I. First-Order Ordinary Differential Equations
6. Applications

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August 2, 2022

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6. Applications

The subject of differential equations was invented along with calculus by Newton and Leibniz in order to solve problems in geometry and physics. It played a central role in the development of Newtonian physics by the Bernoulli family, Euler, and others. It rapidly found applications in biology, finance, and engineering. Now it is applied in almost every discipline that has been quantified. You benefit from these applications every time you use your cell phone, drive your car, fly in an airplane, listen to a weather report, use the internet, take a modern medicine, or do many other daily activities.

6.1. General Guidelines.

Mathematical problems that arise in applications are usually stated as word problems. The problems considered in this chapter will ask questions whose answers require analyzing an initial-value problem. These problems typically will not hand you an initial-value problem. Rather, you will have to construct an initial-value problem from the information given in a word problem. In other words, you must transform a problem stated verbally into a problem that you can analyze by the methods we have been studying. Often this can be done more than one way. The more ways you see to approach a problem, the better you understand it. Here we address how to go about seeking ways to approach these problems.

There is no set of magic steps by which you can approach every word problem. Rather, you should be ready to consider several possible approaches when you are faced with one. To guide your efforts, it is helpful to keep in mind the following three objectives.

(1) *Identify the variables in the problem and the relationships between them.* An appropriately labeled picture that shows the units of each variable is often helpful in this regard. You should note carefully any relations between or constraints on the variables you have introduced. You should be prepared to rethink your choice of variables to help achieve the next objective.

(2) *Reduce the problem to analyzing an initial-value problem.* You should use relations between the various variables to reduce the number of parameters in the differential equation as much as possible. These relations may or may not be stated explicitly in the problem. Similarly, constraints may or may not be stated explicitly in the problem (like the fact that populations should not be negative).

(3) *Solve the resulting problem.* You should pick the method to suit the problem. What method is best will depend on the question being asked. Sometimes a graphical method will be the fastest route to the solution. A numerical method might be best when analytical and graphical methods prove to be difficult.

These are the so-called **IRS** guidelines, which apply to many kinds of word problems: **I**dentify the problem; **R**educe the problem to one that can be analyzed; **S**olve the reduced problem.

The remainder of this chapter will illustrate how these guidelines can help us approach word problems in the context of several applications.

6.2. **Tanks and Mixtures.** These represent a broad class of problems in which a question is asked about the transport of some quantity into and out of a tank or some other volume. The quantity might be a fluid like water, oil, or air, or it might be something carried by a fluid like a polutant or solute. The tank might be any well-defined volume like a pond, lake, or room in a building. These problems generalize to ones involving networks of interconnected tanks, which lie at the heart of many numerical simulations of fluids.

In the following problems we will construct an initial-value problem for the amount Q of some quantity in the tank. The associated ordinary differential equation will have the form

$$\frac{\mathrm{d}Q}{\mathrm{d}t} = \text{RATE IN} - \text{RATE OUT}\,,$$

where RATE IN is the rate the quantity enters the tank while RATE OUT is the rate the quantity exits the tank. Sometimes RATE IN and RATE OUT will be given explicitly in the problem. At other times they will be given in terms of other variables in the problem. Their units should be the units of Q over time.

Example. A tank with a capacity of 250 liters initially contains 200 liters of brine (salt solution) with a salt concentration of 5 grams per liter. At some instant brine with a salt concentration of 0.4 grams per liter begins to flow into the tank at a rate of 3 liters per minute, while the well-stirred mixture flows out at the same rate. How long will it take for the salt concentration in the tank to be reduced to 0.7 grams per liter?

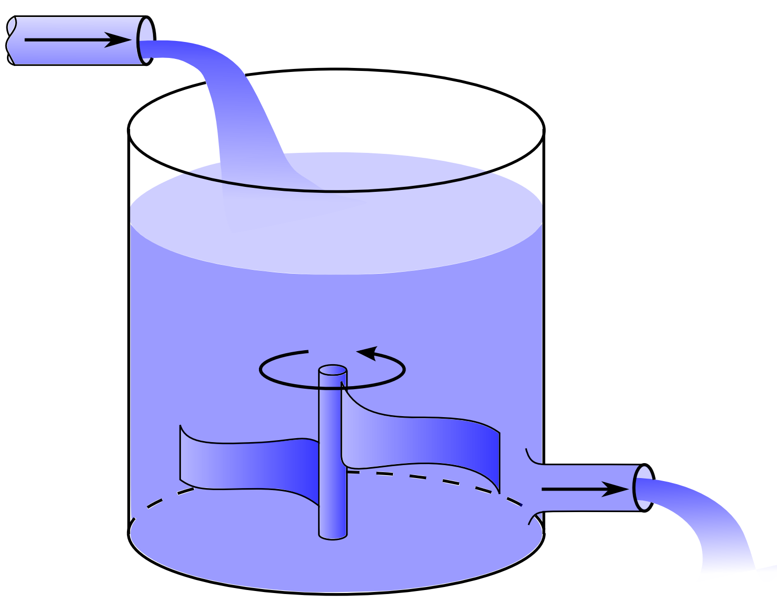


FIGURE 6.1. A mixing tank

Solution. Let $V(t)$ be the volume (lit) of brine in the tank at time t minutes. Let $S(t)$ be the mass (gr) of salt in the tank at time t minutes. Because the mixture is assumed to be well-stirred, the salt concentration of the brine in the tank at time t is $C(t) = S(t)/V(t)$. In particular, this will be the concentration of the brine that flows out of the tank. We have the following picture.

inflow				outflow	
0.4 gr/lit 3 lit/min	$\longrightarrow$	brine $V(t)$ lit salt $S(t)$ gr	$\longrightarrow$	$C(t)$ gr/lit 3 lit/min	$V(0) = 200$ lit, $C(0) = 5$ gr/lit.

We want to find the time T at which $C(T) = 0.7$.

Because brine flows in and out of the tank at the same rate of 3 liters per minute, the volume $V(t)$ of brine in the tank is given by $V(t) = V(0) = 200$. Hence, $C(t) = S(t)/200$. Therefore $S(t)$ satisfies

$$\frac{\mathrm{d}S}{\mathrm{d}t} = \text{RATE IN} - \text{RATE OUT} = 0.4 \cdot 3 - \frac{S}{200} \cdot 3 = 1.2 - \tfrac{3}{200} S\,.$$

We can check that each term in this equation has units of gr/min. Because $S(0) = 200C(0) = 200 \cdot 5 = 1000$, the initial-value problem that governs $S(t)$ is

$$\frac{\mathrm{d}S}{\mathrm{d}t} = 1.2 - 0.015S\,, \qquad S(0) = 1000\,.$$

We want to find T such that $S(T) = 200C(T) = 200 \cdot .7 = 140$.

