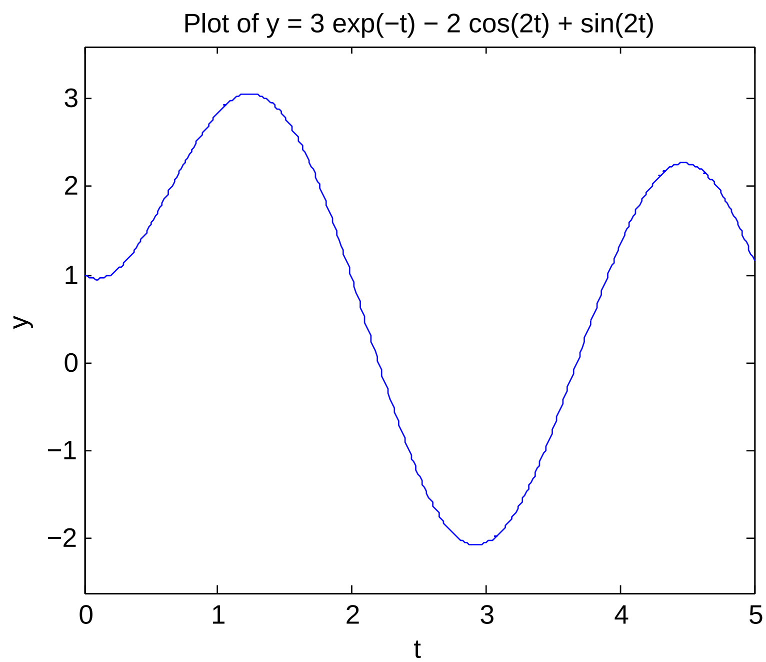
Plot this solution over the time interval $[0, 5]$.

Solution. We can input the explicit expression directly into fplot as a function handle. For example,

```
>> fplot(@(t) 3*exp(−t) − 2*cos(2*t) + sin(2*t), [0 5])
>> xlabel 't', ylabel 'y'
>> title 'Plot of y = 3 exp(−t) − 2 cos(2t) + sin(2t)'
```

The resulting graph is shown below.

Remark. The @(t) indicates to Matlab that the expression following it is a function of t. The combination of @(t) and an expression is called a function handle. Here we used t as the variable, but any other variable will produce the same graph.



Alternative Solution. Modern versions of Matlab allow us to input the explicit expression into fplot as a symbolic expression. For example,

```
>> syms t; sol = 3*exp(−t) − 2*cos(2*t) + sin(2*t);
>> fplot(sol, [0 5])
>> xlabel 't', ylabel 'y'
>> title 'Plot of y = 3 exp(−t) − 2 cos(2t) + sin(2t)'
```

Remark. We will take this approach when plotting members of a family of solutions.

We can also use fplot when the input is a symbolic solution generated by the MATLAB command **dsolve**.

Example. Consider the linear initial-value problem

$$\frac{\mathrm{d}y}{\mathrm{d}t} + y = 5\sin(2t)\,, \qquad y(0) = 1\,.$$

Plot its solution over the time interval $[0, 5]$.

Solution. We can input the initial-value problem into dsolve as character strings. For example,

```
>> sol = dsolve('Dy + y = 5*sin(2*t)', 'y(0) = 1', 't');
>> fplot(sol, [0 5])
>> xlabel 't', ylabel 'y'
>> title 'Solution of Dy + y = 5 sin(2t), y(0) = 1'
```

Remark. The semicolon after the dsolve command suppresses its explicit output.

If $y = Y(t, c)$ is a family of solutions to a differential equation and we want to see how these solutions look over the time interval $[t_L, t_R]$ for several different values of c then we can change the values of c in fplot by having the function handle include c and changing the values of c inside a loop.

Example. Consider the general initial-value problem

$$\frac{\mathrm{d}y}{\mathrm{d}t} + y = 5\sin(2t)\,, \qquad y(0) = c\,.$$

We have found that its solution is

$$y = (c+2)e^{-t} - 2\cos(2t) + \sin(2t)\,.$$

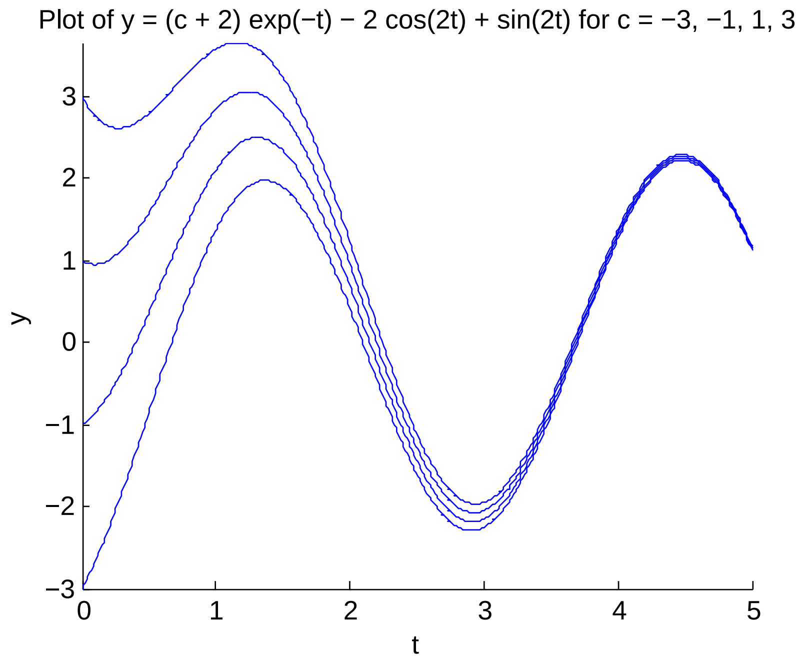
Plot this solution over the time interval $[0, 5]$ for $c = -3$, -1, 1, and 3.

Solution. We can define the family outside the loop as a function handle with two variables, t and c, as

```
>> gensol = @(t,c) (c + 2)*exp(−t) − 2*cos(2*t) + sin(2*t);
>> figure; hold on
>> for cval = −3:2:3
       fplot(@(t) gensol(t,cval), [0 5])
   end
>> axis tight, xlabel 't', ylabel 'y'
>> title 'Plot of y = (c + 2) exp(−t) − 2 cos(2t) + sin(2t) for c = −3, −1, 1, 3'
>> hold off
```

The resulting graph is shown below.

