I. First-Order Ordinary Differential Equations
9. Exact Differential Forms and Integrating Factors

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August 2, 2022

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9. Exact Differential Forms and Integrating Factors

9.1. Implicit General Solutions. Consider a first-order equation in the form

$$\frac{\mathrm{d}y}{\mathrm{d}x} = f(x, y)\,, \tag{9.1}$$

where $f(x, y)$ is continuously differentiable over a region R of the xy-plane. Our basic existence and uniqueness theorem then insures that for every point (x_I, y_I) within the interior of R there exists a unique solution $y = Y(x)$ of the differential equation (9.1) that satisfies the initial condition $Y(x_I) = y_I$ for so long as $(x, Y(x))$ remains within the interior of R.

Let us ask the following question. When are the solutions of the differential equation (9.1) determined by an equation of the form

$$H(x, y) = c \qquad \text{where } c \text{ is some constant?} \tag{9.2}$$

This means that we seek a function $H(x, y)$ defined over the interior of R such that the unique solution $y = Y(x)$ of the differential equation (9.1) that satisfies the initial condition $Y(x_I) = y_I$ is also the unique solution $y = Y(x)$ that satisfies

$$H(x, y) = H(x_I, y_I)\,, \qquad Y(x_I) = y_I\,. \tag{9.3}$$

Such an $H(x, y)$ is called an *integral* of (9.1). Because every solution of (9.1) that lies within the interior of R can be obtained in this way, we call relation (9.2) an *implicit general solution* of (9.1) over R.

The question can now be recast as "When does (9.1) have an integral?" This question is easily answered if we assume that all functions involved are as differentiable as we need. Suppose that an integral $H(x, y)$ exists, and that $y = Y(x)$ is a solution of differential equation (9.1). Then

$$H(x, Y(x)) = H(x_I, Y(x_I))\,,$$

where x_I is any point in the interval of definition of Y. By differentiating this equation with respect to x we find that

$$\partial_x H(x, Y(x)) + Y'(x)\,\partial_y H(x, Y(x)) = 0\,.$$

Therefore, wherever $\partial_y H(x, Y(x)) \neq 0$ we see that

$$Y'(x) = -\frac{\partial_x H(x, Y(x))}{\partial_y H(x, Y(x))}\,.$$

For this to hold for every solution of (9.1), we must have

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{\partial_x H(x, y)}{\partial_y H(x, y)}\,,$$

or equivalently

$$f(x, y) = -\frac{\partial_x H(x, y)}{\partial_y H(x, y)}\,, \tag{9.4}$$

wherever $\partial_y H(x,y) \neq 0$. The question then arises as to whether we can find an $H(x,y)$ such that (9.4) holds for any given $f(x,y)$? It turns out that this cannot always be done. In this chapter we explore how to seek such an $H(x,y)$.

9.2. **Exact Differential Forms.** When seeking an integral $H(x,y)$ for equation (9.1) our starting point will be to recast the equation in a so-called *differential form*

$$M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0\,, \tag{9.5}$$

where $M(x,y)$ and $N(x,y)$ are continuously differentiable over a region R in the xy-plane and

$$f(x,y) = -\frac{M(x,y)}{N(x,y)}\,.$$

There are many ways to do this. Just pick one that looks natural. If we are lucky then there will exist a function $H(x,y)$ such that

$$\partial_x H(x,y) = M(x,y)\,, \qquad \partial_y H(x,y) = N(x,y)\,. \tag{9.6}$$

When this is the case the differential form (9.5) is said to be *exact* over the region R and the function $H(x,y)$ is an integral for the differential equation (9.1).

Leonhard Euler showed that there is a simple test we can apply to find out if we are lucky. It derives from the fact, first proved by Euler, that "mixed partials commute" — namely, the fact that for any $H(x,y)$ that is twice continously differentiable over R we have

$$\partial_y\big(\partial_x H(x,y)\big) = \partial_x\big(\partial_y H(x,y)\big)\,.$$

This fact implies that if (9.6) holds for such an $H(x,y)$ then $M(x,y)$ and $N(x,y)$ satisfy

$$\partial_y M(x,y) = \partial_y\big(\partial_x H(x,y)\big) = \partial_x\big(\partial_y H(x,y)\big) = \partial_x N(x,y)\,.$$

In other words, if the differential form (9.5) is exact then $M(x,y)$ and $N(x,y)$ satisfy

$$\partial_y M(x,y) = \partial_x N(x,y)\,. \tag{9.7}$$

Euler showed that whenever R has no holes the converse holds too. Namely, if the differential form (9.5) satisfies (9.7) for every (x,y) in R then it is exact — i.e. there exists an $H(x,y)$ such that (9.6) holds. Moreover, the problem of finding $H(x,y)$ is reduced to finding two primitives. We illustrate this fact with examples.