This is a nonhomogeneous linear differential equation with normal form

$$\frac{\mathrm{d}S}{\mathrm{d}t} + 0.015S = 1.2\,.$$

Its integrating factor form is

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(e^{0.015t}S\right) = 1.2e^{0.015t}\,.$$

Upon integrating this equation while using the initial condition $S(0) = 1000$ we obtain

$$e^{0.015t}S - 1000 = \frac{1.2}{0.015}\left(e^{0.015t} - 1\right) = 80\left(e^{0.015t} - 1\right)\,,$$

whereby

$$S(t) = 920e^{-0.015t} + 80\,.$$

This solution is graphed below. By setting $S(T) = 140$ we find that T satisfies

$$920e^{-0.015T} + 80 = 140\,.$$

By solving for T we obtain

$$T = \frac{1}{0.015}\log\left(\frac{920}{60}\right) = \frac{200}{3}\log\left(\frac{46}{3}\right) \quad \text{minutes}\,.$$

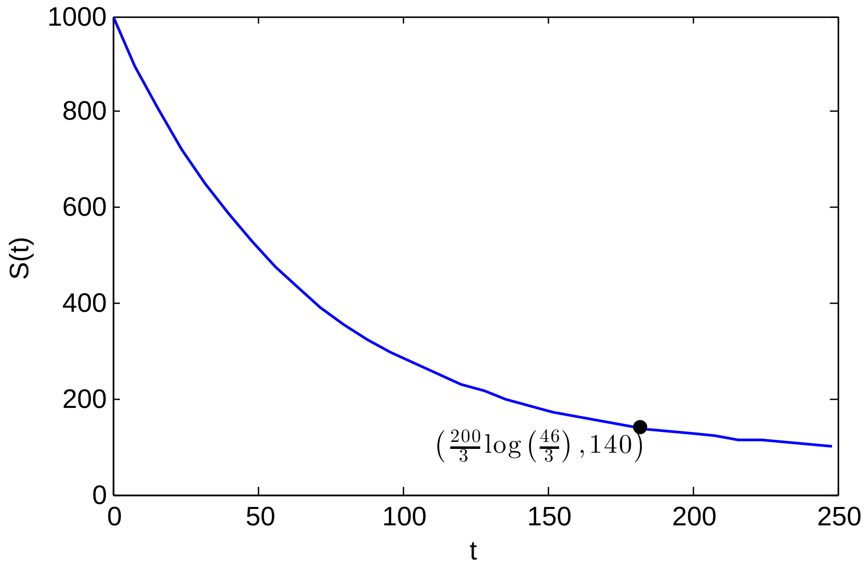


FIGURE 6.2. Graph of the solution to the first salt tank example.

Example. A tank with a capacity of 150 liters and an open top initially contains 120 liters of brine (salt solution) with a salt concentration of 3 grams per liter. At some instant brine with a salt concentration of 0.5 grams per liter begins to flow into the tank at a rate of 4 liters per minute, while the well-stirred mixture flows out at rate of 2 liters per minute. What will be the salt concentration in the tank when it overflows?

Solution. Let $V(t)$ be the volume (lit) of brine in the tank at time t minutes. Let $S(t)$ be the mass (gr) of salt in the tank at time t minutes. Because the mixture is assumed to be well-stirred, the salt concentration of the brine in the tank at time t is $C(t) = S(t)/V(t)$. In particular, this will be the concentration of the brine that flows out of the tank. We have the following picture.

inflow				outflow	
0.5 gr/lit 4 lit/min	$\longrightarrow$	brine $V(t)$ lit salt $S(t)$ gr	$\longrightarrow$	$C(t)$ gr/lit 2 lit/min	$V(0) = 120$ lit, $C(0) = 3$ gr/lit.

We asked to find $C(T)$ at the time T when $V(T) = 150$.

Because brine flows in at 4 liters per minute and out at 2 liters per minute, the volume $V(t)$ of brine in the tank increases at $4-2$ liters per minute. Therefore, because $V(0) = 120$ liters, we see that

$$V(t) = 120 + 2t\,.$$

Because the capacity of the tank is 150 liters, it overflows at the time T that solves $V(T) = 150$, which is $T = 15$. Because the mixture is well-stirred, we have

$$C(t) = \frac{S(t)}{120 + 2t}\,.$$

Therefore $S(t)$ satisfies

$$\frac{\mathrm{d}S}{\mathrm{d}t} = \text{RATE IN} - \text{RATE OUT} = 0.5 \cdot 4 - \frac{S}{120 + 2t} \cdot 2 = 2 - \frac{1}{60 + t} S\,.$$

We can check that each term in this equation has units of gr/min. Because $S(0) = 120C(0) = 120 \cdot 3 = 360$, the initial-value problem that governs $S(t)$ is

$$\frac{\mathrm{d}S}{\mathrm{d}t} = 2 - \frac{1}{60 + t} S\,, \qquad S(0) = 360\,.$$

We are asked to find $C(15)$.

This is a nonhomogeneous linear differential equation with normal form

$$\frac{\mathrm{d}S}{\mathrm{d}t} + \frac{1}{60 + t} S = 2\,, \qquad S(0) = 360\,.$$

Its integrating factor form is

$$\frac{\mathrm{d}}{\mathrm{d}t}\big((60 + t)S\big) = 2(60 + t)\,.$$

Upon integrating this equation while using the initial condition $S(0) = 360$ we obtain

$$(60 + t)S(t) - 60 \cdot 360 = \int_0^t 2(60 + s)\mathrm{d}s = (60 + t)^2 - 60^2\,,$$

whereby

$$S(t) = 60 + t + \frac{18000}{60 + t}\,.$$

Therefore

$$C(t) = \frac{S(t)}{V(t)} = \frac{1}{2} + \frac{9000}{(60 + t)^2}\,,$$

whereby

$$C(15) = \frac{1}{2} + \frac{9000}{75^2} = \frac{1}{2} + \frac{8}{5} = 2.1 \quad \text{gr/lit}\,.$$

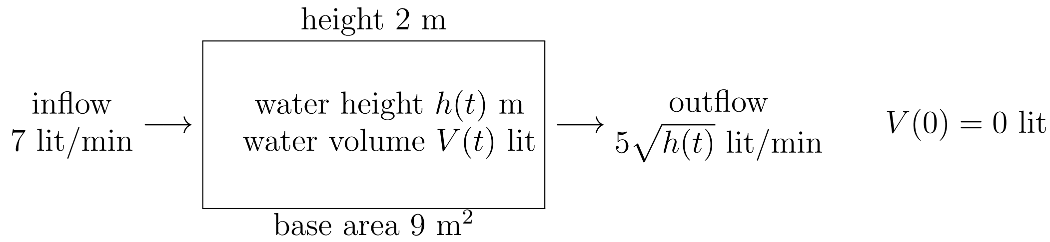
When a tank drains due to gravity, the rate at which it drains at a rate proportional to the square root of the height of the top of the fluid above the drain. More precisely, if the drain is at the bottom of the tank and the top of the fluid is at a height $h(t)$ above the bottom then the volume of fluid flows out of the drain at the rate

$$\text{RATE OUT} = a\sqrt{2gh}\,,$$

where g is the accelleration of gravity ($g = 9.8$ meters per sec^2) and a is the cross sectional area of the stream of water flowing out of the drain. This is the *Torricelli Law.* Torricelli observed a version of it in 1643, well before Newton devised his mechanics.