Remark. Notice that we have plotted the general solution for only four values of c. Plotting the solution for many more values of c can make the resulting graph look cluttered.



We can do a similar thing when the input is a symbolic family of solutions generated by the MATLAB command **dsolve**. In this case we use the MATLAB command **subs** to substitute specific numerical values into the variable c.

Example. Consider the general linear initial-value problem

$$\frac{\mathrm{d}y}{\mathrm{d}t} + y = 5\sin(2t)\,, \qquad y(0) = c\,.$$

Plot its solution over the time interval $[0, 5]$ for $c = -3$, -1, 1, and 3.

Solution. This time the solution uses symbolic expressions instead of function handles, because dsolve produces symbolic expressions as its output.

```
>> gensol = dsolve('Dy + y = 5*sin(2*t)', 'y(0) = c', 't');
>> figure; hold on
>> for cval = −3:2:3
        fplot(subs(gensol, 'c', cval), [0 5])
   end
>> axis tight, xlabel 't', ylabel 'y'
>> title 'Solution of Dy + y = 5 sin(2t), y(0) = c for c = −3, −1, 1, 3'
>> hold off
```

Remark. More examples of how to use fplot to graph explicit solutions of first-order differential equations can be found in our MATLAB book, *Differential Equations with MATLAB* by Hunt, Lipsman, Osborn, and Rosenberg.

5.3. **Contour Plots of Implicit Solutions.** This method can be applied anytime the solutions of a differential equation are implicitly given by an equation of the form

$$H(x, y) = c\,. \tag{5.2}$$

This situation might arise when solving a separable equation for which we can analytically find the primitives $F(x)$ and $G(y)$, but $G(y)$ is so complicated that we cannot analytically compute the inverse function G^{-1}. In that case (5.2) takes on the special form

$$F(x) - G(y) = c\,.$$

We will soon learn other methods that can lead to implicit solutions of the form (5.1) for any function $H(x, y)$ with continuous second partial derivatives. We will assume that $H(x, y)$ has continuous second partial derivatives.

5.3.1. *Rendered by Computer.* Given a value for c, solutions of the differential equation lie on the set in the xy-plane given by

$$\Big\{(x, y) \,:\, H(x, y) = c\Big\}\,.$$

This is the so-called *level set* of $H(x, y)$ associated with c. The idea is to plot one or more level sets of H within a bounded rectangle in the xy-plane that is of interest. When many level sets are used the result is called a *contour plot* of H.

Remark. If we consider $H(x, y)$ to give the height of the graph of H over the xy-plane then a contour plot shows the height of this graph in exactly the same way a contour map shows the elevation of topographical features.

We can produce a contour plot of a function $H(x, y)$ over the rectangle $[x_L, x_R] \times [y_L, y_R]$ by using the MATLAB commands **meshgrid** and **contour** as follows.

>> [X, Y] = meshgrid(x_L:h:x_R,y_L:k:y_R);
>> contour(X, Y, H(X,Y))
>> axis square, xlabel 'x', ylabel 'y'
>> title 'Contour Plot of H(x, y)'

Here h and k are the resolutions of the intervals $[x_L, x_R]$ and $[y_L, y_R]$ respectively, which should have values of the form

$$h = \frac{x_R - x_L}{m}\,, \qquad k = \frac{y_R - y_L}{n}\,, \qquad \text{where } m \text{ and } n \text{ are positive integers.}$$

The meshgrid command creates an array of *grid points* in the rectangle $[x_L, x_R] \times [y_L, y_R]$ given by (x_i, y_j) where

$$x_i = x_L + ih \text{ for } i = 0,\, 1,\, \cdots\, m\,, \qquad y_j = y_L + jk \text{ for } j = 0,\, 1,\, \cdots\, n\,.$$

More precisely, meshgrid creates two arrays; the array X contains x_i in its ij^{th}-entry while the array Y contains y_j in its ij^{th}-entry.

The contour command computes $H(x_i, y_j)$ at each of these grid points, uses these values to construct an approximation to $H(x, y)$ over $[x_L, x_R] \times [y_L, y_R]$ by interpolation, selects nine values for c that are evenly spaced between the minimum and maximum values of this interpolation, and then constructs an approximation to the level set of each c based on this interpolation. Exactly how contour does all this is beyond the scope of this course, so we will not discuss it further. You can learn more about such algorithms in a numerical analysis course. Here all we need to know is that MATLAB produces an approximation to each level set. This appoximation can be improved by making h and k smaller. The resulting contour plot will give us a good idea of what

$H(x,y)$ looks like over the rectangle $[x_L,x_R]\times[y_L,y_R]$ provided h and k are small enough. Typical values for m and n run between 50 and 200.

Example. Consider the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{x}{y}\,.$$

Earlier we showed that this equation has an implicit general solution given by

$$x^2 + y^2 = c\,.$$

Produce a contour plot of $H(x,y) = x^2 + y^2$ in the rectangle $[-5,5]\times[-5,5]$.

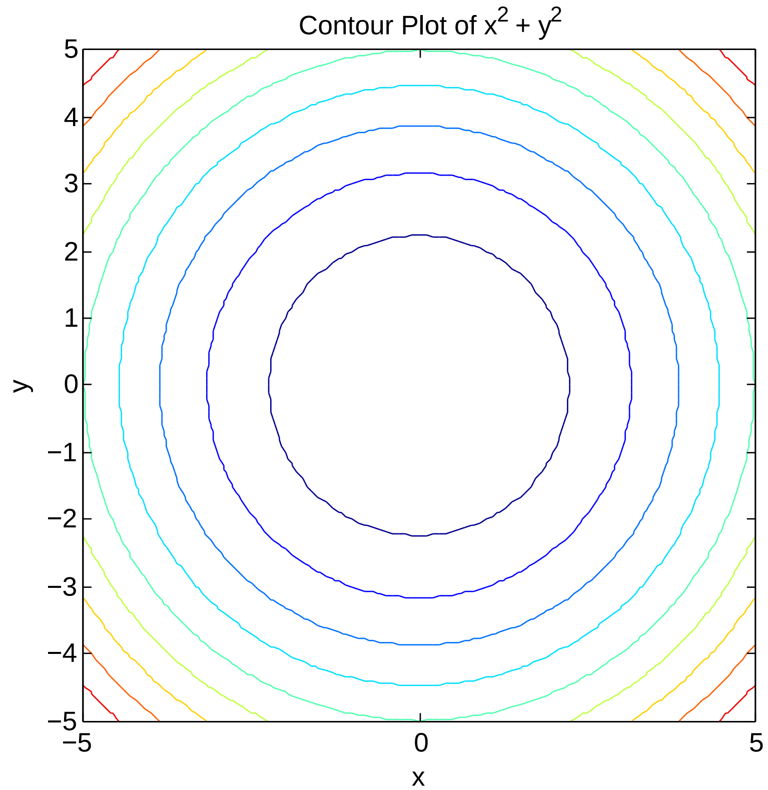
Solution. If we select $m = n = 100$, so that $h = k = .1$, then our MATLAB program becomes

```
>> [X, Y] = meshgrid(−5:0.1:5,−5:0.1:5);
>> contour(X, Y, X.ˆ2 + Y.ˆ2)
>> axis square, xlabel 'x', ylabel 'y'
>> title 'Contour Plot of xˆ2 + yˆ2'
```

