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Second Order Linear Equations and the Airy Functions:

```
Why Special Functions are Really No More Complicated
than Most Elementary Functions

% We shall consider here the most important second order
% ordinary differential equations, namely linear equations.
% The standard format for such an equation is

%            y''(t) + p(t) y'(t) + q(t) y(t) = g(t),

% where y(t) is the unknown function satisfying the
% equation and p, q and g are given functions, all
% continuous on some specified interval. We say that the
% equation is homogeneous if g = 0. Thus:

%            y''(t) + p(t) y'(t) + q(t) y(t) = 0.

% We have studied methods for solving an inhomogeneous
% equation for which you have already solved the
% corresponding homogeneous equation. So we shall
% concentrate on homogeneous equations here. The equation
% is said to be a constant coefficient equation if the
% functions p and q are constant. Again we have studied
% methods for dealing with those. Non-constant coefficient
% equations are more problematic, but alas, they arise
% frequently in nature. In this lesson we shall study
% closely one of the best known examples -- namely,
% Airy's Equation. For the record, the solutions to that
% equation, i.e., the Airy functions, arise in
% diffraction problems in the study of optics,
% and also in relation to the famous Schroedinger equation
% in quantium mechanics.

% Before proceeding, let's recall some basic facts about
% the set of solutions to a linear, homogeneous second
% order differential equation. The most basic fact is that
% the set of solutions forms a two-dimensional vector
```

```
% space. This means that you can find two solutions,
% y_1 and y_2, neither of which is a multiple of the
% other, so that all solutions are given by linear
% combinations of these two:

%      ay_1 + by_2, where a and b are arbitrary constants.

% Just to help you get your bearings, let's mention two
% other two-dimensional vector spaces that you know very
% well:

% The Euclidean plane, where every vector can be
% expressed as
%       a i-> + b j->
% where i-> = (1,0) and j-> = (0,1);

% or all polynomials of degree at most 1
%       a + b x.

% Now returning to second order linear homogenous
% differential equations with constant coefficients, we
% note, by way of examples, that all solutions of

%       y'' + y = 0

% are given by

%       a cos(t) + b sin(t);

% and all solutions of

%       y'' +(1/t)y' + (1/t^2)y = 0

% are given by

%       a cos(ln(t)) + b sin(ln(t)).

% (The latter is an Euler equation.)

% In the first example, the interval of
% definition is the whole real line, whereas in the second,
% we must restrict to t>0. In both instances,
% a and b are arbitrary real numbers.
```

Airy's Equation

This is the equation:

```
%       y'' - ty = 0.

% In this presentation we shall solve it symbolically and
% numerically. We shall also address it graphically.
% Furthermore, if you look at DEwM, Problem 1, p. 157, you
```

```
% will see a qualitative method of dealing with the
% equation, which I shall briefly recall below. Finally,
% although it is not in the course syllabus, one can also
% use the method of series solution to solve Airy's
% Equation. So there are lots of ways to skin this cat.

% Well, let's try dsolve first and see what happens.
clear all
close all

dsolve('D2y - t*y =0', 't')

% How about that, Matlab solves it. But what are those
% functions it reports as the answers? In fact, they are
% special functions, and if I may quote from DEwM, p. 52:

% "By special functions we mean various non-elementary
% functions that mathematicians give names to, often
% because they arise as solutions of particularly important
% differential equations." Recall also that elementary
% functions are "the standard functions of calculus:
% polynomials, exponentials and logarithms,
% trigonometric functions and their inverses, and all
% combinations of these functions through algebraic
% operations and compositions." The simplest
% special function is the error function

%   erf(t) = (2/sqrt(pi))int_0^t exp(-s^2) ds.

% Many special functions have integral formulas like the
% above and/or are specified by a differential equation
% and/or are given by a power series.
```

```
ans =

C2*airy(0, t) + C3*airy(2, t)
```

Graphing Airy functions

In order to do so, we must recognize that there is still some issue in the reporting of symbolic information in Matlab as a consequence of Matlab having "bought" its symbolic solver, i.e., MuPAD. We can see this by typing