Example. Consider a tank with an open top that has a square base with 3 meter sides and a height of 2 meters. The tank is initially empty when water begins to pour into it at a rate of 7 liters per minute. The water also drains from the tank through a hole in its bottom at a rate of $5\sqrt{h}$ liters per minute where h is the height of the water in the tank in meters. Will the tank overflow? If not, how high will it fill? If so, how long does it take to happen?

Solution. Let $V(t)$ be the volume (lit) of water in the tank at time t minutes. We have the following picture.



Because 1 $\mathrm{m}^3 = 1000$ lit and because the area of the square base of the tank is $3^2 = 9$ m^2 while the water height is $h(t)$, we see that

$$V(t) = 1000 \cdot 9 \cdot h(t) = 9000 \cdot h(t) \quad \text{lit}\,.$$

Because the tank is initially empty, it is clear that $h(0) = 0$ and that $h(t)$ is an increasing function of time. We want to determine if $h(T) = 2$ for some T. If not, we want to determine the value $h(t)$ approaches as $t \to \infty$.

Because $V(t)$ satisfies

$$\frac{\mathrm{d}V}{\mathrm{d}t} = \text{RATE IN} - \text{RATE OUT} = 7 - 5\sqrt{h}\,,$$

the initial-value problem that governs $h(t)$ is

$$9000\,\frac{\mathrm{d}h}{\mathrm{d}t} = 7 - 5\sqrt{h}\,, \qquad h(0) = 0\,.$$

We can check that each term in the differential equation has units of lit/min. We want to determine if $h(T) = 2$ for some T. If not, we want to determine the value $h(t)$ approaches as $t \to \infty$.

This equation is autonomous. It has one stationary point at $h = \frac{49}{25}$. Its phase-line portrait is

$$\begin{array}{ccc} + & & - \\ \rightarrow\rightarrow\rightarrow\rightarrow\rightarrow\rightarrow & \bullet & \leftarrow\leftarrow\leftarrow\leftarrow\leftarrow\leftarrow\; h \\ & \frac{49}{25} & \end{array}$$

This portrait shows that if $h(0) = 0$ then $h(t) \to \frac{49}{25}$ as $t \to \infty$. Because $\frac{49}{25} = 1.96 < 2$, the tank does not overflow. Rather, the water approaches a height of 1.96 meters as $t \to \infty$.

6.3. **Population Dynamics.** We now consider models of population dynamics governed by first-order equations of the form

$$\frac{\mathrm{d}p}{\mathrm{d}t} = R(p)p - h(t)\,. \tag{6.1}$$

where $p(t)$ is the size of the population as a function of time, $R(p)$ models the growth rate of the population as a function of p, and $h(t)$ is a harvest rate due to predators. For a population in a closed ecosystem (which will have $h(t) = 0$) the growth rate is simply the birth rate minus the death rate. In more complicated situations the growth rate must also account for migration in to and out of the ecosystem. The harvest rate can model the catch limit in fish populations, the hunting limit in deer populations, or the introduction of any other predator that will reduce the population at a rate that is independent of the size of the population. Obviously such a harvest model will break down if the population is reduced too much. Sometimes it is better to model the introduction of a predator by simply increasing the death rate. We must determine which model to use from the problem.

6.3.1. *Exponential Model.* The simplest such models take $h(t) = 0$ and $R(p) = r$ for some constant r, whereby (6.1) becomes

$$\frac{\mathrm{d}p}{\mathrm{d}t} = rp\,. \tag{6.2}$$

This is the so-called *exponential model* because it has the solution $p(t) = p_I e^{rt}$ when $p(0) = p_I$. This solution grows exponentially when $r > 0$ and decays exponentially when $r < 0$.

Sometimes we will have to figure out the value of r from other information in the problem. For example, if we are told that a population triples every five years then we are being told that $p(t+5) = 3p(t)$. Then we can figure out r by setting

$$3 = \frac{p(t+5)}{p(t)} = \frac{p_I e^{r(t+5)}}{p_I e^{rt}} = e^{r5}\,,$$

whereby $r = \frac{1}{5}\log(3)$ per year. Alternatively, we are being told that $p(t) = p_I 3^{\frac{1}{5}t}$. Because $3^{\frac{1}{5}t} = e^{\frac{1}{5}\log(3)t}$, we can read off that $r = \frac{1}{5}\log(3)$ per year.

If a harvest $h(t) > 0$ is introduced into an exponential model then it takes the form

$$\frac{\mathrm{d}p}{\mathrm{d}t} = rp - h(t)\,. \tag{6.3}$$

This is a nonhomogeneous linear equation that we know how to solve.

Example. In the absence of predators the population of mosquitoes in a certain area would increase at a rate proportional to its population and would double every three weeks. There are 250,000 mosquitoes in the area initially when a flock of birds arrives that eats 80,000 mosquitoes per week. How many mosquitoes remain after two weeks?

Solution. Let $M(t)$ be the number of mosquitoes at time t weeks. Doubling every three weeks means the population grows like $2^{\frac{1}{3}t} = e^{\frac{1}{3}\log(2)t}$, which implies a growth rate of $\frac{1}{3}\log(2)$ per week. The rate at which the mosquitoes reproduce thereby is $\frac{1}{3}\log(2)M(t)$ while the rate at which they are eaten is 80,000. Therefore the initial-value problem that M satisfies is

$$\frac{\mathrm{d}M}{\mathrm{d}t} = \tfrac{1}{3}\log(2)M - 80,000\,, \qquad M(0) = 250,000\,.$$

We can check that each term in the differential equation has units of mosquitoes/week. We are asked to find $M(2)$.

This is a nonhomogeneous linear differential equation with normal form

$$\frac{\mathrm{d}M}{\mathrm{d}t} - \tfrac{1}{3}\log(2)M = -80,000\,.$$

Its integrating factor form is

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(e^{-\frac{1}{3}\log(2)t}M\right) = -80,000e^{-\frac{1}{3}\log(2)t}\,.$$

Upon integrating this equation while using the initial condition $M(0) = 250,000$ we obtain

$$e^{-\frac{1}{3}\log(2)t}M(t) - 250,000 = -80,000\int_0^t e^{-\frac{1}{3}\log(2)s}\,\mathrm{d}s = \frac{80,000}{\frac{1}{3}\log(2)}\left(e^{-\frac{1}{3}\log(2)t} - 1\right)\,.$$

Hence, we find that

$$\begin{aligned} M(t) &= \frac{80,000}{\frac{1}{3}\log(2)} + \left(250,000 - \frac{80,000}{\frac{1}{3}\log(2)}\right)e^{\frac{1}{3}\log(2)t} \\ &= \frac{240,000}{\log(2)} + \left(250,000 - \frac{240,000}{\log(2)}\right)2^{\frac{1}{3}t}\,. \end{aligned}$$

Therefore after two weeks there are

$$M(2) = \frac{240,000}{\log(2)} + \left(250,000 - \frac{240,000}{\log(2)}\right)4^{\frac{1}{3}} \quad \text{mosquitoes}\,.$$

This is about 193,500 mosquitoes, but the answer should be left in the exact form above. It would be nice if the birds stuck around for several more weeks.