Remark. The dots that appear in X.ˆ2 + Y.ˆ2 tell MATLAB that we want to square each entry in the arrays X and Y rather than square the arrays themselves by matrix multiplication.

Because the minimum and maximum of x^2+y^2 over the rectangle $[-5,5]\times[-5,5]$ are 0 and 50 respectively, the values of c MATLAB will choose are $5l$ where $l = 0, 1, 2, \cdots, 10$. The resulting graph is shown below. It shows a point at the origin plus the intersections of the rectangle $[-5,5]\times[-5,5]$ with ten concentric circles centered at the origin that have radii $\sqrt{5}$, $\sqrt{10}$, $\sqrt{15}$, $2\sqrt{5}$, 5, $\sqrt{30}$, $\sqrt{35}$, $2\sqrt{10}$, $3\sqrt{5}$, and $5\sqrt{2}$.

No solutions of the differential equation can cross the line $y = 0$. The circles in the contour plot show the graphs of different solutions above and below the x-axis. It shows that every solution has a bounded interval of definition. Moreover, these solutions have derivatives that blow-up at the endpoints of this interval but they do not blow-up themselves.



Remark. A very similar contour plot can be produced using the MATLAB command **ezcontour**. You can find out how from on-line MATLAB documentation. When used in its simplest form, "ezcontour" does not give as much control as the combination of "meshgrid" and "contour."

Remark. The previous example was so simple that we could have generated a good sketch of the resulting contour plot by hand. When programming it is a good idea to start with such simple examples in order to verify that the basic coding works. The subsequent examples are more substantial.

If we are investigating an initial-value problem with initial data (x_I, y_I) inside a recangle $[x_L, x_R] \times [y_L, y_R]$ then the only level set that we want to plot is the one corresponding to $c = H(x_I, y_I)$. This level set might show more solutions than the one we seek, so be careful! We can plot just this level set by the following.

>> c = $H(x_I, y_I)$;
>> [X, Y] = meshgrid(x_L:h:x_R,y_L:k:y_R);
>> contour(X, Y, H(X,Y), [c c])
>> axis square, xlabel 'x', ylabel 'y'
>> title 'Level Set for H(x, y) = $H(x_I, y_I)$'

Example. Consider the initial-value problem

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{e^x \cos(t)}{1+x}\,, \qquad x(0) = -2\,.$$

Graph its solution over the time interval $[-4, 4]$.

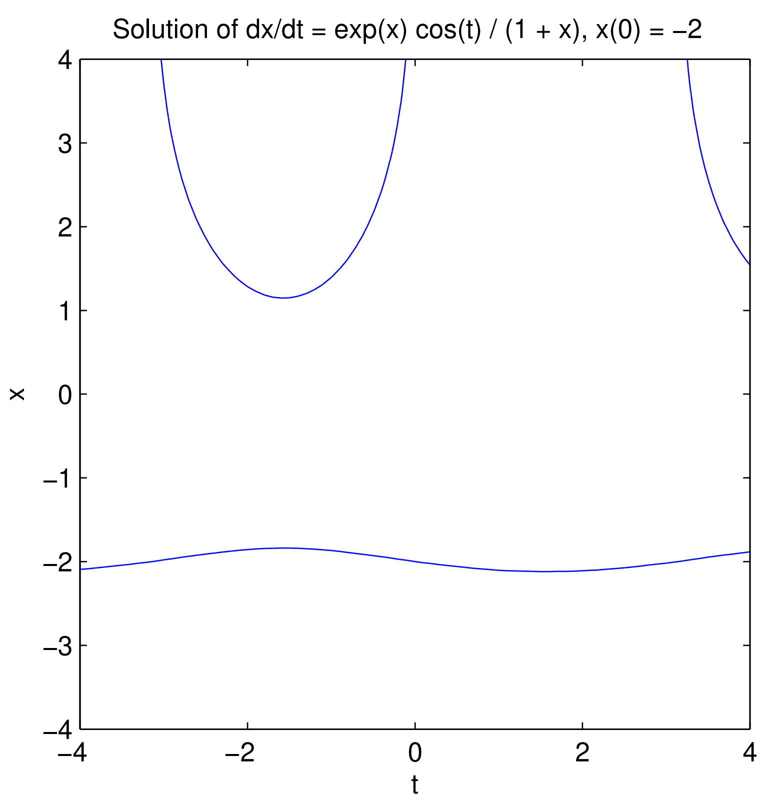
Solution. In Section 3.5 we used the fact this equation is separable to show that the solution of this initial-value problem satisfies

$$\sin(t) = -(2+x)e^{-x}\,.$$

The analysis in Section 3.5 showed moreover that there is a unique oscillatory solution $x = X(t) < -1$ to this equation with interval of definition $(-\infty, \infty)$. However, other solutions lie on this level set in the region $x > -1$. We can plot the level set by the following.

```
>> [T, X] = meshgrid(−4:0.1:4,−4:0.1:4);
>> contour(T, X, sin(T) + (2 + X).*exp(−X), [0 0])
>> axis square, xlabel ’t’, ylabel ’x’
>> title ’Solution of dx/dt = exp(x) cos(t) / (1 + x), x(0) = −2’
```

The resulting plot is shown below. The lower curve is the graph of the solution. The upper curves lie on the level set but have nothing to do with the solution of this initial-value problem.