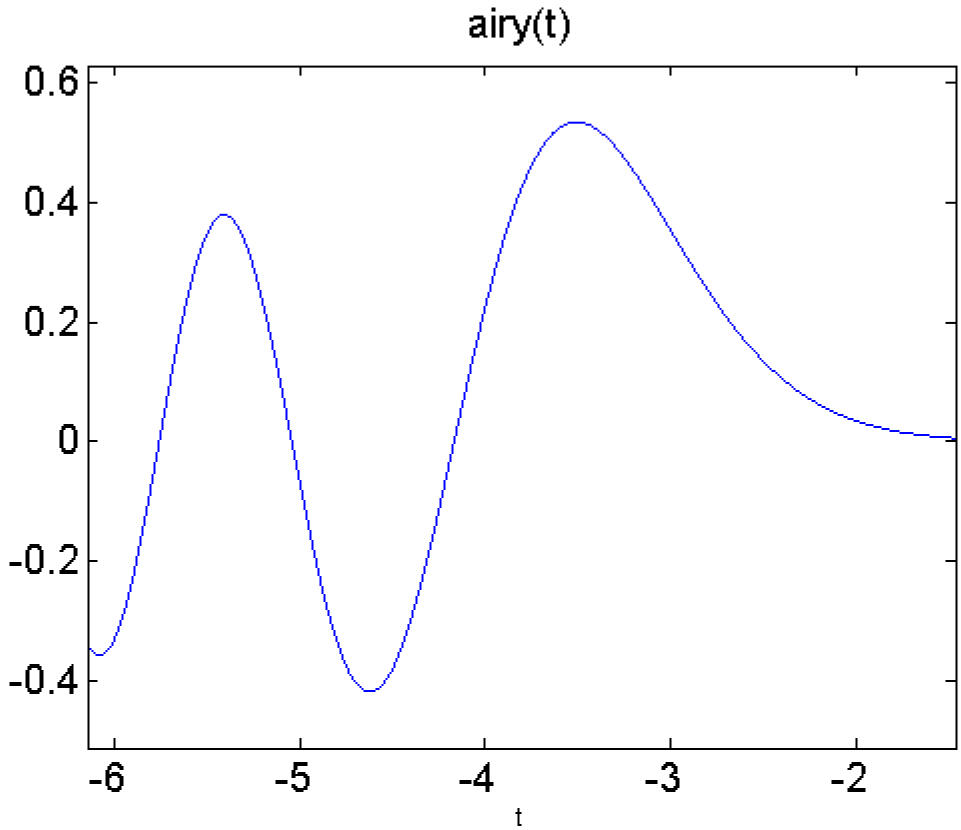
```
help airy
```

```
 AIRY   Airy functions.
    W = AIRY(Z) is the Airy function, Ai(Z), of the elements of Z.

    W = AIRY(K,Z) returns various Airy functions specified by K:
        0 - (default) is the same as AIRY(Z)
        1 - returns the derivative, Ai'(Z)
```

```
    2 - returns the Airy function of the second kind, Bi(Z)
    3 - returns the derivative, Bi'(Z)

W = AIRY(K,Z,SCALE) returns a scaled AIRY(K,Z) specified by SCALE:
    0 - (default) is that same as AIRY(K,Z)
    1 - returns AIRY(K,Z) scaled by EXP(2/3.*Z.^(3/2)) for K = 0,1,
        and scaled by EXP(-ABS(2/3.*REAL(Z.^(3/2)))) for K = 2,3.

AIRY no longer returns a second output. Use W = AIRY(K,Z) instead.

[W,IERR] = AIRY(K,Z) also returns an array of error flags.
    ierr = 1   Illegal arguments.
    ierr = 2   Overflow.  Return Inf.
    ierr = 3   Some loss of accuracy in argument reduction.
    ierr = 4   Complete loss of accuracy, z too large.
    ierr = 5   No convergence.  Return NaN.

The relationship between the Airy and modified Bessel functions is:

    Ai(z) = 1/pi*sqrt(z/3)*K_1/3(zeta)
    Bi(z) = sqrt(z/3)*(I_-1/3(zeta)+I_1/3(zeta))
    where zeta = 2/3*z^(3/2)

See also BESSELH, BESSELI, BESSELJ, BESSELK, BESSELY.

Overloaded methods:
   sym/airy

Reference page in Help browser
   doc airy
```