6.3.2. *Logistic Model.* The ability of a individual to survive and reproduce depends upon the environment experienced by that individual. The assumption in the exponential model that $R(p)$ is constant assumes that this environment is uneffected by the number of individuals present. However once the population grows large enough there might be competition between its individuals for resources, which should reduce the growth rate. The simplest model that captures this effect takes $R(p)$ to be a linear function of p,

$$R(p) = r - ap\,, \qquad \text{for some positive constants } a \text{ and } r\,.$$

When there is no harvesting then model (6.1) becomes

$$\frac{\mathrm{d}p}{\mathrm{d}t} = (r - ap)p\,. \tag{6.4}$$

This is called the *logistic model*. It was introduced in 1838 by Pierre-François Verhulst because it gives a better fit to observed population data than the exponential model (6.2).

Phase-Line Portrait. The logistic model is autonomous and can be solved analytically. However, there are many questions that can be addressed with a phase-line portrait. The stationary points of the equation are $p = 0$ and $p = \frac{r}{a}$. The phase-line portrait is as follows.

$$-\qquad\qquad\qquad +\qquad\qquad\qquad -$$

$$\leftarrow\leftarrow\leftarrow\leftarrow\leftarrow\leftarrow \bullet \rightarrow\rightarrow\rightarrow\rightarrow\rightarrow\rightarrow \bullet \leftarrow\leftarrow\leftarrow\leftarrow\leftarrow\leftarrow \; h$$
$$\qquad 0 \qquad\qquad\qquad \tfrac{r}{a}$$

This portrait shows that if $p(t)$ is any solution with $p(0) > 0$ then $p(t) \to \frac{r}{a}$ as $t \to \infty$. In other words, the stationary point $\frac{r}{a}$ is attracting (asymptotically stable). This stationary point is called the *carrying capacity* of the ecosystem. It is the size of the population that the ecosystem naturally supports. Notice that any solution with $0 < p(0) < k$ has the interval of definition $(-\infty, \infty)$, while we expect that any solution with $k < p(0)$ has an interval of definition of the form (t_L, ∞) for some $t_L < 0$.

Explicit Solution. If we let $k = r/a$ denote the carrying capacity of the ecosystem then the logistic model (6.4) can be expressed as

$$\frac{\mathrm{d}p}{\mathrm{d}t} = r\left(1 - \frac{p}{k}\right)p\,.$$

Its stationary solutions are $p = 0$ and $p = k$. Its nonstationary solutions can be found by our recipe for autonomous equations. The phase-line portrait showed that solutions cannot cross the stationary point at $p = 0$, so solutions that are initially positive will remain positive. Such a solution is governed implicitly by

$$\begin{aligned} rt &= \int \frac{1}{\left(1 - \dfrac{p}{k}\right)p}\,\mathrm{d}p = \int \frac{k}{(k-p)\,p}\,\mathrm{d}p = \int \frac{1}{p} + \frac{1}{k-p}\,\mathrm{d}p \\ &= \log(p) - \log(|k-p|) + c = \log\left(\frac{p}{|k-p|}\right) + c\,. \end{aligned}$$

Here we do not need absolute values on p because we are considering positive solutions.

If we impose the initial condition $p(0) = p_I > 0$ then we find that

$$0 = \log\left(\frac{p_I}{|k-p_I|}\right) + c\,,$$

so that $c = -\log(p_I/|k-p_I|)$. The implicit solution then becomes

$$rt = \log\left(\frac{p}{|k-p|}\right) - \log\left(\frac{p_I}{|k-p_I|}\right) = \log\left(\frac{p\,(k-p_I)}{p_I\,(k-p)}\right)\,,$$

where we have dropped the absolute values because we know from the phase-line portrait that $k - p$ and $k - p_I$ will have the same sign. This can be exponentiated to find

$$e^{rt} = \frac{p\,(k-p_I)}{p_I\,(k-p)}\,,$$

whereby $p_I\,(k-p)\,e^{rt} = p\,(k-p_I)$, which can be solved to obtain the explicit solution

$$p(t) = \frac{k\,e^{rt}\,p_I}{k + (e^{rt}-1)p_I}\,.$$

Notice that if $0 < p_I < k$ then the interval of definition is $(-\infty, \infty)$ while if $k < p_I$ then the interval of definition is $(\log(1 - k/p_I)/r, \infty)$. This is consistent with what we expected from our earlier analysis of the phase-line portrait.

Remark. The above explicit solution depends on three parameters: the initial value p_I, the bare growth rate r, and the carrying capacity k. Therefore the solution is uniquely determined by the initial condition plus two additional conditions. For example, if the population is measured at three times: $t = 0$, $t = T$, and $t = 2T$ and is found to be p_I, p_T and p_{2T} respectively then the parameters r and k can be determined from the additional conditions $p(T) = p_T$ and $p(2T) = p_{2T}$. By using the formula for $p(t)$ derived above, these conditions are

$$p_T = \frac{k\, e^{rT}\, p_I}{k + (e^{rT} - 1)p_I}\,, \qquad p_{2T} = \frac{k\, e^{r2T}\, p_I}{k + (e^{r2T} - 1)p_I}\,.$$

By reversing the last step in the derivation of $p(t)$, we see that these conditions are equivalent to the equations

$$e^{rT} = \frac{p_T\,(k - p_I)}{p_I\,(k - p_T)}\,, \qquad e^{r2T} = \frac{p_{2T}\,(k - p_I)}{p_I\,(k - p_{2T})}\,.$$

Upon using the first equation to eliminate e^{rT} from the second equation we find that

$$\left(\frac{p_T\,(k - p_I)}{p_I\,(k - p_T)}\right)^2 = \frac{p_{2T}\,(k - p_I)}{p_I\,(k - p_{2T})}\,.$$

After clearing the denominators we obtain a quadratic equation that can be solved to determine k. We can then recover r from

$$r = \frac{1}{T}\log\left(\frac{p_T\,(k - p_I)}{p_I\,(k - p_T)}\right)\,.$$

Example. A population of squirrels is introduced onto a big island in the middle of a large lake. The population is found to be three times the initial population after three years and five times the initial population after six years. Use the logistic model to estimate the carrying capacity for squirrels on the island as a multiple of the initial population.

Solution. Let p_0 be the initial population of squirrels. Let $k = mp_0$ be the carrying capacity for the squirrels on the island. We are asked to find m. Because the population of squirrels is $3p_0$ after three years and is $5p_0$ after six years, we know that $m > 5$. The solution $p(t)$ of the logistic model satisfies

$$e^{rt} = \frac{p(t)\,(mp_0 - p_0)}{p_0\,(mp_0 - p(t))} = \frac{p(t)}{p_0}\,\frac{m - 1}{m - \frac{p(t)}{p_0}}\,.$$

By evaluating this at $t = 3$ and $t = 6$ we find that

$$e^{r3} = 3\,\frac{m - 1}{m - 3}\,, \qquad e^{r6} = 5\,\frac{m - 1}{m - 5}\,.$$

Upon using the first equation to eliminate e^{r3} from the second equation we find that

$$\left(3\,\frac{m - 1}{m - 3}\right)^2 = 5\,\frac{m - 1}{m - 5}\,.$$

This reduces to

$$9(m - 1)(m - 5) = 5(m - 3)^2\,,$$

which reduces to $m^2 - 6m = 0$. Hence $m = 6$, because we know $m > 5$. Therefore the logistic model predicts that the carrying capacity for squirrels on the island is six times the initial population.