Example. Solve the initial-value problem

$$\frac{\mathrm{d}y}{\mathrm{d}x} + \frac{e^x y + 2x}{2y + e^x} = 0\,, \quad y(0) = 0\,.$$

Solution. Express this equation in the differential form

$$(e^x y + 2x)\,\mathrm{d}x + (2y + e^x)\,\mathrm{d}y = 0\,.$$

Because

$$\partial_y(e^x y + 2x) = e^x \qquad = \qquad \partial_x(2y + e^x) = e^x\,,$$

this differential form satisfies (9.3) everywhere in the xy-plane and thereby is *exact*. Therefore we can find $H(x,y)$ such that

$$\partial_x H(x,y) = e^x y + 2x\,, \qquad \partial_y H(x,y) = 2y + e^x\,. \tag{9.8}$$

We can now integrate either equation, and plug the result into the other equation to obtain a second equation to integrate.

If we first integrate the first equation in (9.8) then we find that

$$H(x,y) = \int (e^x y + 2x)\,\mathrm{d}x = e^x y + x^2 + h(y)\,.$$

Here we are integrating with respect to x while treating y as a constant. The function $h(y)$ is the "constant of integration". The partial derivative of this expression with respect to y is

$$\partial_y H(x,y) = e^x + h'(y)\,,$$

which when plugged into the second equation in (9.8) gives

$$e^x + h'(y) = \partial_y H(x,y) = 2y + e^x\,.$$

This reduces to $h'(y) = 2y$. Notice that this equation for $h'(y)$ only depends on y. Take $h(y) = y^2$, so that $H(x,y) = e^x y + x^2 + y^2$ is an integral of the differential equation. Therefore an implicit general solution is

$$H(x,y) = e^x y + x^2 + y^2 = c\,.$$

The initial condition $y(0) = 0$ implies that

$$c = e^0 \cdot 0 + 0^2 + 0^2 = 0\,.$$

Therefore

$$y^2 + e^x y + x^2 = 0\,.$$

The quadratic formula then yields

$$y = \frac{-e^x + \sqrt{e^{2x} - 4x^2}}{2}\,,$$

where the positive square root is taken so that solution satisfies the initial condition. This is a solution wherever $e^{2x} > 4x^2$. □

Alternative Solution. If we first integrate the second equation in (9.8) then we find that

$$H(x,y) = \int (2y + e^x)\,\mathrm{d}y = y^2 + e^x y + h(x)\,.$$

Here we are integrating with respect to y while treating x as a constant. The function $h(x)$ is the "constant of integration". The partial derivative of this expression with respect to x is

$$\partial_x H(x,y) = e^x y + h'(x)\,,$$

which when plugged into the first equation in (9.8) gives

$$e^x y + h'(x) = \partial_x H(x,y) = e^x y + 2x\,.$$

This reduces to $h'(x) = 2x$. Notice that this equation for $h'(x)$ only depends on x. Taking $h(x) = x^2$, so $H(x,y) = e^x y + x^2 + y^2$, we see that a general solution satisfies

$$e^x y + x^2 + y^2 = c\,.$$

Because this is the same relation for a general solution that we had found previously, the evaluation of c is done as before. □

The points to be made here are the following:

- In principle we can integrate either equation in (9.8) first.
- If we integrate with respect to x first then the "constant of integration" $h(y)$ will depend on y and the equation for $h'(y)$ should only depend on y.
- If we integrate with respect to y first then the "constant of integration" $h(x)$ will depend on x and the equation for $h'(x)$ should only depend on x.
- In either case, if our equation for h' involves both x and y then we have made a mistake!

Sometimes the differential equation will be given to us already in differential form. In that case, use that form as the starting point.