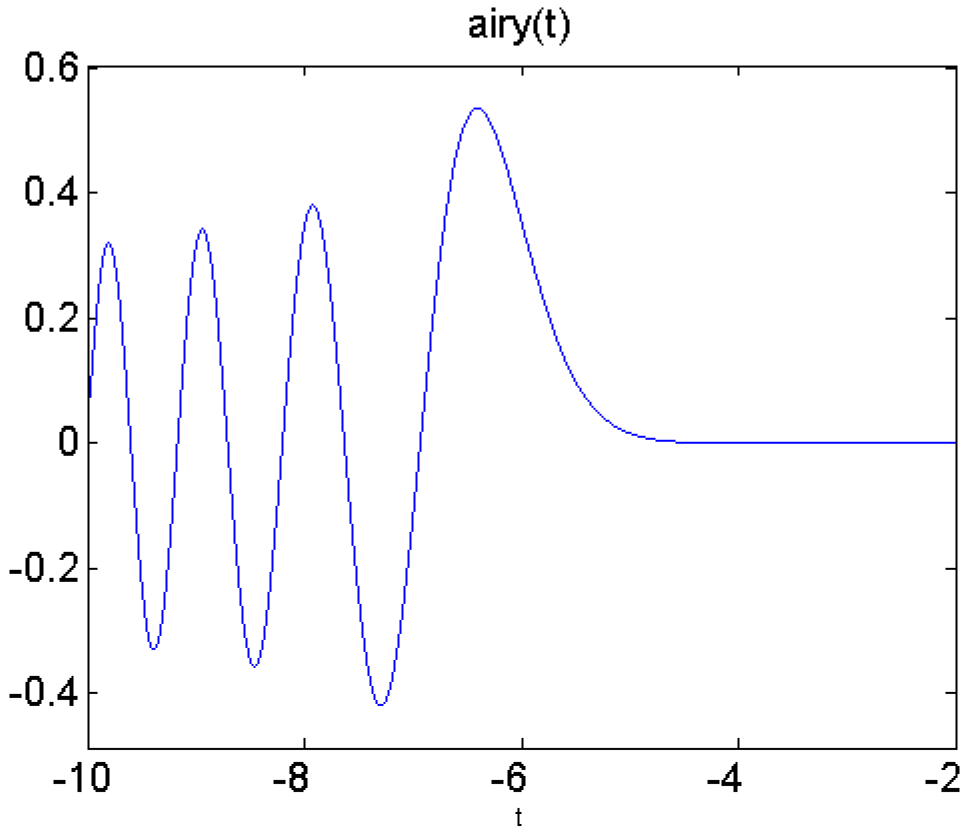
Note that this is one of those cases wherein there is some translation issue. In any event, the function that is reported as airy(0,t) or more simply airy(t), is just the normal Airy function Ai(t) and the other Airy function reported as airy(2,t) is the Airy function Bi(t). More on these below. But first let's graph these functions.

```
figure;
ezplot('airy(t)')
set(get(gca, 'Title'),'Fontsize', 15)
set(gca,'XTickLabel',get(gca, 'XTickLabel'),'FontSize',15)
% set(gcf, 'Position', [1 1 1320 1020])
```