Remark. In practice the values for r and k in a logistic model are seldom obtained by fitting data at just two additional times. Rather, statistical methods are used that give a "best fit" to data at many times.

Example. A population of fish is introduced into a large lake. Suppose that the bare growth rate r and carrying capacity k for the logistic model have been determined by a careful fit to data. If we use the logisitc model

$$\frac{\mathrm{d}p}{\mathrm{d}t} = r\left(1 - \frac{p}{k}\right)p - h\,, \tag{6.5}$$

then how large can the harvest rate h be and still sustain the fish population?

Solution. The stationary points of this equation satisfy

$$0 = r\left(1 - \frac{p}{k}\right)p - h\,.$$

If we divide this equation by rk and complete the square, it becomes

$$\begin{aligned} 0 &= \left(1 - \frac{p}{k}\right)\frac{p}{k} - \frac{h}{rk} \\ &= \frac{p}{k} - \left(\frac{p}{k}\right)^2 - \frac{h}{rk} \\ &\frac{1}{4} - \frac{h}{rk} - \left(\frac{p}{k} - \frac{1}{2}\right)^2 . \end{aligned}$$

This has a real solution if and only if $\frac{1}{4} - \frac{h}{rk} \geq 0$. Therefore to sustain the fish population we must have

$$h \leq \frac{rk}{4}\,,$$

in which case the stable stationary point is

$$p = k\left(\frac{1}{2} + \sqrt{\frac{1}{4} - \frac{h}{rk}}\right).$$

This is the adjusted carrying capacity.

Remark. Due to the inherent uncertainty is the logistic model, it is wise to pick a harvest rate h below the bound found above. For example, if we take

$$h = \frac{9rk}{100}\,,$$

then the adjusted carrying capacity is $.9k$.

Remark. The logistic model is the simplest model for which the growth rate $R(p)$ is a decreasing function of p. However, as populations get smaller their growth rate might also decline. This can be due to the fact that a smaller herd might offer easier prey to predators, or that individuals in a more dispersed population might have a harder time finding mates. The simplest model that also captures this effect takes $R(p)$ to be a quadratic function of p,

$$R(p) = r + bp - ap^2\,, \qquad \text{for some constants } r,\ b \text{ and } a \text{ with } a > 0 \text{ and } b > 0\,.$$

When there is no harvesting then model (6.1) becomes

$$\frac{\mathrm{d}p}{\mathrm{d}t} = \left(r + bp - ap^2\right) p\,.$$

The growth rate $R(p)$ will be positive for some $p > 0$ whenever

$$-\frac{b^2}{4a} < r\,.$$

In that case the differential equation has one positive stationary point when $r \geq 0$ and two positive stationary points when $r < 0$.

6.4. **Motion.** A moving object with fixed mass m is governed by the Newton law of motion $ma = F$, where a is its acceleration and F is the net force acting on it. The net force is the sum of the gravitational force and the drag force. The velocity $v(t)$ of the object is governed by the Newtonian law of motion

$$m\frac{\mathrm{d}v}{\mathrm{d}t} = \text{gravitational force} + \text{drag force} = F_{\text{grav}} + F_{\text{drag}}\,, \tag{6.6}$$

The gravitational force will pull the object downward. The drag force acts in the direction opposite to that of the velocity of the object. If the object is going up then the drag force acts downward while if the object is going down the drag force acts upward.

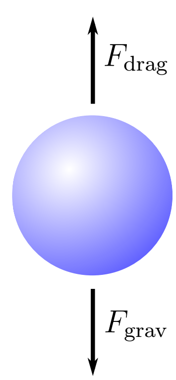


FIGURE 6.3. The action of the forces F_{drag} and F_{grav} on a falling object.

For objects near the surface of the Earth that move distances that are short compared to the radius of the Earth, we can approximate the gravitational force as the constant mg where g is the so-called gravitaional acceleration given by $g = 9.8\ \text{m}/\text{sec}^2$. Therefore, if we neglect the density of the medium through which the object is moving, we can use the approximation

$$F_{\text{grav}} = mg\,. \tag{6.7}$$

Here we have used the convention that downward velocities are positive, which is natual for problems in which the object is falling. This is an excellent approximation for an object moving through the atmosphere when the density of the object is much greater than that of the air.

Remark. More generally, if an object is moving through a denser medium (like water or oil) then we can use the approximation

(6.8a) $$F_{\rm grav} = \alpha m g \,,$$

where α is expressed in terms of the density of the object $\rho_{\rm obj}$ and the density of the medium $\rho_{\rm med}$ as

(6.8b) $$\alpha = \frac{\rho_{\rm obj} - \rho_{\rm med}}{\rho_{\rm obj} + \rho_{\rm med}} \,.$$

This nondimensional ratio α is called the *Atwood number.* It takes values between -1 and 1.

- It is near 1 when $\rho_{\rm med}$ is much smaller than $\rho_{\rm obj}$.
- It is negative when $\rho_{\rm med}$ is greater than $\rho_{\rm obj}$.
- It is near -1 when $\rho_{\rm med}$ is much greater than $\rho_{\rm obj}$.

When α is negative gravity will cause the the object to rise because it is lighter than the surrounding medium. For example, a ping pong ball released under water will rise. In that case the Atwood number is very close to -1.

The drag force $F_{\rm drag}$ is harder to approximate. While many processes can contribute to drag, we consider just two here.

- **Viscous Drag.** As the object moves it heats the surrounding medium due to viscous forces. It thereby loses some of its kinetic energy to this process, which slows it down. We will model the viscous drag as
$$F_{\rm visc} = -bv \,,$$
where b is a nonnegative constant.
- **Turbulent Drag.** As the object falls it can create a wake in the surrounding medium. In that case it loses some of its kinetic energy to this process, which also slows it down. We will model the turbulent drag as
$$F_{\rm turb} = -c|v|v \,,$$
where c is a nonnegative constant.

Therefore our general model for drag has the form

(6.9) $$F_{\rm drag} = F_{\rm visc} + F_{\rm turb} = -bv - c|v|v \,.$$

Remark. A simple model for viscous drag sets

(6.10a) $$F_{\rm visc} = -\rho_{\rm med}\nu_{\rm med} L v \,,$$

where $\rho_{\rm med}$ is the density of the medium, $\nu_{\rm med}$ is the kinematic viscosity of the medium, and L is a length scale. Stokes showed that $L = 6\pi R$ for a sphere of radius R. However there is no such simple formula for general objects. Often L is determined by experiment. If values for ρ_{med}, ν_{med}, and L are needed for a problem then they will be given to you.

Remark. A simple model of turbulent drag through air sets

$$F_{\text{turb}} = -\tfrac{1}{2}\rho_{\text{air}} A|v|v\,, \tag{6.10b}$$

where ρ_{air} is the density of the air and A is the aerodynamic cross-section, which has units of area. The value of ρ_{air} varies significantly with temperature, pressure, and humidity; for dry air at 20 °C and one atmosphere pressure ρ_{air} is about 1.2 kg/m^3. The value of A depends upon the shape of the object, the air density, and the velocity v in an extremely complicated way. When the speed of the falling object is far below the speed of sound then the dominant dependence is on the shape of the object. Often A is determined by experiment. If values for ρ_{air} and A are needed for a problem then they will be given to you.

