I. First-Order Ordinary Differential Equations
10. Special Equations and Substitution

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August 2, 2022

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10. Special Equations and Substitution

So far we have developed analytical methods for linear equations, separable equations, and equations that can be expressed as an exact differential form. There are other first-order equations that can be solved by analytical methods. Here we present a few of them. In each case a substitution will transform the problem into a form that we know how to solve.

10.1. Linear Argument Equations.

These equations can be put into the form

$$\frac{\mathrm{d}y}{\mathrm{d}x} = k(ax+by)\,, \tag{10.1}$$

where $k(z)$ is a differentiable function over an interval (z_L, z_R) while a and b are constants with $b \neq 0$. Upon setting $z = ax + by$ we see that

$$\frac{\mathrm{d}z}{\mathrm{d}x} = \frac{\mathrm{d}}{\mathrm{d}x}(ax+by) = a + b\,\frac{\mathrm{d}y}{\mathrm{d}x} = a + b\,k(ax+by) = a + b\,k(z)\,.$$

Therefore z satisfies the autonomous equation

$$\frac{\mathrm{d}z}{\mathrm{d}x} = a + b\,k(z)\,. \tag{10.2}$$

If we can solve this equation for z then the solution of (10.1) is obtained by setting

$$y = \frac{z - ax}{b}\,.$$

Solutions of (10.2) are given implicitly by

$$G(z) = x + c\,, \qquad \text{where} \quad G'(z) = \frac{1}{a + b\,k(z)}\,. \tag{10.3}$$

Of course, we will not be able to find an explicit primitive $G(z)$ for every $k(z)$. And when we can find $G(z)$, often we will not be able to solve (10.3) for z as an explicit function of x.

Example. Solve the equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = (x+y)^2\,.$$

Solution. This has the form (10.1) with $k(z) = z^2$ and $a = b = 1$. Rather than remember the form (10.2), it is easier to remember the substitution $z = x + y$ and rederive (10.2). Indeed, we see that

$$\frac{\mathrm{d}z}{\mathrm{d}x} = \frac{\mathrm{d}}{\mathrm{d}x}(x+y) = 1 + \frac{\mathrm{d}y}{\mathrm{d}x} = 1 + (x+y)^2 = 1 + z^2\,.$$

Solutions of this autonomous equation satisfy

$$x = \int \frac{1}{1+z^2}\,\mathrm{d}z = \tan^{-1}(z) + c\,.$$

This equation can be solved explicitly to obtain $z = \tan(x - c)$. Because $y = z - x$, a family of solutions to the original equation is

$$y = \tan(x - c) - x\,.$$

□

10.2. **Dilation Invariant Equations.** These equations can be put into the form

$$\frac{\mathrm{d}y}{\mathrm{d}x} = x^{p-1} k\Big(\frac{y}{x^p}\Big) \,, \qquad \text{for some } p \neq 0 \,, \tag{10.4}$$

where $k(z)$ is a differentiable function over $(-\infty, \infty)$. A first-order equation in the form

$$\frac{\mathrm{d}y}{\mathrm{d}x} = f(x, y) \,,$$

can be put into the form (10.4) if and only if $f(x, y)$ satisfies the dilation symmetry

$$f(\lambda x, \lambda^p y) = \lambda^{p-1} f(x, y) \qquad \text{for every } \lambda \neq 0 \,. \tag{10.5}$$

Indeed, if $f(x, y)$ satisfies has this symmetry then by choosing $\lambda = 1/x$ we see that

$$f(x, y) = \lambda^{1-p} f(\lambda x, \lambda^p y) = x^{p-1} f\bigg(\frac{1}{x}\, x \,,\, \frac{1}{x^p}\, y\bigg) = x^{p-1} f\Big(1, \frac{y}{x^p}\Big) \,,$$

whereby $k(z) = f(1, z)$. When $f(x, y)$ satisfies (10.5) with $p = 1$ then equation (10.4) is said to be *homogeneous*. This notion of homogeneous should not be confused with the notion of homogeneous that arises in the context of linear equations.