We can plot the level sets for the c values c_1, c_2, and c_3 on a single graph by the following.

```
>> [X, Y] = meshgrid(x_L:h:x_R,y_L:k:y_R);
>> contour(X, Y, H(X,Y), [c_1 c_2 c_3])
>> axis square, xlabel 'x', ylabel 'y'
>> title 'Level Sets for H(x, y) = c_1, c_2, c_3'
```

Example. Consider the general initial-value problem

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{e^x \cos(t)}{1+x}\,, \qquad x(0) = x_I\,.$$

Graph its solutions over the time interval $[-4, 4]$ for $x_I = -2$, 0, 1, and 2.

Solution. In Section 3.5 we showed that the solution of this initial-value problem satisfies

$$\sin(t) = (2 + x_I)e^{-x_I} - (2 + x)e^{-x}\,.$$

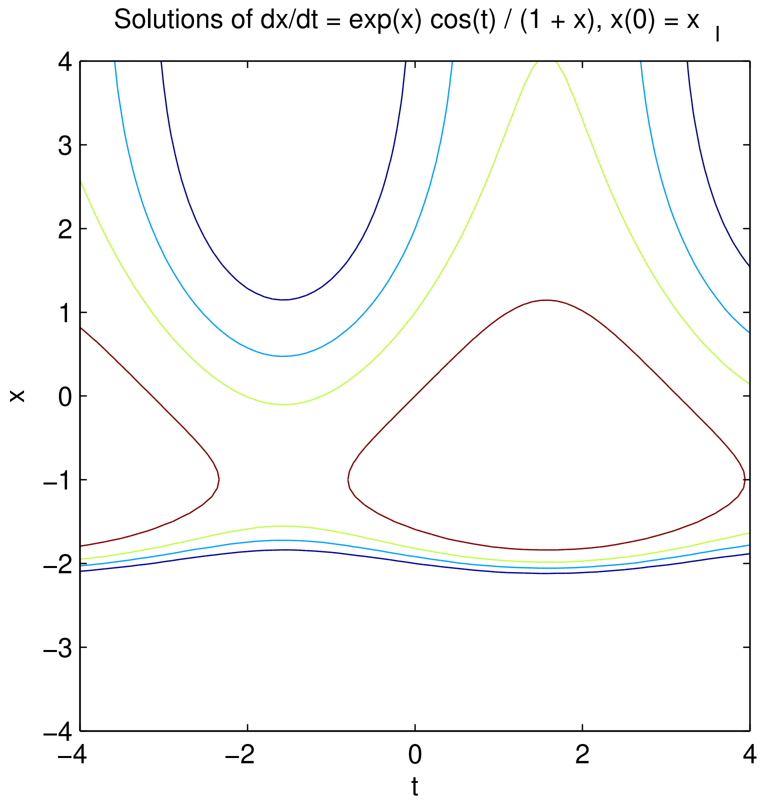
This equation has the form

$$\sin(t) + (2 + x)e^{-x} = c\,,$$

where the values of c corresponding to $x_I = -2$, 0, 1, and 2 are $c = 0$, 2, $3/e$ and $4/e^2$ respectively. We can plot these level sets by the following.

```
>> [T, X] = meshgrid(−4:0.1:4,−4:0.1:4);
>> contour(T, X, sin(T) + (2 + X).*exp(−X), [0 2 3/exp(1) 4/exp(2)])
>> axis square, xlabel 't', ylabel 'x'
>> title 'Solutions of dx/dt = exp(−x) cos(t) / (1 + x), x(0) = x_I'
```

The resulting plot is shown below. Remember that no solutions of this equation can cross the line $x = -1$. The curves that cross this line show the graphs of different solutions above and below this line. The plot shows that some solutions of this equation exist for all time and some solutions have bounded intervals of definition. Moreover, it shows that some of the solutions with bounded intervals of definition blow-up at the endpoints of the interval while others have derivatives that blow-up at the endpoints of the interval but do not blow-up themselves. In the next subsection we will show that the separation between these different behaviors is given by components of the level sets corresponding to the values $c = 1$ and $c = -1 + e$.



5.3.2. *Critical Points and Values.* Some level sets are more important than others. The contour plot produced in the last example of the previous subsection missed level sets that separated solutions with different behaviors. Here we show how to identify such level sets. We begin with a review of some ideas from multivariable calculus.

Recall from multivariable calculus that the *critical points* of H are those points in the xy-plane where the gradient of H vanishes — i.e. those points (x, y) where

$$\partial_x H(x, y) = 0\,, \quad \text{and} \quad \partial_y H(x, y) = 0\,.$$

The value of $H(x, y)$ at a critical point is called a *critical value* of H.

Example. Find all critical points and critical values of $H(x, y) = x^2 + y^2$.

Solution. Because $\partial_x H(x, y) = 2x$ and $\partial_y H(x, y) = 2y$, we see that (x, y) is a critical point of $H(x, y)$ if $2x = 0$ and $2y = 0$. Therefore the only critical point of $H(x, y)$ is $(0, 0)$ and the only critical value is $H(0, 0) = 0^2 + 0^2 = 0$.

Remark. There is nothing special about x and y in our definitions. Any two variables can play the roles of the independent variables. We illustrate this in the next example.

Example. Find all critical points and critical values of $H(t, x) = \sin(t) + (2 + x)e^{-x}$.

Solution. Because $\partial_t H(t, x) = \cos(t)$ and $\partial_x H(t, x) = -(1 + x)e^{-x}$, we see that (t, x) is a critical point of H if

$$\cos(t) = 0\,, \quad \text{and} \quad -(1 + x)e^{-x} = 0\,.$$

The first equation implies that $t = n\pi + \frac{\pi}{2}$ for some integer n, while the second implies that $x = -1$. Therefore the critical points of H are

$$(n\pi + \tfrac{\pi}{2}, -1)\,, \qquad \text{where } n \text{ is any integer}\,,$$

and the associated critical values are

$$H(n\pi + \tfrac{\pi}{2}, -1) = \sin(n\pi + \tfrac{\pi}{2}) + (2-1)e^1 = (-1)^n + e\,, \qquad \text{where } n \text{ is any integer}\,.$$

We see that there are an infinite number of critical points of $H(t,x)$, one specified by each integer n, but only two critical values: $1+e$ when n is even and $-1+e$ when n is odd.