We will first consider the case of an object moving through the atmosphere. In this case the viscous drag can typically be neglected. The resulting equation of motion is then a first-order differential equation in the form

$$\frac{\mathrm{d}v}{\mathrm{d}t} = g - k|v|v\,, \qquad \text{where} \quad k = \frac{\rho_{\text{air}} A}{2m} > 0\,. \tag{6.11}$$

Each term in this differential equation has units of acceleration (= speed/time = length/time^2.) This equation is autonomous. Its right-hand side is differentiable with respect to v because $\partial_v(|v|v) = 2|v|$, so that either Theorem 3.1 or Theorem 4.1 can be applied to it. Its only stationary solution is

$$v_\infty = \sqrt{\frac{g}{k}}\,. \tag{6.12}$$

Its phase-line portrait is

$$\begin{array}{c} \quad + \qquad\qquad\qquad - \\ \rightarrow\rightarrow\rightarrow\rightarrow\rightarrow\rightarrow \bullet \leftarrow\leftarrow\leftarrow\leftarrow\leftarrow\leftarrow\; v \\ v_\infty \end{array}$$

This portrait shows that the stationary point v_∞ is attracting, and that $v(t) \to v_\infty$ as $t \to \infty$ for every nonstationary solution. The velocity v_∞ is called the terminal velocity because it is the velocity being approached by the falling object until it hits the ground.

Remark. The terminal velocity depends upon the aerodynamic cross-section A as $A^{-\frac{1}{2}}$. Skydivers control the rate of their descent by changing their aerodynamic cross-section. They are close to terminal velocity after falling about ten seconds. They can reach speeds of 90 m/s in a bullet-like position or slow to about 50 m/s by spreading their arms and legs. With a parachute deployed they slow to 3-4 m/s, depending on the design of the parachute.

Example. A skydiver of mass 60 kg jumps from an airplane and assumes a position with an aerodynamic cross-section of 0.1 m^2 in air with a density of 1.2 kg/m^3. What is her terminal velocity? What fraction of her terminal velocity does she reach after 10 s? How far has she fallen after 10 s?

Solution. Let $v(t)$ be her downward velocity at t seconds. Because she is always falling during the ten seconds, we know that $v(t) \geq 0$ and have the following picture.

$$\begin{array}{c} \text{gravitational} \\ \text{acceleration} \\ = 9.8 \text{ m/s}^2 \end{array} \longrightarrow \boxed{\begin{array}{c} \text{downward} \\ \text{velocity } v(t) \text{ m/s} \end{array}} \longrightarrow \begin{array}{c} \text{drag} \\ \text{acceleration} \\ = -kv^2 \text{ m/s}^2 \end{array} \qquad v(0) = 0\,.$$

We first seek v_∞ and $v(10)/v_\infty$, where v_∞ is her terminal velocity.

The initial-value problem satisfied by $v(t)$ is

$$\frac{\mathrm{d}v}{\mathrm{d}t} = g - kv^2\,, \qquad v(0) = 0\,,$$

where $g = 9.8$ m/s^2 and

$$k = \frac{\rho_{\text{air}} A}{m} = \frac{1.2 \cdot 0.1}{60} = \frac{1}{500} = .002 \text{ 1/m}\,.$$

We can check that each term in the differential equation has units of m/s^2. Therefore her terminal velocity is

$$v_\infty = \sqrt{\frac{g}{k}} = \sqrt{9.8 \cdot 500} = \sqrt{4900} = 70 \text{ m/s}\,.$$

The next step is to compute $v(t)$. The inital-value problem can be expressed as

$$\frac{\mathrm{d}v}{\mathrm{d}t} = k\left(v_\infty^2 - v^2\right)\,, \qquad v(0) = 0\,,$$

where $k = .002$ and $v_\infty = 70$. This differential equation is autonomous, so an implicit solution can be found by using a partial fraction decomposition as

$$\begin{aligned} kt &= \int \frac{1}{v_\infty^2 - v^2}\,\mathrm{d}v = \int \frac{1}{(v_\infty + v)(v_\infty - v)}\,\mathrm{d}v = \int \frac{\frac{1}{2v_\infty}}{v_\infty + v} + \frac{\frac{1}{2v_\infty}}{v_\infty - v}\,\mathrm{d}v \\ &= \frac{1}{2v_\infty}\log(v_\infty + v) - \frac{1}{2v_\infty}\log(v_\infty - v) + c = \frac{1}{2v_\infty}\log\left(\frac{v_\infty + v}{v_\infty - v}\right) + c\,. \end{aligned}$$

Here we do not need absolute values inside the log because we know from the phase-line portrait that $v(t)$ will increase from 0 to the terminal velocity v_∞. The initial condition $v(0) = 0$ implies that

$$k \cdot 0 = \frac{1}{2v_\infty}\log\left(\frac{v_\infty + 0}{v_\infty - 0}\right) + c = \frac{1}{2v_\infty}\log(1) + c = 0 + c\,,$$

whereby $c = 0$. By exponentiating the implicit solution we find that

$$e^{2v_\infty kt} = \frac{v_\infty + v}{v_\infty - v}\,,$$

which can be solved for v to obtain the explicit solution

$$v(t) = v_\infty \frac{e^{2v_\infty kt} - 1}{e^{2v_\infty kt} + 1} = 70\,\frac{e^{0.28t} - 1}{e^{0.28t} + 1}\,.$$

Therefore the fraction of her terminal velocity reached after ten seconds is

$$\frac{v(10)}{70} = \frac{e^{2.8} - 1}{e^{2.8} + 1} \,.$$

This is about 0.88535 of her terminal velocity, but the answer should be left in the exact form given above.

If we let $y(t)$ be distance she has fallen after t seconds then

$$\begin{aligned} y(10) &= \int_0^{10} v(t)\,\mathrm{d}t = \int_0^{10} 70\,\frac{e^{0.28t} - 1}{e^{0.28t} + 1}\,\mathrm{d}t = 70 \int_0^{10} \frac{2e^{0.28t}}{e^{0.28t} + 1} - 1\,\mathrm{d}t \\ &= 70\left[\frac{2}{0.28}\log(e^{0.28t} + 1) - t\right]\Bigg|_0^{10} = 70\,\left[\frac{1}{0.14}\log\left(\frac{e^{2.8} + 1}{2}\right) - 10\right] \\ &= 500\log\left(\frac{e^{2.8} + 1}{2}\right) - 700\,. \end{aligned}$$

This is about 383 meters, but the answer should be left in the exact form given above.