Example. Find an implicit general solution to the differential equation

$$(xy^2 + y + e^x)\,\mathrm{d}x + (x^2y + x)\,\mathrm{d}y = 0\,.$$

Solution. Because

$$\partial_y(xy^2 + y + e^x) = 2xy + 1 \qquad = \qquad \partial_x(x^2y + x) = 2xy + 1\,.$$

this differential form satisfies (9.7) everywhere in the xy-plane and thereby is exact. Therefore we can find $H(x, y)$ such that

$$\partial_x H(x, y) = xy^2 + y + e^x\,, \qquad \partial_y H(x, y) = x^2y + x\,.$$

By integrating the second equation we obtain

$$H(x, y) = \int (x^2y + x)\,\mathrm{d}y = \tfrac{1}{2}x^2y^2 + xy + h(x)\,.$$

The partial derivative of this expression with respect to x is

$$\partial_x H(x, y) = xy^2 + y + h'(x)\,,$$

which when plugged into the first equation in (9.8) gives

$$xy^2 + y + h'(x) = \partial_x H(x, y) = xy^2 + y + e^x\,,$$

which yields $h'(x) = e^x$. (Notice that this only depends on x!) Take $h(x) = e^x$, so that $H(x, y) = \frac{1}{2}x^2y^2 + xye^x$. Therefore an implicit general solution is

$$\tfrac{1}{2}x^2y^2 + xy + e^x = c\,.$$

□

In the last example we could just as easily have integrated the equation for $\partial_x H(x, y)$ first and plugged the resulting expression into the equation for $\partial_y H(x, y)$. The next example shows that it can be helpful to first integrate whichever equation for $H(x, y)$ is easier to integrate.

Example. Find an implicit general solution to the differential equation

$$\big(x^2\cos(x)e^y + 2x\sin(x)e^y + e^x\cos(y)\big)\,\mathrm{d}x + \big(x^2\sin(x)e^y - e^x\sin(y)\big)\,\mathrm{d}y = 0\,.$$

Solution. Because

$$\begin{aligned}\partial_y\big(x^2\cos(x)e^y + 2x\sin(x)e^y + e^x\cos(y)\big) &= x^2\cos(x)e^y + 2x\sin(x)e^y - e^x\sin(y)\,,\\ \partial_x\big(x^2\sin(x)e^y - e^x\sin(y)\big) &= x^2\cos(x)e^y + 2x\sin(x)e^y - e^x\sin(y)\,,\end{aligned}$$

this differential form satisfies (9.3) and thereby is exact. Therefore we can find $H(x,y)$ such that

$$\begin{aligned}\partial_x H(x,y) &= x^2\cos(x)e^y + 2x\sin(x)e^y + e^x\cos(y)\,,\\ \partial_y H(x,y) &= x^2\sin(x)e^y - e^x\sin(y)\,.\end{aligned}$$

Now notice that it is more apparent how to integrate the bottom equation in y than how to integrate the top equation in x. (Recall that integrating terms like $x^2\cos(x)$ requires two integration-by-parts.) So integrating the bottom equation we obtain

$$H(x,y) = \int \big(x^2\sin(x)e^y - e^x\sin(y)\big)\,\mathrm{d}y = x^2\sin(x)e^y + e^x\cos(y) + h(x)\,.$$

The partial derivative of this expression with respect to x is

$$\partial_x H(x,y) = x^2\cos(x)e^y + 2x\sin(x)e^y + e^x\cos(y) + h'(x)\,,$$

which when plugged into the top equation gives

$$\begin{aligned}&x^2\cos(x)e^y + 2x\sin(x)e^y + e^x\cos(y) + h'(x)\\ &= \partial_x H(x,y) = x^2\cos(x)e^y + 2x\sin(x)e^y + e^x\cos(y)\,,\end{aligned}$$

which yields $h'(x) = 0$. Taking $h(x) = 0$, so that $H(x,y) = x^2\sin(x)e^y + e^x\cos(y)$, we see that a general solution is given by

$$x^2\sin(x)e^y + e^x\cos(y) = c\,.$$

□

Remark. Of course, had we seen that $x^2\cos(x)+2x\sin(x)$ is the derivative of $x^2\sin(x)$ then we could have just as easily started by integrating the $\partial_x H(x,y)$ equation with respect to x in the previous example. But such insights do not always arrive when we need them.