We will assume that each critical point of H is *nondegenerate*. This means that at each critical point of H the Hessian matrix $\boldsymbol{\partial}^2 H(x,y)$ of second partial derivatives has a nonzero determinant — i.e. at every critical point (x,y) we have

$$\det\big(\boldsymbol{\partial}^2 H(x,y)\big) \neq 0\,, \qquad \text{where} \quad \boldsymbol{\partial}^2 H(x,y) = \begin{pmatrix} \partial_{xx}H(x,y) & \partial_{xy}H(x,y) \\ \partial_{yx}H(x,y) & \partial_{yy}H(x,y) \end{pmatrix}\,.$$

Nondegenerate critical points of H are *isolated*. This means that each critical point of H is contained within a rectangle $(a,b)\times(c,d)$ that contains no other critical points. This is a consequence of the *Implicit Function Theorem* of multivariable calculus.

Nondegenerate critical points of H can be classified as follows.

- If $\det\big(\boldsymbol{\partial}^2 H(x,y)\big) > 0$ and $\partial_{xx}H(x,y) > 0$ then (x,y) is a local minimizer of H.
- If $\det\big(\boldsymbol{\partial}^2 H(x,y)\big) > 0$ and $\partial_{xx}H(x,y) < 0$ then (x,y) is a local maximizer of H.
- If $\det\big(\boldsymbol{\partial}^2 H(x,y)\big) < 0$ then (x,y) is a saddle point of H.

We illustrate these facts from multivariable calculus with two examples.

Example. Classify all critical points of $H(x,y) = x^2 + y^2$.

Solution. We have already shown that $(0,0)$ is the only critical point of $H(x,y)$. Because the Hessian matrix is

$$\boldsymbol{\partial}^2 H(x,y) = \begin{pmatrix} \partial_{xx}H(x,y) & \partial_{xy}H(x,y) \\ \partial_{yx}H(x,y) & \partial_{yy}H(x,y) \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}\,,$$

we see that

$$\det\big(\boldsymbol{\partial}^2 H(0,0)\big) = \det\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} = 2\cdot 2 - 0 \cdot 0 = 4\,.$$

Because $\det\big(\boldsymbol{\partial}^2 H(0,0)\big) = 4 > 0$ and $\partial_{xx}H(0,0) = 2 > 0$ we see that $(0,0)$ is a nondegenerate critical point of H and that it is a local minimizer.

Example. Classify all critical points of $H(t,x) = \sin(t) + (2+x)e^{-x}$.

Solution. We have already shown that the critical points of $H(t,x)$ are

$$(n\pi + \tfrac{\pi}{2}, -1)\,, \qquad \text{where } n \text{ is any integer}\,,$$

Because the Hessian matrix is

$$\boldsymbol{\partial}^2 H(t,x) = \begin{pmatrix} \partial_{tt}H(t,x) & \partial_{tx}H(t,x) \\ \partial_{xt}H(t,x) & \partial_{xx}H(t,x) \end{pmatrix} = \begin{pmatrix} -\sin(t) & 0 \\ 0 & xe^{-x} \end{pmatrix}\,,$$

we see that

$$\boldsymbol{\partial}^2 H(n\pi + \tfrac{\pi}{2}, -1) = \begin{pmatrix} -(-1)^n & 0 \\ 0 & -1e^1 \end{pmatrix} = \begin{pmatrix} -(-1)^n & 0 \\ 0 & -e \end{pmatrix},$$

whereby

$$\det\bigl(\boldsymbol{\partial}^2 H(n\pi + \tfrac{\pi}{2}, -1)\bigr) = \det\begin{pmatrix} -(-1)^n & 0 \\ 0 & -e \end{pmatrix} = -(-1)^n \cdot (-e) - 0 \cdot 0 = (-1)^n e\,.$$

Because $\det\bigl(\boldsymbol{\partial}^2 H(n\pi + \frac{\pi}{2}, -1)\bigr) = (-1)^n e \neq 0$ for every n we see that every critical point of $H(t, x)$ is nondegenerate. Moreover,

- $(n\pi + \frac{\pi}{2}, -1)$ is a local maximizer when n is even because
$$\det\bigl(\boldsymbol{\partial}^2 H(n\pi + \tfrac{\pi}{2}, -1)\bigr) = e > 0\,, \quad \text{and} \quad \partial_{tt} H(n\pi + \tfrac{\pi}{2}, -1) = -1 < 0\,;$$
- $(n\pi + \frac{\pi}{2}, -1)$ is a saddle point when n is odd because
$$\det\bigl(\boldsymbol{\partial}^2 H(n\pi + \tfrac{\pi}{2}, -1)\bigr) = -e < 0\,.$$

The local maximizers are the critical points with critical value $1 + e$ while the saddle points the critical points with critical value $-1 + e$.

We are now ready to apply these notions to identifying important level sets for contour plots. Given a value for c, solutions of the differential equation lie on the level set of $H(x, y)$ in the xy-plane given by

$$\Bigl\{(x, y) \,:\, H(x, y) = c\Bigr\}\,.$$

Whenever this set has at least one point in it, the *Implicit Function Theorem* of multivariable calculus tells us the following.

- If c is not a critical value then its level set will look like one or more curves in the xy-plane that never meet. These curves will either be loops that close on themselves or extend to infinity. They will not have endpoints.
- If c is a critical value then its level set will look like one or more local extremizers of H plus some curves in the xy-plane that might meet only at saddle points. These curves will either be loops that close on themselves, extend to infinity, or have an endpoint at a saddle point.

This suggests that important values of c to consider when making a contour plot are the critical values of $H(x, y)$ associated with *saddle points.*

5.4. **Direction Fields.** This is the crudest tool in our toolbox. Its virtue is that it can be applied to almost any first-order equation

$$\frac{\mathrm{d}y}{\mathrm{d}t} = f(t, y)\,. \tag{5.3}$$

We assume that the function $f(t, y)$ is defined over a set S in the ty-plane such that

- f is continuous over S,
- f is differentiable with respect to y over S,
- $\partial_y f$ is continuous over S.

Moreover, we assume that every point in a rectangle $[t_L, t_R] \times [y_L, y_R]$ is in the interior of S. Then by Theorem 4.1 every point (t_I, y_I) in $[t_L, t_R] \times [y_L, y_R]$ has a unique curve $(t, Y(t))$ passing through it such that $y = Y(t)$ is a solution of (5.3). This curve can be extended to the largest time interval $[a, b]$ such that $(t, Y(t))$ remains within the rectangle $[t_L, t_R] \times [y_L, y_R]$.