When the object is moving through a very viscous fluid (like honey) then there will be no tubulent wake. In that case the turbulent drag can be negelcted. The resulting equation of motion is then a first-order differential equation in the form

$$\text{(6.13)} \qquad \frac{\mathrm{d}v}{\mathrm{d}t} = \alpha g - rv\,, \qquad \text{where} \quad r = \frac{\rho_{\mathrm{med}}\nu_{\mathrm{med}}L}{m} > 0\,.$$

Each term in this differential equation has units of acceleration (= speed/time = length/time^2.) This equation is linear and autonomous. Its only stationary solution is

$$\text{(6.14)} \qquad v_\infty = \frac{\alpha g}{r}\,.$$

Its phase-line portrait is

$$\begin{array}{ccc} + & & - \\ \rightarrow\rightarrow\rightarrow\rightarrow\rightarrow\rightarrow & \bullet & \leftarrow\leftarrow\leftarrow\leftarrow\leftarrow\leftarrow\ v \\ & v_\infty & \end{array}$$

This portrait shows that the stationary point v_∞ is attracting, and that $v(t) \to v_\infty$ as $t \to \infty$ for every nonstationary solution. The velocity v_∞ is called the terminal velocity because it is the velocity being approached by the falling object until it is stopped.

The solution of (6.13) that satisfies the initial condition $v(0) = v_o$ is found to be

$$v(t) = e^{-rt}v_o + (1 - e^{-rt})v_\infty\,.$$

This is consistent with the phase-line portrait shown above.

6.5. **Loans.** A loan problem can be viewed as a tank problem where the loan balance is being drained away by the borrower. To keep things simple, we consider fixed-rate loans with continuously compounded interest and with continuous payments by the borrower at a constant rate. The term of the loan is the time period over which the loan is paid off. If we let $B(t)$ be the balance of the loan at time t years then $B(t)$ will satisfy

$$\frac{\mathrm{d}B}{\mathrm{d}t} = rB - P\,,$$

where r is the per annum interest rate and P is the per annum payment rate. If the initial loan amount is known to be B_I then we would impose the initial condition $B(0) = B_I$. However, if we are trying to find B_I or P, given r and the term T of the loan in years then we would impose the initial condition $B(T) = 0$.

Example. A car buyer has 4000$ for a down payment and can afford to make continuous payments for a loan at a constant rate of no more than 250$ per month. If five-year fixed-rate loans are available at an interest rate of 5% per year compounded continuously, what is the price of the most expensive car that the buyer can afford?

Solution. Let $B(t)$ be the balance of the loan at time t years. Because 1 year = 12 months, we see $P = 12 \cdot 250 = 3000$ \$/yr. Because $1 = 100\%$, we see $r = 5/100 = 0.05$ 1/yr. The term $T = 5$ yr. We have the following picture.

$$\begin{array}{c} \text{interest} \\ 5\% \text{ per yr} \\ (= 0.05 \text{ 1/yr}) \end{array} \longrightarrow \boxed{\text{balance } B(t)\ \$} \longrightarrow \begin{array}{c} \text{payments} \\ 250 \text{ \$/mon} \\ (= 3000 \text{ \$/yr}) \end{array} \qquad B(5) = 0\ \$\,.$$

The price of the most expensive car the buyer can afford will be $4000 + B(0)$ dollars.

The initial-value problem satisfied by $B(t)$ is

$$\frac{\mathrm{d}B}{\mathrm{d}t} = 0.05B - 3000\,, \qquad B(5) = 0\,.$$

We can check that each term in the differential equation has units of \$/yr. We want to find $4000 + B(0)$.

This is a nonhomogeneous linear differential equation with normal form

$$\frac{\mathrm{d}B}{\mathrm{d}t} - 0.05B = -3000\,.$$

Its integrating factor form is

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(e^{-0.05t}B\right) = -3000e^{-0.05t}\,.$$

Upon integrating this equation while using the initial condition $B(5) = 0$ we obtain

$$e^{-0.05t}B(t) = -3000\int_5^t e^{-0.05s}\,\mathrm{d}s = \frac{3000}{0.05}\left(e^{-0.05t} - e^{-0.05\cdot 5}\right) = 60000\left(e^{-\frac{1}{20}t} - e^{-\frac{1}{4}}\right)\,.$$

By setting $t = 0$ above we find that the car buyer can afford a car costing at most

$$4000 + 60000\left(1 - e^{-\frac{1}{4}}\right) \quad \text{dollars}\,.$$

This is about 17,272\$, but the answer should be left in the exact form above.

Example. A student borrows 6000\$ at an interest rate of 6% compounded continuously. The student wants to pay off the loan in five years by making payments continuously at a constant rate of P dollars per year. What should P be?

Solution. Let $B(t)$ be the balance of the loan at time t years. Because $1 = 100\%$, we see $r = 6/100 = 0.06$ 1/yr. The term $T = 5$ yr. We have the following picture.

interest 6% per yr (= 0.06 1/yr) ⟶ [balance $B(t)$ \$] ⟶ payments P \$/yr

$$B(0) = 6000\ \$\,,\qquad B(5) = 0\ \$\,.$$

One of the last two conditions will determine P.

The initial-value problem satisfied by $B(t)$ is

$$\frac{\mathrm{d}B}{\mathrm{d}t} = 0.06B - P\,, \qquad B(5) = 0\,.$$

We can check that each term in the differential equation has units of \$/yr. We want to find P such that $B(0) = 6000$.

This is a nonhomogeneous linear differential equation with normal form

$$\frac{\mathrm{d}B}{\mathrm{d}t} - 0.06B = -P\,.$$

Its integrating factor form is

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(e^{-0.06t}B\right) = -Pe^{-0.06t}\,.$$

Upon integrating this equation while using the initial condition $B(5) = 0$ we obtain

$$e^{-0.06t}B(t) = \frac{P}{0.06}\left(e^{-0.06t} - e^{-0.06\cdot 5}\right) = -P\int_5^t e^{-\frac{3}{50}s}\,\mathrm{d}s = \frac{50P}{3}\left(e^{-\frac{3}{50}t} - e^{-\frac{3}{10}}\right)\,.$$

By setting $t = 0$ we see that $B(0) = \frac{50P}{3}(1 - e^{-\frac{3}{10}})$. Then $B(0) = 6000$ implies that the student has to pay off the loan at a rate of

$$P = \frac{6000}{\frac{50}{3}(1 - e^{-\frac{3}{10}})} = \frac{360}{1 - e^{-\frac{3}{10}}} \quad \text{dollars per year}\,.$$

This is about 1,389\$ per year, but the answer should be left in the exact form above.

Exercises on Applications

(1) A tank with a capacity of 500 gal contains 200 gal of water with 100 lb of salt in solution. Water containing 1 lb of salt per gallon is entering the tank at a rate of 3 gal/min, and the mixture is allowed to flow out of the tank at a rate of 2 gal/min. Set up, but do not solve, the differential equation describing the rate of change in pounds of salt of the mixture before the tank overflows. Please simplify you equation and include all units (Hint: The amount of solution in the tank depends on time).

Short Answer
Solution

(2) A 500 liter tank initially has 200 liters of solution with a concentration of salt of 5g/L. Salt water with a concentration of 20g/L is added at a rate of 3 L/min while the well mixed solution is allowed to leave at a rate of 2 L/min. This process will continue until the tank is filled.

Please give the differential equation, initial condition, and interval of definition which models the amount of salt in the above system.