We can transform (10.4) into a separable equation by setting $z = y/x^p$. By using (10.4) we see that

$$\begin{aligned} \frac{\mathrm{d}z}{\mathrm{d}x} = \frac{1}{x^p} \frac{\mathrm{d}y}{\mathrm{d}x} - p\, \frac{y}{x^{p+1}} &= \frac{1}{x^p}\, x^{p-1} k\Big(\frac{y}{x^p}\Big) - p\, \frac{y}{x^{p+1}} \\ &= \frac{1}{x}\bigg(k\Big(\frac{y}{x^p}\Big) - \frac{y}{x^p}\bigg) = \frac{1}{x}\big(k(z) - pz\big) \,. \end{aligned}$$

Therefore z satisfies the separable equation

$$\frac{\mathrm{d}z}{\mathrm{d}x} = \frac{k(z) - pz}{x} \,. \tag{10.6}$$

If we can solve this equation for z then the solution of (10.4) is obtained by setting $y = zx^p$.

Solutions of (10.6) are given implicitly by

$$\log(|x|) = G(z) + c \,, \qquad \text{where} \quad G'(z) = \frac{1}{k(z) - pz} \,. \tag{10.7}$$

Of course, we will not be able to find an explicit primitive $G(z)$ for every $k(z)$. And when we can find $G(z)$, often we will not be able to solve (10.7) for z as an explicit function of x.

Example. Solve the equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{y + \sqrt{x^2 + y^2}}{x} \qquad \text{for } x > 0 \,.$$

Solution. This equation can be expressed as

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{y}{x} + \sqrt{1 + \frac{y^2}{x^2}} \,.$$

It thereby has the dilation invariant form (10.4) with $p = 1$ and $k(z) = z + \sqrt{1+z^2}$. Rather than remember the form (10.6), it is easier to remember the substitution $z = y/x$ and rederive (10.6). Indeed, we see that

$$\frac{\mathrm{d}z}{\mathrm{d}x} = \frac{\mathrm{d}}{\mathrm{d}x}\frac{y}{x} = \frac{1}{x}\frac{\mathrm{d}y}{\mathrm{d}x} - \frac{y}{x^2} = \frac{1}{x}\left(\frac{y}{x} + \sqrt{1+\frac{y^2}{x^2}}\right) - \frac{y}{x^2} = \frac{1}{x}\sqrt{1+\frac{y^2}{x^2}} = \frac{\sqrt{1+z^2}}{x}\,.$$

Solutions of this separable equation satisfy

$$\log(x) = \int \frac{1}{\sqrt{1+z^2}}\,\mathrm{d}z = \sinh^{-1}(z) + c\,.$$

This equation can be solved explicitly to obtain

$$z = \sinh\big(\log(x) - c\big) = \frac{e^{\log(x)-c} - e^{-\log(x)+c}}{2} = \frac{1}{2}\left(\frac{x}{e^c} - \frac{e^c}{x}\right)\,.$$

Because $y = xz$, a family of solutions to the original equation is

$$y = \frac{x^2 - e^{2c}}{2e^c}\,.$$

□

10.3. **Bernoulli Equations.** These equations can be put into the form

$$\frac{\mathrm{d}y}{\mathrm{d}t} = a(t)y - b(t)y^{1+m}\,, \tag{10.8}$$

where $a(t)$ and $b(t)$ are continous over an interval (t_L, t_R) while m is a constant. If $m = 0$ this reduces to a homogeneous linear equation, which can be solved the method of Section 2.2. So here we will treat only the case $m \neq 0$. If $m = -1$ then equation (10.8) is a nonhomogeneous linear equation, which can be solved the method of Section 2.3.

Remark. Jacob Bernoulli wrote down such equations in a 1695 letter to Gottfried Leibniz. The next year Leibniz showed that they can be transformed into a nonhomogeneous linear equation for every $m \neq 0$ by a simple substitution. Bernoulli then showed that they can be transformed into a separable equation by another simple substitution.

Leibniz transformed (10.8) into a linear equation by setting $z = y^{-m}$. We see that

$$\begin{aligned}\frac{\mathrm{d}z}{\mathrm{d}t} = -my^{-m-1}\frac{\mathrm{d}y}{\mathrm{d}t} &= -my^{-m-1}\big(a(t)y - b(t)y^{1+m}\big)\\ &= -m\,a(t)y^{-m} + m\,b(t)\\ &= -m\,a(t)z + m\,b(t)\,.\end{aligned}$$

Therefore z satisfies the nonhomogeneous linear equation

$$\frac{\mathrm{d}z}{\mathrm{d}t} + m\,a(t)z = m\,b(t)\,. \tag{10.9}$$

If we can solve this equation for z then a solution of (10.8) is obtained by setting $y = z^{-\frac{1}{m}}$.