If we cannot find an explicit or implicit solution of (5.3) then we cannot plot the curve $(t, Y(t))$ by the methods of the previous two sections. However, if $f(t, y)$ meets the criteria given above then we know by Theorem 4.1 that the curve exists and has a tangent vector given by

$$\frac{\mathrm{d}}{\mathrm{d}t}\big(t\,,\,Y(t)\big) = \big(1\,,\,Y'(t)\big) = \big(1\,,\,f(t, Y(t))\big)\,.$$

In other words, the unique solution that goes through any point (t, y) in the rectangle $[t_L, t_R] \times [y_L, y_R]$ has the tangent vector $(1, f(t, y))$. A *direction field* for equation (5.3) over the rectangle $[t_L, t_R] \times [y_L, y_R]$ is a plot that shows the direction of this tangent vector with an arrow at each point of a grid in the rectangle $[t_L, t_R] \times [y_L, y_R]$. The idea is that these arrows might give us a correct picture of how the orbits move inside the rectangle.

5.4.1. *Rendered by Computer.* We can produce such a direction field by using the MATLAB commands **meshgrid** and **quiver** as follows.

>> [T, Y] = meshgrid(t_L:h:t_R,y_L:k:y_R);
>> S = f(T, Y);
>> L = sqrt(1 + S.ˆ2);
>> quiver(T, Y, 1./L, S./L, ℓ)
>> axis tight, xlabel 't', ylabel 'y'
>> title 'Direction Field for dy/dt = f(t, y)'

Here h and k are the grid spacings for the intervals $[t_L, t_R]$ and $[y_L, y_R]$ respectively, which should have values of the form

$$h = \frac{t_R - t_L}{m}\,, \qquad k = \frac{y_R - y_L}{n}\,, \qquad \text{where } m \text{ and } n \text{ are positive integers.}$$

The meshgrid command creates an array of *grid points* in the rectangle $[t_L, t_R] \times [y_L, y_R]$ given by (t_i, y_j) where

$$t_i = t_L + ih \text{ for } i = 0,\, 1,\, \cdots\, m\,, \qquad y_j = y_L + jk \text{ for } j = 0,\, 1,\, \cdots\, n\,.$$

More precisely, meshgrid creates two arrays; the array T contains t_i in its ij^{th}-entry while the array Y contains y_j in its ij^{th}-entry. Next, an array S is computed that contains the slope $f(t_i, y_j)$ in its ij^{th}-entry. Then an array L is computed that contains the length of the tangent vector $(1, f(t_i, y_j))$ in its ij^{th}-entry.

Finally, the quiver command plots an array of arrows of length ℓ so that the ij^{th}-arrow is centered at the grid point (t_i, y_j) and is pointing in the direction of the unit tangent vector

$$\left(\frac{1}{\sqrt{1 + f(t_i, y_j)^2}}\,,\,\frac{f(t_i, y_j)}{\sqrt{1 + f(t_i, y_j)^2}}\right)\,.$$

The length ℓ should be smaller than h or k so that the plotted arrows will not overlap. Typically m and n will be about 20 to insure there will be enough arrows to give a complete picture of the direction field, but not so many that the plot becomes cluttered.

Remark. Often it is hard to figure out how the orbits move from the arrows in a direction field. Therefore they should be used only as a tool of last resort. They should never be used for autonomous equations because phase-line portraits are much easier to use.

Example. Describe the solutions of the equation

$$\frac{dy}{dt} = \sqrt{1 + t^2 + y^2}\,.$$

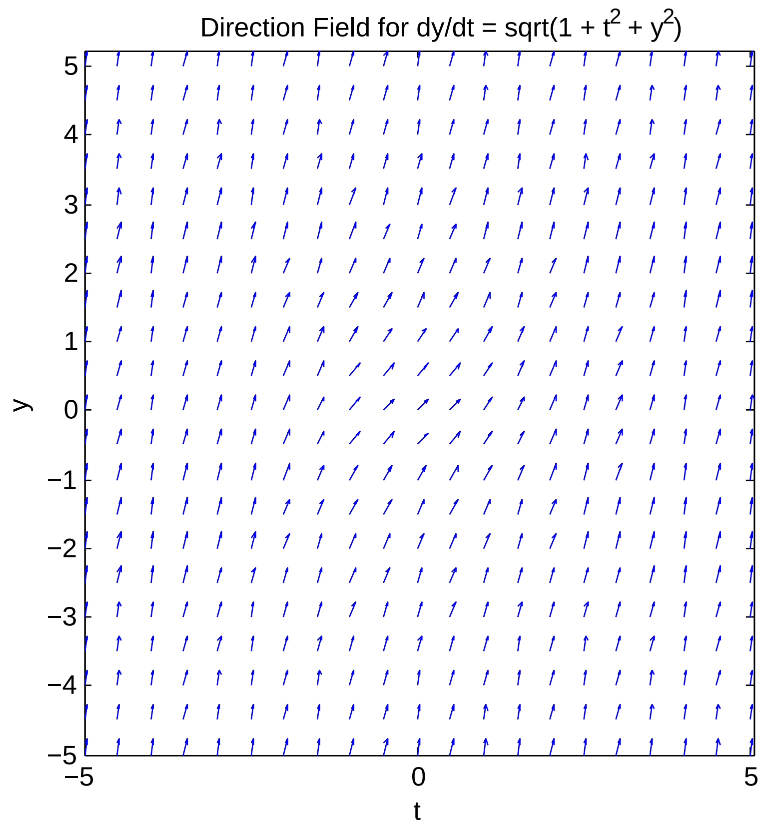
Solution. This equation is not linear or separable, so it cannot be solved by the analytic methods we have studied. Earlier we showed that this equation meets the criteria of Theorem 4.1, so there is a unique solution that passes through every point in the ty-plane. Therefore the only method we have discussed that applies to this equation is direction fields.

Before applying the method of direction fields, we should see what information can be seen directly from the equation. Its right-hand side satifies $\sqrt{1 + t^2 + y^2} \geq 1$. This means its solutions will all be increasing functions of t. If our direction fields are not consistent with this observation then we will know we have made a mistake in our MATLAB program.