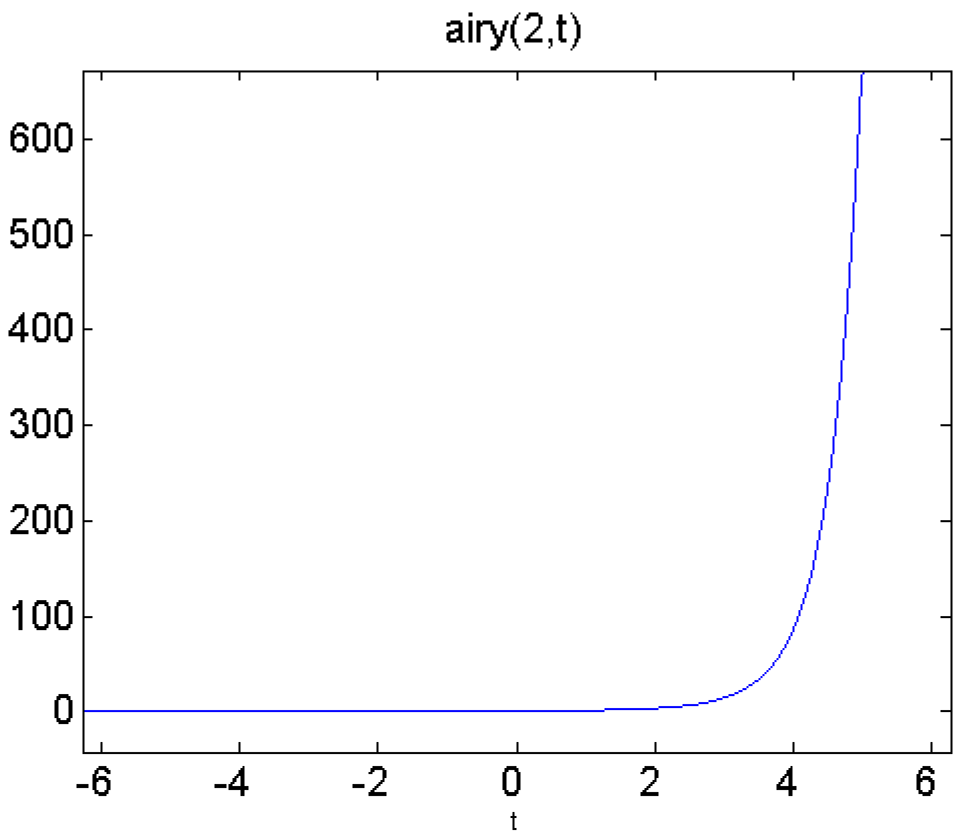
and on a bigger interval

```
figure; % set(gcf, 'Position', [1 1 1920 1420])
ezplot('airy(t)', [-10 10])
set(get(gca, 'Title'),'Fontsize', 15)
set(gca,'XTickLabel',get(gca, 'XTickLabel'),'FontSize',15)

% Looks like a mildly damped oscillation to the left, but
% with increasing frequency; and
% a rapid decay to zero to the right.
```



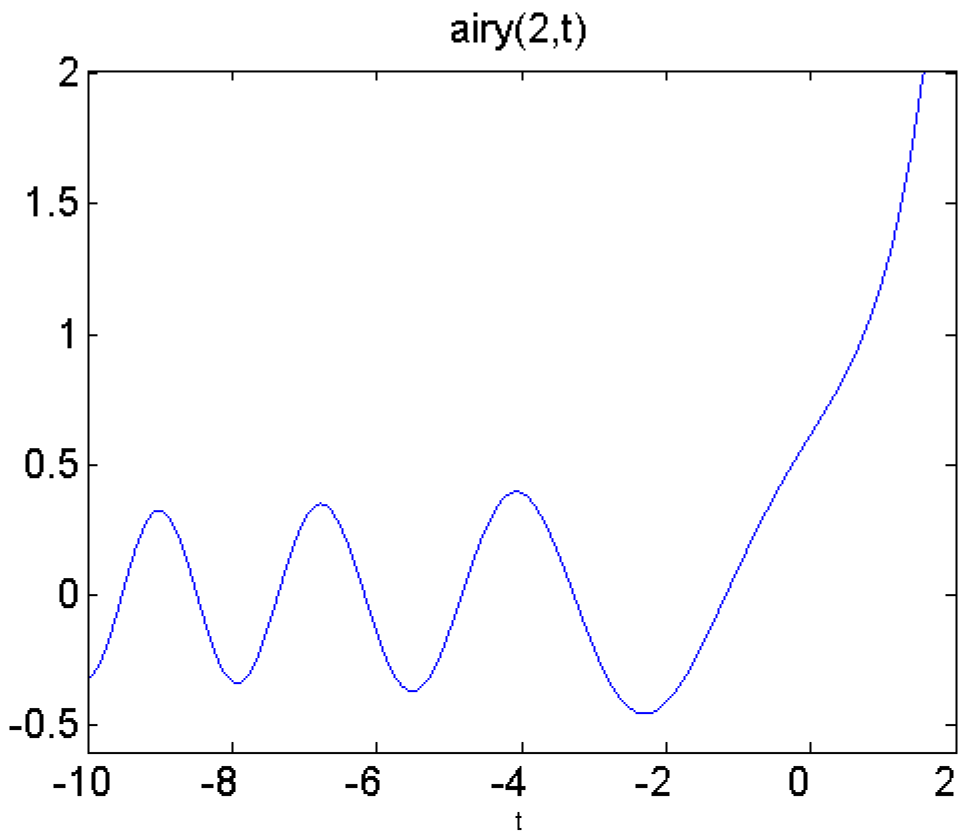
And the second solution

```
figure; % set(gcf, 'Position', [1 1 1920 1420])
ezplot('airy(2,t)')
set(get(gca, 'Title'),'Fontsize', 15)
set(gca,'XTickLabel',get(gca, 'XTickLabel'),'FontSize',15)
```