We will now derive formulas for $H(x,y)$ in terms of definite integrals that apply whenever R is a rectangle in the xy-plane and (x_I, y_I) is any point that lies within the interior of R. These formulas will encode the two steps given above. They thereby show that those steps can always be carried out in this setting. We consider a differential form

$$M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0\,, \tag{9.9}$$

where $M(x,y)$ and $N(x,y)$ are continuously differentiable over R and satisfy

$$\partial_y M(x,y) = \partial_x N(x,y)\,. \tag{9.10}$$

Now seek $H(x,y)$ such that

$$\partial_x H(x,y) = M(x,y)\,, \qquad \partial_y H(x,y) = N(x,y)\,. \tag{9.11}$$

By integrating the first equation with respect to x we obtain

$$H(x,y) = \int_{x_I}^{x} M(r,y)\,\mathrm{d}r + h(y)\,.$$

We now take the partial derivative of this expression with respect to y, use (9.10) to assert that $\partial_y M(r,y) = \partial_r N(r,y)$, and apply the First Fundamental Theorem of Calculus to obtain

$$\begin{aligned}\partial_y H(x,y) &= \int_{x_I}^{x} \partial_y M(r,y)\,\mathrm{d}r + h'(y) \\ &= \int_{x_I}^{x} \partial_r N(r,y)\,\mathrm{d}r + h'(y) = N(x,y) - N(x_I,y) + h'(y)\,.\end{aligned}$$

When this is plugged into the second equation we get

$$N(x,y) - N(x_I,y) + h'(y) = \partial_y H(x,y) = N(x,y)\,.$$

This reduces to $h'(y) = N(x_I,y)$, which only depends on y because x_I is a number. Let

$$h(y) = \int_{y_I}^{y} N(x_I,s)\,\mathrm{d}s\,.$$

An implicit general solution of (9.5) thereby is $H(x,y) = c$, where $H(x,y)$ is given by

$$H(x,y) = \int_{x_I}^{x} M(r,y)\,\mathrm{d}r + \int_{y_I}^{y} N(x_I,s)\,\mathrm{d}s\,. \tag{9.12a}$$

Notice that $H(x_I,y_I) = 0$.

If the second equation in (9.11) had been integrated first then we would have found that an implicit general solution of (9.9) is $H(x,y) = c$, where $H(x,y)$ is given by

$$H(x,y) = \int_{x_I}^{x} M(r,y_I)\,\mathrm{d}r + \int_{y_I}^{y} N(x,s)\,\mathrm{d}s\,. \tag{9.12b}$$

Formulas (9.12 give two expressions for the same function $H(x,y)$. Rather than memorize these formulas, it is better if we simply learn the steps underlying them.

Remark. In the two examples given previously the rectange R was the entire xy-plane. This will be the case whenever $M(x,y)$ and $N(x,y)$ appearing in the differential form (9.9) are continuously differentiable over the entire xy-plane and satisfy (9.10).

Remark. Our recipe for separable equations can be viewed as a special case of our recipe for exact differential forms. Consider the separable first-order ordinary differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = f(x)g(y)\,.$$

It can be put into the separated differential form

$$f(x)\,\mathrm{d}x - \frac{1}{g(y)}\,\mathrm{d}y = 0\,.$$

This differential form is exact because

$$\partial_y f(x) = 0 \qquad = \qquad \partial_x\left(\frac{1}{g(y)}\right) = 0\,.$$

Therefore we can find $H(x,y)$ such that

$$\partial_x H(x,y) = f(x)\,, \qquad \partial_y H(x,y) = \frac{1}{g(y)}\,.$$

Indeed, we find that

$$H(x,y) = F(x) - G(y)\,, \qquad \text{where} \quad F'(x) = f(x) \quad \text{and} \quad G'(y) = \frac{1}{g(y)}\,.$$

Therefore an implicit general solution is $F(x) - G(y) = c$. This is precisely the recipe for solving separable equations that we derived earlier.

9.3. **Integrating Factors.** Suppose we had considered the differential form

$$M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0\,, \tag{9.13}$$

and found that is not exact. Just because we were unlucky at first, we will not give up! Recall that this differential form has the same solutions as the differential form

$$M(x,y)\rho(x,y)\,\mathrm{d}x + N(x,y)\rho(x,y)\,\mathrm{d}y = 0\,, \tag{9.14}$$

where $\rho(x,y)$ any *nonzero* function. Indeed, both (9.13) and (9.14) are differential forms associated with the first-order differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = f(x,y)\,, \qquad \text{where} \quad f(x,y) = -\frac{M(x,y)}{N(x,y)} = -\frac{M(x,y)\rho(x,y)}{N(x,y)\rho(x,y)}\,.$$

Therefore we can seek a nonzero function $\rho(x,y)$ that makes the differential form (9.14) exact! This means that $\rho(x,y)$ must satisfy

$$\partial_y\big[M(x,y)\rho(x,y)\big] = \partial_x\big[N(x,y)\rho(x,y)\big]\,.$$

Expanding the above partial derivatives using the product rule, we see that ρ must satisfy

$$M(x,y)\partial_y\rho + [\partial_y M(x,y)]\rho = N(x,y)\partial_x\rho + [\partial_x N(x,y)]\rho\,. \tag{9.15}$$

This is a first-order linear partial differential equation for ρ. Finding its general solution is equivalent to finding the general solution of the original ordinary differential equation. Fortunately, we do not need this general solution. All we need is one nonzero solution. Such a ρ is called an *integrating factor* for the differential form (9.13).