We can produce a direction field for this equation in the rectangle $[-5, 5] \times [-5, 5]$ with a grid spacing of 0.5 (by taking $m = n = 20$) and arrows of length 0.35 as follows.

```
>> [T, Y] = meshgrid(−5:0.5:5,−5:0.5:5);
>> S = sqrt(1 + T.ˆ2 + Y.ˆ2);
>> L = sqrt(1 + S.ˆ2);
>> quiver(T, Y, 1./L, S./L, 0.35)
>> axis tight, xlabel 't', ylabel 'y'
>> title 'Direction Field for dy/dt = sqrt(1 + tˆ2 + yˆ2)'
```

The resulting plot is below. In this case it is easy to figure out how the orbits move from the direction field.



5.4.2. *Sketched by Hand.* When $f(t, y)$ is sufficiently simple it is easy to sketch a direction field of equation (5.3) by hand using the *method of isoclines.* Here "sufficiently simple" means that we are able to sketch by hand the sets in the ty-plane that satisfy the equation $f(t, y) = m$ for several values of m. Recall that the *level set* of $f(t, y)$ associated with a given value m is the set in the ty-plane given by

$$\Big\{ (t, y) \; : \; f(t, y) = m \Big\} .$$

Whenever this set has at least one point in it, the *Implicit Function Theorem* of multivariable calculus tells us the following.

- If m is not a critical value then its level set will look like one or more curves in the ty-plane that never meet. These curves will either be loops that close on themselves or extend to infinity. They will not have endpoints.
- If m is a critical value and each of its associated critical points is nondegenerate then its level set will look like one or more local extremizers of f plus some curves in the ty-plane that might meet at only saddle points. These curves will either be loops that close on themselves, extend to infinity, or have an endpoint at a saddle point.

The significance of these level sets is as follows. Let (t_o, y_o) be any point on the level set associated with m. Let $Y(t)$ be the unique solution of the differential equation (5.3) that passes through this point — i.e. the solution that satisfies the initial condition $Y(t_o) = y_o$. Then

$$Y'(t_o) = f\big(t_o, Y(t_o)\big) = f(t_o, y_o) = m\,.$$

In other words, at every point (t_o, y_o) on this level set the curve $(t, Y(t))$ passing through it has the tangent vector $(1, m)$. This means that every solution $Y(t)$ of the differential equation has the property that $Y'(t) = m$ whenever $(t, Y(t))$ is on this level set. Because every solution of the differential equation has the same slope on these level sets, they are called *isoclines* — a name that means "same slope".

At every point in the ty-plane that lies on the isocline associated with m the direction field is an arrow with slope m pointing in the direction of increasing time. This arrow shows the direction of the tangent vector $(1, m)$. The idea is that if the isoclines are easy to identify for enough values of m then sketching a few appropriate arrows along each of them might give us a good idea of what the direction field looks like everywhere nearby. Let us illustrate this approach by examples.

Example. Sketch the direction field of the equation

$$\frac{\mathrm{d}y}{\mathrm{d}t} = \sqrt{1 + t^2 + y^2}\,.$$

Solution. The isocline associated with any $m > 1$ satisfies

$$1 + t^2 + y^2 = m^2\,.$$

This is the circle centered at the origin with radius $\sqrt{m^2 - 1}$. By setting m equal to $\sqrt{2}$, $\sqrt{5}$, and $\sqrt{10}$ we see that the direction field arrows on the circle of radius 1 have slope $\sqrt{2}$, the arrows on the circle of radius 2 have slope $\sqrt{5}$, and the arrows on the circle of radius 3 have slope $\sqrt{10}$. By sketching a few such arrows on each of these circle we get a fairly clear idea of what the direction field looks like.

Example. Sketch the direction field of the equation

$$\frac{\mathrm{d}y}{\mathrm{d}t} = (y - t^2)^3\,.$$

Solution. The isocline associated with any m satisfies

$$y - t^2 = m^{\frac{1}{3}}\,.$$

This is the parabola $y = t^2 + m^{\frac{1}{3}}$. By setting m equal to -8, -1, 0, 1, and 8 we see that the direction field arrows on the parabola $y = t^2 - 2$ have slope -8, the arrows on the parabola $y = t^2 - 1$ have slope -1, the arrows on the parabola $y = t^2$ have slope 0, the arrows on the parabola $y = t^2 + 1$ have slope 1, and the arrows on the parabola $y = t^2 + 2$ have slope 8. By sketching a few such arrows on each of these parabolas we get a fairly clear idea of what the direction field looks like.

Exercises on Graphical Methods

For problems #1–#5a, describe the behavior of the solutions of the autonomous equations by drawing a phase-line portrait. Be sure to comment on the nature of the stationary points.

(1) $\dfrac{\mathrm{d}y}{\mathrm{d}t} = y^2 - 1$

Short Answer
Solution

(2) $\dot{y} = 3y^3 + 6y^2$

Short Answer
Solution

(3) $\dfrac{\mathrm{d}y}{\mathrm{d}t} = -y^2(1-y)(3-y)$

Short Answer
Solution

(4) $\dfrac{\mathrm{d}z}{\mathrm{d}t} = (z-3)^2(z+3)^2$

Short Answer
Solution

(5) (a) $\dfrac{\mathrm{d}y}{\mathrm{d}t} = e^y$
(b) The equation in part (a) is separable; solve it. Comment on whether the explicit solution agrees with your phase line.

Short Answer
Solution

(6) (a) Suppose $f(t)$ is the solution to

$$\frac{\mathrm{d}y}{\mathrm{d}t} = (y-2)(y-1)^2(y+1)(y+3)^3$$

with $f(0) = -2$. What is $\lim_{t\to\infty} f(t)$?
(b) Same differential equation, but change the initial condition to $f(5) = 0$. What is $\lim_{t\to-\infty} f(t)$?
(c) Change again to $f(-3) = 2$. What is $\lim_{t\to\infty} f(t)$?

Short Answer
Solution

For problems #7–#9, solve the given initial value problem, then plot the solution you get over the interval given.

(7) $\dot{w} + w = 2e^t, \;\; w(0) = 3, \;\; t \in [-5, 5]$

Short Answer
Solution

(8) $x' - x = t^2, \;\; x(0) = 1, \;\; t \in [-4, 4]$

Short Answer

Solution

(9) $y' - 2y = 5\sin(x)$, $y(0) = 0$, $t \in [-5, 2]$

Short Answer
Solution

For problems #10–#12, find the general solution to the differential equation, then plot the solutions for the suggested values of q. Do the various solutions all have the same end behavior (i.e., their limits as $t \to \infty$)?