This time I will readjust the interval in order to see the oscillation

```
figure; % set(gcf, 'Position', [1 1 1920 1420])
ezplot('airy(2,t)', [-10 2])
set(get(gca, 'Title'),'Fontsize', 15)
set(gca,'XTickLabel',get(gca, 'XTickLabel'),'FontSize',15)

% So both solutions manifest decreasing oscillations
% toward minus infinity, apparently with increasing
% frequencies, but as t -> +infinity, one solution
% grows without bound and the other decays to zero.
```



Some Qualitative Analysis

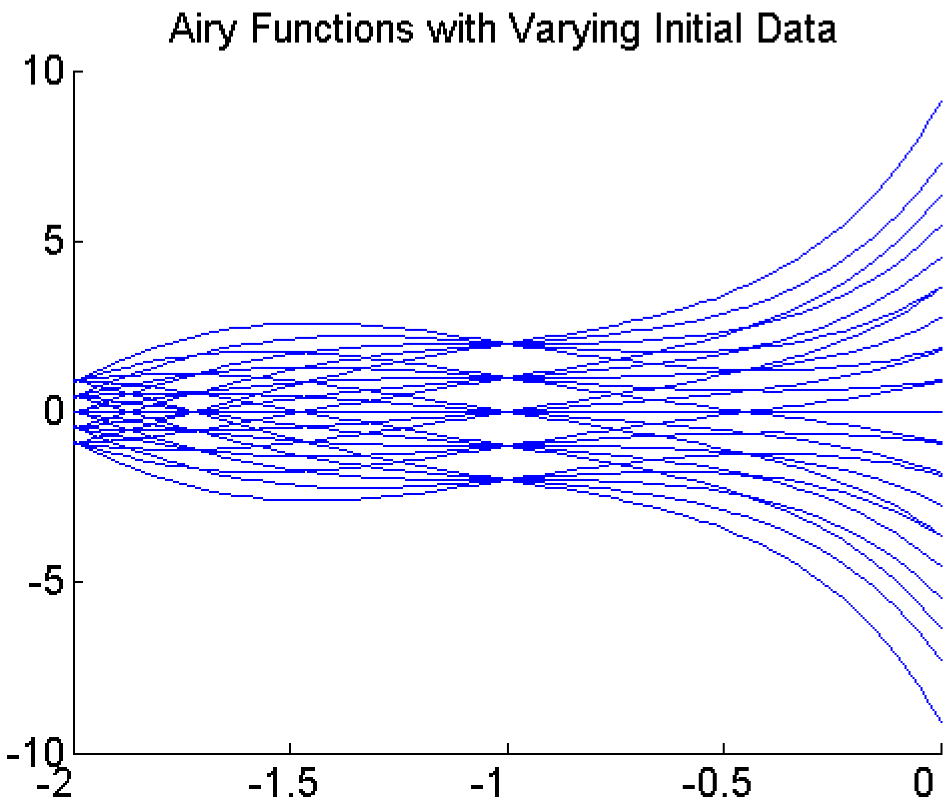
Now I recall Problem 1 in PSD in DEwM. It leads the reader through the following reasoning. For a large negative value of t, say t = -K^2, Airy's Equation resembles

```
%       y'' + K^2 y = 0,

% whose solutions are sinusoidal oscillations with
% frequency K. Thus the oscillatory behavior on the
% negative axis of the Airy functions is not surprising.
% Moreover, as t moves toward -infty, and so has
% larger absolute value (K is getting bigger), then
% the frequency is increasing. This analysis does not
% allow us to conclude anything about the amplitude.

% On the other hand, for t large postive, say near K^2,
% the equation resembles

%       y'' - K^2 y = 0,

% whose solutions are e^(Kt) and e^(-Kt). Thus the growth
% at positive infinity is expected; and exactly as the
% functions

%       a e^t + b e^(-t)
```

```
% have exactly "one direction" in which the function
% decays at +infinity, whereas in all other directions
% the solutions grow quickly, it is again not surprising
% that the same behavior is manifested by the Airy
% functions.

% Indeed, the basic Airy function airy(t) = airy(0,t) is
% exactly that special choice of the Airy functions.
```