Remark. The strategy of seeking an integrating factor is due to Leonhard Euler. With it, he was able to solve many equations that had been unsolved before.

A trick that sometimes yields a solution of (9.15) is to assume either that ρ is only a function of x, or that ρ is only a function of y. When ρ is only a function of x then $\partial_y\rho = 0$ and (9.15) reduces to the first-order linear ordinary differential equation

$$\frac{\mathrm{d}\rho}{\mathrm{d}x} = \frac{\partial_y M(x,y) - \partial_x N(x,y)}{N(x,y)}\,\rho\,.$$

This equation will be consistent with our assumption that ρ is only a function of x when the fraction on its right-hand side is independent of y. In that case we can integrate the equation to find the integrating factor

$$\rho(x) = e^{A(x)}\,, \qquad \text{where} \quad A'(x) = \frac{\partial_y M(x,y) - \partial_x N(x,y)}{N(x,y)}\,.$$

Similarly, when ρ is only a function of y then $\partial_x \rho = 0$ and (9.15) reduces to the first-order linear ordinary differential equation

$$\frac{\mathrm{d}\rho}{\mathrm{d}y} = \frac{\partial_x N(x,y) - \partial_y M(x,y)}{M(x,y)}\,\rho\,.$$

This equation will be consistent with our assumption that ρ is only a function of y when the fraction on its right-hand side is independent of x. In that case we can integrate the equation to find the integrating factor

$$\rho(y) = e^{B(y)}\,, \qquad \text{where} \quad B'(y) = \frac{\partial_x N(x,y) - \partial_y M(x,y)}{M(x,y)}\,.$$

This will be the only method for finding integrating factors that we will use in this course.

Remark. Rather than memorize the above formulas for $\rho(x)$ and $\rho(y)$ in terms of primitives, it is strongly recommend that you simply follow the steps by which they were derived. Namely, we seek an integrating factor ρ that satisfies

$$\partial_y[M(x,y)\,\rho] = \partial_x[N(x,y)\,\rho]\,.$$

We then expand the partial derivatives using the product rule as

$$M(x,y)\partial_y\rho + [\partial_y M(x,y)]\,\rho = N(x,y)\partial_x\rho + [\partial_x N(x,y)]\,\rho\,,$$

and combine the ρ terms. If the resulting equation reduces to an equation that only depends on x when we set $\partial_y \rho = 0$ then there is an integrating factor $\rho(x)$. On the other hand, if the equation reduces to an equation that only depends on y when we set $\partial_x \rho = 0$ then there is an integrating factor $\rho(y)$. We will illustrate this approach with the following examples.

Example. Find an implicit general solution to the differential equation

$$(2e^x + y^3)\,\mathrm{d}x + 3y^2\,\mathrm{d}y = 0\,.$$

Solution. This differential form is not exact because

$$\partial_y(2e^x + y^3) = 3y^2 \qquad \neq \qquad \partial_x(3y^2) = 0\,.$$

Therefore we seek an integrating factor ρ such that

$$\partial_y[(2e^x + y^3)\rho] = \partial_x[(3y^2)\rho]\,.$$

Expanding the partial derivatives using the product rule gives

$$(2e^x + y^3)\partial_y\rho + 3y^2\rho = 3y^2\partial_x\rho\,.$$

Notice that if $\partial_y\rho = 0$ then this equation reduces to the homogeneous, linear ordinary differential equation $\rho = \partial_x\rho$. By the methods of Chapter 2 we see that $\rho(x) = e^x$ is a nonzero solution. Therefore we can use $\rho(x) = e^x$ as an integrating factor. (See how easy that was!)