Equation (10.9) can be solved by the recipe of Section 2.3. Let $A(t)$ and $B(t)$ satisfy

$$A'(t) = m\,a(t)\,, \qquad\qquad B'(t) = m\,e^{A(t)}b(t)\,.$$

Then a general solution of (10.9) is given by

$$z = e^{-A(t)}B(t) + e^{-A(t)}c\,, \qquad \text{where } c \text{ is an arbitrary constant}\,. \tag{10.10}$$

Therefore a solution of (10.8) is given by

$$y = \left(e^{-A(t)}B(t) + e^{-A(t)}c\right)^{-\frac{1}{m}}\,, \qquad \text{where } c \text{ is an arbitrary constant}\,. \tag{10.11}$$

Remark. Because equation (10.9) is linear, if $a(t)$ and $b(t)$ are continuous over a time interval (t_L, t_R) then its solution z will exist over (t_L, t_R) and be given by (10.10). However, formula (10.11) for the solution y of (10.8) can break down for several reasons.

Example. Solve the logistic model for populations

$$\frac{\mathrm{d}p}{\mathrm{d}t} = (r - ap)p\,.$$

Remark. Earlier we solved this using our autonomous equation recipe. Here we treat it as a Bernoulli equation.

Solution. The equation has the form

$$\frac{\mathrm{d}p}{\mathrm{d}t} = rp - ap^2\,,$$

which is the Bernoulli form (10.8) with $a(t) = r$, $b(t) = a$, and $m = 1$. If we apply formula (10.10) with $A(t) = rt$ and

$$B(t) = \int e^{rt}a\,\mathrm{d}t = \frac{a}{r}e^{rt} + c\,,$$

then we obtain

$$p = \frac{1}{\dfrac{a}{r} + e^{-rt}c}\,, \qquad \text{where } c \text{ is an arbitrary constant}\,.$$

This solution breaks down where the denominator vanishes. □

Remark. Bernoulli transformed (10.8) into a separable equation by setting

$$z = e^{-A(t)}y\,, \qquad \text{where} \quad A'(t) = a(t)\,.$$

We see that

$$\begin{aligned}
\frac{\mathrm{d}z}{\mathrm{d}t} &= \frac{\mathrm{d}}{\mathrm{d}t}\left(e^{-A(t)}y\right) = e^{-A(t)}\frac{\mathrm{d}y}{\mathrm{d}t} - e^{-A(t)}a(t)y \\
&= e^{-A(t)}\left(a(t)y - b(t)y^{1+m}\right) - e^{-A(t)}a(t)y = -e^{-A(t)}b(t)y^{1+m} \\
&= -b(t)e^{mA(t)}\left(e^{-A(t)}y\right)^{1+m} = -b(t)e^{mA(t)}z^{1+m}\,.
\end{aligned}$$

Therefore z satisfies the separable equation

$$\frac{\mathrm{d}z}{\mathrm{d}t} = -b(t)e^{mA(t)}z^{1+m}\,. \tag{10.12}$$

If we can solve this equation for z then a solution of (10.8) is obtained by setting $y = e^{A(t)}z$.

Equation (10.12) is separable and can be solved by the recipe of Section 3.2. Because $m \neq 0$, it has the separated form

$$-\frac{m}{z^{1+m}}\,\mathrm{d}z = mb(t)e^{mA(t)}\,\mathrm{d}t\,.$$

Therefore an implicit general solution is

$$\frac{1}{z^m} = F(t) + c\,, \qquad \text{where} \quad F'(t) = mb(t)e^{mA(t)}\,.$$

An explicit general solution is $z = (F(t)+c)^{-\frac{1}{m}}$ whenever this expression makes sense.