(10) $y' + y = 2e^t$, $y(0) = q$, $q = -2, -1, 0, 1$, $t \in [-5, 5]$

Short Answer
Solution

(11) $\dot{y} - y = t^2$, $y(0) = q$, $q = -6, -2, 2, 6$, $t \in [-4, 4]$

Short Answer
Solution

(12) $y' - 2y = 5\sin(x)$, $y(0) = q$, $q = -3, -1, 1, 3$, $t \in [-5, 2]$

Short Answer
Solution

(13) Produce a contour plot of $H(x, y) = \sin(xy)$ on $[-2, 2] \times [-2, 2]$.

Short Answer
Solution

(14) Produce a contour plot of $H(x, y) = x^2 + y^3 + 2xy + 1$ on $[-1.5, 1.5) \times [-1.5, 1.5]$.

Short Answer
Solution

(15) Consider the initial value problem

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{2t + 2te^x}{e^x}, \quad x(1) = 0\,.$$

(a) Separate the equation and show that the solution to this differential equation satisfies $\ln(1 + e^x) = t^2 + \ln(2) - 1$.

(b) Use a contour plot to graph this solution on $[0.75, 5]$. [*Hint.* While the range for x is $[0.75, 5]$, the range for t should be between -3 and 25 or so in order to get the full picture. Take this into account when you build your `meshgrid`.]

(c) As a check, solve the equation from part (a) and get an explicit formula for x, then graph it on $[0.75, 5]$. [hopefully they match!]

Short Answer
Solution

(16) (a) Check that solutions to the equation

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{e^{2x}}{1 + t^2}$$

satisfy $e^{-2x} = -2\arctan(t) + c$.

(b) What values of the constant c correspond to the initial conditions $x(0) = 0$, $\ln(2)$, and 1?

(c) Produce a contour plot of the solutions of the three initial value problems considered in part (b). Plot them on $[0, 1]$. [*Hint.* The vertical range should be $[0, 5]$ or so. The inverse tangent function is implemented in MATLAB by the command `atan`.]

Solution

For problems #17–19, perform the following tasks.

(a) Separate the differential equation to get an implicit relationship that x and t satisfy, then

(b) determine what values of the constant of integration will give the requested initial conditions, and finally

(c) plot those solutions on the ranges of t and x specified.

(17) $\dfrac{\mathrm{d}x}{\mathrm{d}t} = \dfrac{t\cos(t)}{1+2x}$, $x(0) = 1,\ \frac{3}{2},\ 2,\ \ x \in [-2, 2],\ t \in [-5, 5]$

Solution

(18) $\dfrac{\mathrm{d}x}{\mathrm{d}t} = \dfrac{x(\log(x))^2}{t^2}$, $x(1) = e,\ e^2,\ e^3\ \ x \in [-3, 1],\ t \in [0, 5]$

Solution

(19) $\dfrac{\mathrm{d}x}{\mathrm{d}t} = \dfrac{4tx}{1+x}$, $x(1) = 1,\ 3,\ 5\ \ x \in [-3, 3],\ t \in [0, 10]$

Solution

(20) Plot a direction field for the equation $y' = cos(x + y)$ on $[-2, 2] \times [-2, 2]$.

Solution

(21) Plot a direction field for the differential equation $y' = -\frac{t}{y}$ on $[-5, 5] \times [-5, 5]$. The directions in the field should seem to rotate around the origin if you have the correct picture; how do you know that's the right behavior? [*Note.* If you want practice drawing direction fields by hand, this is a good one to work on.]

Solution

(22) Consider the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{2\cos(x) - 3y}{x + y + 1}\,.$$

(a) Where could solutions of this equation fail to exist, or fail to be unique?

(b) Plot the direction field for this equation on $[0, 10] \times [-5, 5]$. Do you see some evidence supporting your answer to part (a)?

Solution

(23) Let $f(x)$ be the solution to the initial value problem $y' = e^{-x^2}$, $y(0) = 0$. [This function is essentially the **error function**, up to a constant.] Is $\lim_{x\to\infty} f(x)$ finite or is it infinite? Explain how the direction field of the differential equation offers evidence of this.

Solution

(24) $\dfrac{\mathrm{d}x}{\mathrm{d}t} = x^2 - \mu\, x$

Is the stability different if μ changes sign? Plot the phase lines for $\mu < 0$ and $\mu > 0$.

Short Answer
Solution

(25) $\dfrac{\mathrm{d}u}{\mathrm{d}t} = (5-u)^{\mu}$

Find $u(t)$ as $t \to \infty$. For what values of μ is it different?

Short Answer
Solution

Navigation to Other Chapters

This page may not work with every browser-driven pdf viewer. When it does work then it will enable you to link directly to any chapter. When it does not work then you can link to any chapter through the main webpage.

Ordinary Differential Equations

0. Course Introduction and Overview

I. First-Order Ordinary Differential Equations

1. Introduction to First-Order Equations
2. Linear Equations
3. Separable Equations
4. General Theory
5. Graphical Methods
6. Applications
7. Numerical Methods
8. Second-Order Equations Reducible to First-Order Ones
9. Exact Differential Forms and Integrating Factors
10. Special Equations and Substitution

II. Higher-Order Linear Ordinary Differential Equations

1. Introduction to Higher-Order Linear Equations
2. Homogenous Equations: General Methods and Theory
3. Supplement: Linear Algebraic Systems and Deteminants
4. Homogenous Equations with Constant Coefficients
5. Nonhomogeneous Equations: General Methods and Theory
6. Nonhomogeneous Equations with Constant Coefficients
7. Nonhomogeneous Equations with Variable Coefficients
8. Application: Mechanical Vibrations
9. Laplace Transform Method

III. First-Order Systems of Ordinary Differential Equations

1. Introduction to First-Order Systems
2. Linear Systems: General Methods and Theory
3. Supplement: Matrices and Vectors
4. Linear Systems: Matrix Exponentials
5. Linear Systems: Eigen Methods
6. Linear Systems: Laplace Transform Methods
7. Linear Planar Systems
8. Autonomous Planar Systems: Integral Methods
9. Autonomous Planar Systems: Nonintegral Methods
10. Application: Population Dynamics