Because e^x is an integrating factor, we *know* that

$$(2e^x + y^3)e^x\,\mathrm{d}x + 3y^2 e^x\,\mathrm{d}y = 0 \quad \text{is exact}\,.$$

(Of course, we should check that this is exact. If it is not then we made a mistake in finding ρ!) Therefore we can find $H(x,y)$ such that

$$\partial_x H(x,y) = 2e^{2x} + y^3 e^x\,, \qquad \partial_y H(x,y) = 3y^2 e^x\,.$$

By integrating the second equation we see that $H(x,y) = y^3 e^x + h(x)$. When this expression for $H(x,y)$ is plugged into the first equation we obtain

$$y^3 e^x + h'(x) = \partial_x H(x,y) = 2e^{2x} + y^3 e^x\,,$$

which yields $h'(x) = 2e^{2x}$. Upon taking $h(x) = e^{2x}$, so that $H(x,y) = y^3 e^x + e^{2x}$, a general solution satisfies

$$y^3 e^x + e^{2x} = c\,.$$

In this case the general solution can be given explicitly as

$$y = (ce^{-x} - e^x)^{\frac{1}{3}}\,,$$

where c is an arbitrary constant. □

Example. Find an implicit general solution to the differential equation

$$2xy\,\mathrm{d}x + (2x^2 - e^y)\,\mathrm{d}y = 0\,.$$

Solution. This differential form is not exact because

$$\partial_y(2xy) = 2x \qquad \neq \qquad \partial_x(2x^2 - e^y) = 4x\,.$$

Therefore we seek an integrating factor ρ such that

$$\partial_y[(2xy)\rho] = \partial_x[(2x^2 - e^y)\rho]\,.$$

Expanding the partial derivatives using the product rule gives

$$2xy\partial_y\rho + 2x\rho = (2x^2 - e^y)\partial_x\rho + 4x\rho\,.$$

Combining the ρ terms then yields

$$2xy\partial_y\rho = (2x^2 - e^y)\partial_x\rho + 2x\rho\,.$$

Notice that if $\partial_x\rho = 0$ then this equation reduces to homogeneous, linear ordinary differential equation $y\partial_y\rho = \rho$. Its normal form is $\partial_y\rho = \rho/y$, which is undefined at $y = 0$. By the methods of Chapter 2 we see that $\rho(y) = y$ is a nonzero solution. Therefore we can use $\rho(y) = y$ as an integrating factor. (See how easy that was!)

Because y is an integrating factor, we *know* that

$$2xy^2\,\mathrm{d}x + (2x^2 - e^y)y\,\mathrm{d}y = 0 \quad \text{is exact}\,.$$

Therefore we can find $H(x,y)$ such that

$$\partial_x H(x,y) = 2xy^2\,, \qquad \partial_y H(x,y) = 2x^2 y - e^y y\,.$$

By integrating the first equation we see that $H(x,y) = x^2y^2 + h(y)$. When this expression for $H(x,y)$ is plugged into the second equation we obtain

$$2x^2 y + h'(y) = \partial_y H(x,y) = 2x^2 y - e^y y\,,$$

which yields $h'(y) = -e^y y$. Upon taking $h(y) = e^y(1-y)$, so that $H(x,y) = x^2y^2 + e^y(1-y)$, a general solution satisfies

$$x^2y^2 + e^y(1-y) = c\,.$$

In this case we cannot solve for y explicitly. □

Remark. Sometimes it might not be evident whether we should set $\partial_y\rho = 0$ or $\partial_x\rho = 0$ when searching for the integrating factor ρ. The next example illustrates such a case.

Example. Find an implicit general solution to the differential equation

$$(4xy + 3y^3)\,\mathrm{d}x + (x^2 + 3xy^2)\,\mathrm{d}y = 0\,.$$

Solution. This differential form is *not exact* because

$$\partial_y(4xy + 3y^3) = 4x + 9y^2 \qquad \neq \qquad \partial_x(x^2 + 3xy^2) = 2x + 3y^2\,.$$

Therefore we seek an *integrating factor* ρ such that

$$\partial_y[(4xy + 3y^3)\rho] = \partial_x[(x^2 + 3xy^2)\rho]\,.$$

Expanding the partial derivatives gives

$$(4xy + 3y^3)\partial_y\rho + (4x + 9y^2)\rho = (x^2 + 3xy^2)\partial_x\rho + (2x + 3y^2)\rho\,.$$

Combining the ρ terms yields

$$(4xy + 3y^3)\partial_y\rho + (2x + 6y^2)\rho = (x^2 + 3xy^2)\partial_x\rho\,.$$

At this point it might not be evident whether we should set $\partial_y\rho = 0$ or $\partial_x\rho = 0$. However, the picture becomes clear once we notice that $(x + 3y^2)$ is a common factor of the last two terms. Hence, if we set $\partial_y\rho = 0$ then this becomes

$$2(x + 3y^2)\rho = (x + 3y^2)x\partial_x\rho\,,$$

which reduces to $2\rho = x\partial_x\rho$. This yields the integrating factor $\rho = x^2$.