10.4. **Substitution.** The idea behind each of the foregoing examples is to transform the original differential equation into an equation with a form that we know how to solve. In general, let the original equation be

$$\frac{\mathrm{d}y}{\mathrm{d}t} = f(t,y)\,. \tag{10.13}$$

Upon setting $z = Z(t,y)$ and using (10.13) we see that

$$\frac{\mathrm{d}z}{\mathrm{d}t} = \partial_t Z(t,y) + \partial_y Z(t,y)\,\frac{\mathrm{d}y}{\mathrm{d}t} = \partial_t Z(t,y) + \partial_y Z(t,y)\,f(t,y)\,.$$

We then assume the relation $z = Z(t,y)$ can be inverted to obtain $y = Y(t,z)$, and substitute this result into the above to find

$$\frac{\mathrm{d}z}{\mathrm{d}t} = \partial_t Z\big(t, Y(t,z)\big) + \partial_y Z\big(t, Y(t,z)\big)\,f\big(t, Y(t,z)\big)\,.$$

We thereby obtain the transformed equation

$$\frac{\mathrm{d}z}{\mathrm{d}t} = g(t,z)\,, \tag{10.14}$$

where $g(t,z)$ is given in terms of $f(t,y)$, $Z(t,y)$, and $Y(t,z)$ by

$$g(t,z) = \partial_t Z\big(t, Y(t,z)\big) + \partial_y Z\big(t, Y(t,z)\big)\,f\big(t, Y(t,z)\big)\,.$$

This relation can be inverted to give $f(t,y)$ in terms of $g(t,z)$, $Y(t,z)$, and $Z(t,y)$ as

$$f(t,y) = \partial_t Y\big(t, Z(t,y)\big) + \partial_z Y\big(t, Z(t,y)\big)\,g\big(t, Z(t,y)\big)\,.$$

Therefore if we can solve equation (10.14) then we can solve equation (10.13), and vice versa.

Sections 10.1 through 10.3 each provide examples of this general approach.

Example. In Section 10.1 the linear argument equation (10.1) was transformed into the autonomous equation (10.2) with the substitution $z = Z(x,y)$ and its inverse $y = Y(x,z)$ given by

$$Z(x,y) = ax + by\,, \qquad Y(x,z) = \frac{z - ax}{b}\,.$$

Example. In Section 10.2 the dilation invariant equation (10.4) was transformed into the separable equation (10.6) with the substitution $z = Z(x, y)$ and its inverse $y = Y(x, z)$ given by

$$Z(x, y) = \frac{y}{x^p}, \qquad Y(x, z) = x^p z\,.$$

Examples. In Section 10.3 the Bernoulli equation (10.8) was treated two ways. It was transformed into the nonhomogeneous linear equation (10.9) with the substitution $z = Z(t, y)$ and its inverse $y = Y(t, z)$ given by

$$Z(t, y) = y^{-m}, \qquad Y(t, z) = z^{-\frac{1}{m}}\,.$$

It was also transformed into the separable equation (10.12) with the substitution $z = Z(t, y)$ and its inverse $y = Y(t, z)$ given by

$$Z(t, y) = e^{-A(t)} y\,, \qquad Y(t, z) = e^{A(t)} z\,, \qquad \text{where } A'(t) = a(t)\,.$$

EXERCISES ON SPECIAL EQUATIONS AND SUBSTITUTION

In each of the following problems use one of the substitutions discussed in this chapter to find a general solution to the differential equation.

(1)

$$\frac{dy}{dt} = \frac{1}{y+t}$$

Short Answer
Solution

(2)

$$\frac{dy}{dt} = (4t+9y)^2$$

Short Answer
Solution

(3)

$$y' = \frac{1}{\sqrt{y+t}}$$

Short Answer
Solution

(4)

$$\dot{y} = e^{2t+y}$$

Short Answer
Solution

(5)

$$\frac{dy}{dt} = \tan^2(y+t)$$

Short Answer
Solution

(6)

$$y' = \frac{y\ln(y/x^2)}{x}$$

Short Answer
Solution

(7)

$$\frac{dy}{dx} = \frac{e^{xy}}{x^2} - \frac{y}{x}$$

(HINT: Let $z = xy$)

Short Answer
Solution

(8)

$$y' = t^2 \cos(y/t^3) + 3y/t$$

Short Answer
Solution

(9)

$$\frac{dy}{dx} = \frac{\sqrt{x^2 + y^2}}{x}$$

Short Answer
Solution

Solve each of the following Bernoulli equations using (a) Leibniz's substitution and (b) Bernoulli's substitution.

(10)

$$y' = 2y + ty^2$$

Short Answer
Solution

(11)

$$y' = y + t^2/y$$

Short Answer
Solution

(12)

$$y' = \frac{t - y^3}{ty^2}$$

Short Answer
Solution

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