Short Answer
Solution

(3) A 900 gallon tank containing 600 gallons of salt water with a concentration of $\frac{1}{3}$ pounds of salt for every gallon of water. Fresh water is added at a rate of 3 gal/min, while the well mixed solution is emptied out at a rate of 1 gal/min. Find an equation for $S(t)$ which describes the amount of salt in the tank for the first 5 hours of this process (i.e. Let $S(t)$ be the number of pounds of salt in the tank after t minutes, until the moment before the tank would start to overflow in this process).

Short Answer
Solution

(4) A 500 liter tank initially has 100 liters of fresh water. Salt water with a concentration of 30g/L is added at a rate of 5 L/min while the well mixed solution is allowed to leave at a rate of 1 L/min. This process will continue until the tank is filled. Find a function which will model this system over the interval $[0, t^*]$ where t^* is the time at which the tank overflows.

Short Answer
Solution

(5) Suppose a dye solution is made by adding dye with a concentration of 1 lb/gal to a 20 gallon tank of fresh water at a rate of 2 gal/min, while the well mixed solution is emptied at the same rate. Find the function $D(t)$ which describes the amount of solution in pounds after t minutes.

Short Answer
Solution

(6) A tank with a capacity of 500 gallons originally contains 200 gallons of water with 50 lbs of salt in solution. Water containing 0.5 lbs of salt per gallon then

enters the tank at a rate of 4 gal/min while the well mixed solution exits the tank at a rate of 1 gal/min.

a) How long before the tank begins to overflow?

b) If t^* minutes is your answer for part (a) write down an equation which models the amount of salt, $S(t)$ lbs, in the tank for $0 \leq t \leq t^*$.

Short Answer
Solution

(7) Suppose you buy a house for \$ 300,000. You initially put \$ 30,000 as a down payment. So you take out a 15-yr loan for \$ 270,000 with interest compounded continuously with an annual percentage rate of of 3%. What will your monthly payments be (do not take into account any additional escrow accounts)?

Short Answer
Solution

(8) Suppose you want to buy a house and the current APR for a 30-yr fixed rate mortgage is 4 %. If you can only afford to pay \$ 2,000 a month, what is the largest loan you can afford?

Short Answer
Solution

(9) Suppose your car costs \$18000. What annual interest rate must you have if you are paying \$400 a month and pay off the loan in 5 years, when the interest is compounded continuously?

Short Answer
Solution

(10) Suppose you wish to buy a car which costs \$24,000. The car dealership charges a 5 % annual interest rate which is compounded continuously. You set up the loan so that you are paying \$ 400 a month. Let $A(t)$ denote the amount owed after t years. Set up the appropriate initial value problem, find an explicit solution for $A(t)$, and find how long it will take to pay off the loan.

Short Answer
Solution

(11) Suppose you get a mortgage for \$200,000, which is compounded continuously at an annual percentage rate of 2.5 %. Suppose that you are paying \$1500 a month. How long before you have paid off your loan?

Short Answer
Solution

(12) Suppose a ball weighing 0.2 kg is falling. Suppose that the drag constant for the ball is $k = 0.0004 \quad 1/m$. Suppose that the ball is dropped, so that the initial velocity is 0.

(a) What is the terminal velocity v_∞ of the ball?

(b) Suppose the ball hits the ground after 8 seconds. What fraction of the terminal velocity is reached when the ball hits the ground?

(c) How far did the ball fall?

Suppose for the remaining parts that there is a parachute on the ball from the previous problem that opens after 2 seconds. The parachute increases the drag constant to 0.04 $1/m$.

(d) What is the speed of the ball the instant the chute is opened?

(e) What is the terminal velocity of the ball with the chute opened?

(f) Does the ball accelerate or decelerate when the chute is opened (i.e is the acceleration positive, negative or zero)?

Solution

(13) A 160 lb person has a mass of about 5 slugs. Suppose such a person jumps from a plane has a drag constant of 0.0004 1/ft. Suppose the individual jumps from a height of 10000 ft.

(a) Find the terminal velocity of the individual.

(b) How long before the individual has reached 90 % of their terminal velocity?

(c) If they do not have a parachute, how long before they hit the ground?

(d) Suppose that with a parachute the drag constant increases to 0.05 1/ft. Find the terminal velocity of the individual with a parachute.

(e) If the individual starts at 90 % of their terminal velocity from part (a) how long will the chute need to be open for the speed to be reduced to within 5 ft/sec of your answer in part (d)?

(f) If the individual needs to be within 5 ft/sec of your answer in (d) to walk away uninjured, then how long should the skydiver wait to open their chute to have the most time free falling. (For simplicity assume it takes 4 seconds for the chute to open completely and that the change in velocity is negligible during that time)?

Solution

(14) Suppose that in 2011 approximately 140 million people are born a year while only 57 million die a year. The population in 2011 reached 7 billion people.

(a) Suppose that over time the percentage of people who are born and die a year is consistent. Find a function that expresses an estimate of the world's population p, t years after 2011.

(b) What is the doubling time for the world's population using our model from part (a)?

(c) Suppose the world's resources limit this growth. If the carrying capacity of the world is P, give the logistic growth model for the world's population p, t years after 2011 (assume that the growth rate far away from $p(t) = P$ is approximately the same as the growth rate in 2011).

Solution

(15) Each year fruit flies come and devastate a gardener's crop of blackberries. Suppose that during the Summer months the flies population can double every week.

(a) If the initial population is 200 flies at the beginning of Summer, and nothing is done, find the population of the flies at the end of Summer (13 weeks later).

(b) Suppose that 2 weeks into Summer the gardener notices the problem, and puts out a trap that kills 250 flies a week. Find the population of flies 11 weeks later.

(c) If the gardener puts out the trap one week earlier, how much would the final population of flies be reduced?

Solution

(16) In this problem we work through the details of our model for a falling object. In this model we assume that the force due to drag is $F_{drag} = -mkv^2$ where m is the mass of the object and k is the drag constant.

(a) Newton's second law says that $ma = mg - F_{drag}$ so we have $v' = g - kv^2$. Find the stationary points of this system.

(b) We call the positive stationary point the terminal velocity v_∞. Briefly justify that we can rewrite the equation as $v' = k(v_\infty^2 - v^2)$.

(c) Use partial fractions rederive the solution $v(t) = v_\infty \frac{e^{2v_\infty kt}-1}{e^{2v_\infty kt}+1}$ when we impose the initial condition $v(0) = 0$.

(d) Justify why the distance travelled when falling for t^* seconds is given by $\int_0^{t^*} v(t)dt$.

Solution

(17) In this problem we discuss the derivations for the logistic growth model.

(a) Suppose that the enviroment for our population has a maximum capacity of K (this is sometimes called the carrying capacity). Also suppose that so long as the population is significantly less than K that the rate of growth is r. Justify why the model $p' = rp(1 - p/k)$ satisfies these assumptions.

(b) Let p_I be the initial population for a population satisfying the model described in part (a). Rederive the solution

$$p(t) = \frac{Kp_I e^{rt}}{K + p_I(e^{rt} - 1)}.$$

(c) Suppose that the carrying capacity for a population is K, but that the population needs to be at least J to have growth In this situation $0 < J < K$. Give an autonomous model for the population which has these properties. (HINT: You will want $0, J,$ and K to be stationary points for our equation). You do not need to solve the equation.

Solution

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