Because x^2 is an integrating factor, the differential form

$$(4xy + 3y^3)x^2\,\mathrm{d}x + (x^2 + 3xy^2)x^2\,\mathrm{d}y = 0 \quad \text{is exact}\,.$$

Therefore we can find $H(x,y)$ such that

$$\partial_x H(x,y) = 4x^3y + 3x^2y^3\,, \qquad \partial_y H(x,y) = x^4 + 3x^3y^2\,.$$

Integrating the first equation with respect to x yields

$$H(x,y) = x^4y + x^3y^3 + h(y)\,,$$

whereby

$$\partial_y H(x,y) = x^4 + 3x^3y^2 + h'(y)\,.$$

Plugging this expression for $\partial_y H(x,y)$ into the second equation gives

$$x^4 + 3x^3y^2 + h'(y) = x^4 + 3x^3y^2\,,$$

which yields $h'(y) = 0$. Taking $h(y) = 0$, an implicit general solution is therefore given by

$$x^4y + x^3y^3 = c\,.$$

Solving for y explicitly requires the cubic formula, which you are not expected to know. $\square$

Remark. Integrating factors for the linear equations can be viewed as a special case of the foregoing method. Consider the linear first-order ordinary differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = a(x)y + f(x)\,.$$

It can be put into the differential form

$$-\big(a(x)y + f(x)\big)\,\mathrm{d}x + \mathrm{d}y = 0\,.$$

This differential form is generally not exact because when $a(x) \neq 0$ we have

$$\partial_y\big(-a(x)y - f(x)\big) = -a(x) \qquad \neq \qquad \partial_x 1 = 0\,.$$

Therefore we seek an integrating factor ρ such that

$$\partial_y\big[\big(-a(x)y - f(x)\big)\,\rho\big] = \partial_x\rho\,.$$

Expanding the partial derivatives by the product rule gives

$$-\big(a(x)y + f(x)\big)\,\partial_y\rho - a(x)\rho = \partial_x\rho\,.$$

Notice that if $\partial_y\rho = 0$ then this equation reduces to $-a(x)\rho = \partial_x\rho$, whereby an integrating factor is $\rho(x) = e^{-A(x)}$ where $A'(x) = a(x)$.

Because $e^{-A(x)}$ is an integrating factor, we *know* that

$$e^{-A(x)}\big(-a(x)y - f(x)\big)\,\mathrm{d}x + e^{-A(x)}\,\mathrm{d}y = 0 \quad \text{is exact}\,.$$

Therefore we can find $H(x,y)$ such that

$$\partial_x H(x,y) = e^{-A(x)}\big(-a(x)y - f(x)\big)\,, \qquad \partial_y H(x,y) = e^{-A(x)}\,.$$

By integrating the second equation we see that $H(x,y) = e^{-A(x)}y + h(x)$. When this expression for $H(x,y)$ is plugged into the first equation we obtain

$$-e^{-A(x)}a(x)y + h'(x) = \partial_x H(x,y) = e^{A(x)}\big(-a(x)y - f(x)\big)\,,$$

which yields $h'(x) = -e^{-A(x)}f(x)$. Therefore a general solution is $H(x,y) = c$ with $H(x,y)$ given by

$$H(x,y) = e^{-A(x)}y - B(x)\,, \qquad \text{where} \quad A'(x) = a(x) \quad \text{and} \quad B'(x) = e^{-A(x)}f(x)\,.$$

This can be solved to obtain the explicit general solution

$$y = e^{A(x)}c + e^{A(x)}B(x)\,.$$

This is equivalent to the recipe for solving linear equations that we derived previously.

Exercises on Exact Differential Forms and Integrating Factors

Determine whether the following differential form is exact or not. If it is not exact then find an integrating factor ρ that transforms it into an exact differential form.

(1) $2xy\,\mathrm{d}x + (x^2 - e^y)\,\mathrm{d}y = 0$

Solution

(2) $(xy^2 + 3x^2y)\,\mathrm{d}x + (x+y)x^2\,\mathrm{d}y = 0$

Solution

(3) $(2xy + 2x^2y + y^2)\,\mathrm{d}x + (x^2 + y)\,\mathrm{d}y = 0$

Solution

(4) $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{-y}{(2x - ye^y)}$

Solution

(5) $(2x+3) + 2(y-2)\frac{\mathrm{d}y}{\mathrm{d}x} = 0$

Solution

(6) $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{-(2xy^2+2y)}{(2x^2y+2x)}$

Solution

Find an implicit general solution to the following differential equations.
(Make sure to check whether the differential form is exact or not, first!)

(7) $(3x^2 - 2xy + 2)\,\mathrm{d}x + (6y^2 - x^2 + 3)\,\mathrm{d}y = 0$

Short Answer
Solution

(8) $\left(\frac{y}{x} + 6x\right)\,\mathrm{d}x + (\log(x) - 2)\,\mathrm{d}y = 0\,, \qquad x > 0$

Short Answer
Solution

(9) $(x + 2y)\,\mathrm{d}x + (2x - y)\,\mathrm{d}y = 0$

Short Answer
Solution

(10) $(3xy - y^2)\,\mathrm{d}x + (x^2 - xy)\,\mathrm{d}y = 0$

Short Answer
Solution

(11) $(y^2 - xy)\,\mathrm{d}x + (3xy - x^2)\,\mathrm{d}y = 0$

Short Answer
Solution

(12) $-(2x + y)\,\mathrm{d}x = (x + 9y)\,\mathrm{d}y$

Short Answer
Solution

(13) $\left(\frac{x^3}{y^2} + \frac{2}{y}\right)\,\mathrm{d}x + \left(\frac{2x}{y^2} + y\right)\,\mathrm{d}y = 0$

Short Answer

Solution

(14) $(3x - 2y)\,\mathrm{d}x + (y - 2x)\,\mathrm{d}y = 0$

Short Answer
Solution

(15) $\dfrac{\mathrm{d}y}{\mathrm{d}x} = e^{2x} + y - 2$

Short Answer
Solution

(16) $(ye^{2xy})\,\mathrm{d}x + (xe^{2xy} + y)\,\mathrm{d}y = 0$

Short Answer
Solution

(17) $(e^y \cot(x) + 2x\csc(x))\,\mathrm{d}x + e^y\,\mathrm{d}y = 0$

Short Answer
Solution

For #18-20, solve the initial-value problems.
(You may leave your answers in implicit form)

(18) $(3x - 2y)\,\mathrm{d}x + (y - 2x)\,\mathrm{d}y = 0 \qquad y(1) = 3\,.$

Short Answer
Solution

(19) $(2xy - 9x^2)\,\mathrm{d}x + (2y + x^2 + 1)\,\mathrm{d}y = 0$

Short Answer
Solution

(20) $ye^{2xy}\,\mathrm{d}x + (xe^{2xy} + y)\,\mathrm{d}y = 0\,, \qquad y(1) = 0\,.$

Short Answer
Solution

(21) $(x + 2y)\,\mathrm{d}x + (2x - y)\,\mathrm{d}y = 0\,, \qquad y(2) = 2\,.$

Short Answer
Solution

(22) Show that a separated differential form (i.e. $\dfrac{1}{g(y)}\,\mathrm{d}y = f(x)\,\mathrm{d}x$) is exact. Assume $g(y) \neq 0$ for every y. Find an implicit general solution using $f(x)$ and $g(y)$.
(Note: Let $F(x)$ denote a primitive of $f(x)$ and $G(y)$ denote a primitive of $\dfrac{1}{g(y)}$.)

Solution

(23) Consider a differential form $M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0$ such that $\dfrac{\partial_y M(x,y) - \partial_x N(x,y)}{N(x,y)}$ is a function of only x. Show that there exists an integrating factor $\rho(x)$ that satisfies

$$\rho' = \frac{\partial_y M(x,y) - \partial_x N(x,y)}{N(x,y)}\,\rho\,.$$

Solution

(24) Consider a differential form $M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0$ such that $\dfrac{\partial_x N(x,y) - \partial_y M(x,y)}{M(x,y)}$ is a function of only y. Show that there exists an integrating factor $\rho(y)$ that satisfies

$$\rho' = \frac{\partial_x N(x,y) - \partial_y M(x,y)}{M(x,y)}\,\rho\,.$$

Solution

(25) What value of α makes this exact? $(\alpha\,x^3 + \alpha\,y^2 x)\,\mathrm{d}x + (4yx^2 + y^4)\,\mathrm{d}y = 0$

Short Answer

Solution

(26) Solve the previous problem using the value of α that makes the differential form exact.

Short Answer

Solution

(27) What value of α,β make this exact? $(\frac{\alpha\,y}{x} + \beta\,x)\,\mathrm{d}x + (log(\beta\,x) + 6)\,\mathrm{d}y = 0$

Short Answer

Solution

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