Numerical solutions to yield a graphical presentation

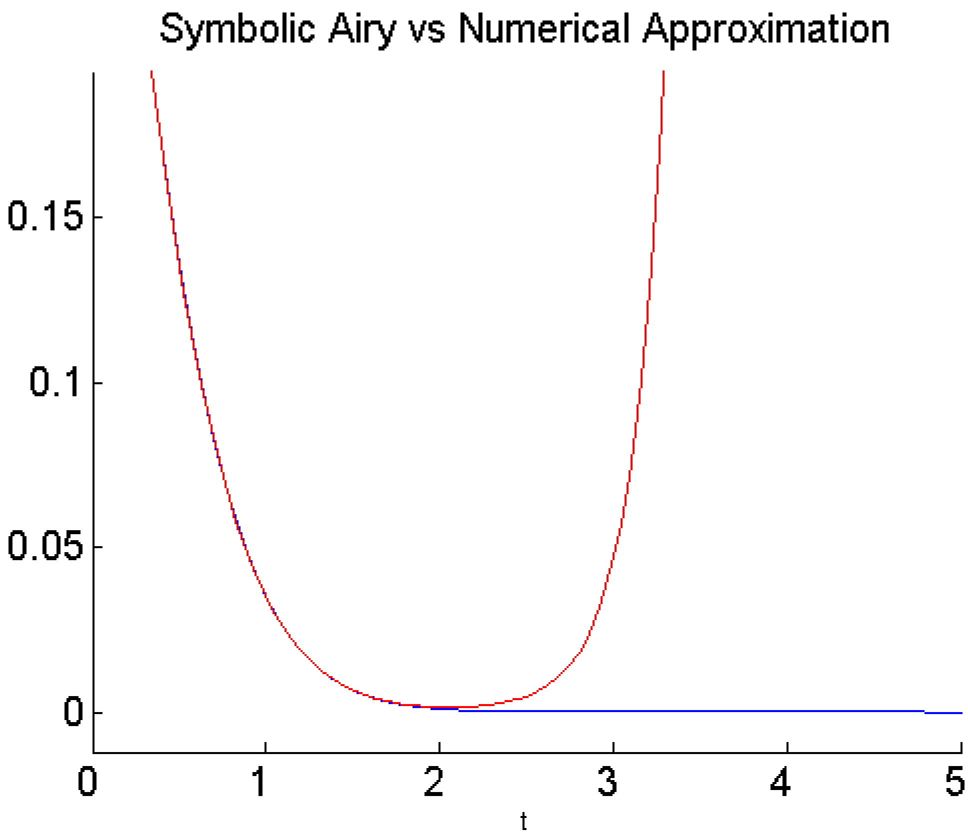
```
% Now we imitate the code on p. 142 of DEwM. As we saw
% above, there are two arbitrary constants to be specified
% in the choice of an Airy function. That corresponds to
% the fact that the second order Airy equation requires
% two pieces of initial data to determine a specific
% solution. Thus drawing a representative set of solutions
% does not, like in the case of first order equations,
% yield a set of non-intersecting curves. Still, sometimes
% the pictures are striking and reveal the general nature
% of solutions rather dramatically -- see e.g., the graph
% on p. 142 of DEwM.

rhs = @(t, y) [y(2); t*y(1)];
figure; % set(gcf, 'Position', [1 1 1920 1420])
hold on
for y0 = -2:2
    for yp0 = -1:0.5:1
        [tfor, yfor] = ode45(rhs, [0 2], [y0, yp0]);
        [tbak, ybak] = ode45(rhs, [0 -2], [y0, yp0]);
        plot(tfor, yfor(:,1))
        plot(tbak, ybak(:,1))
    end
end
title('Airy Functions with Varying Initial Data', 'FontSize', 15)
set(gca,'XTickLabel',get(gca, 'XTickLabel'),'FontSize',15)
```



Stability

We have not considered stability of second order equations, but it is not hard to envision what we would mean -- small perturbations in the initial data -- both position and velocity -- should lead to only small perturbations in the solution curve over the long term. Given what we have learned about the Airy functions, do you think the Airy equation is stable?

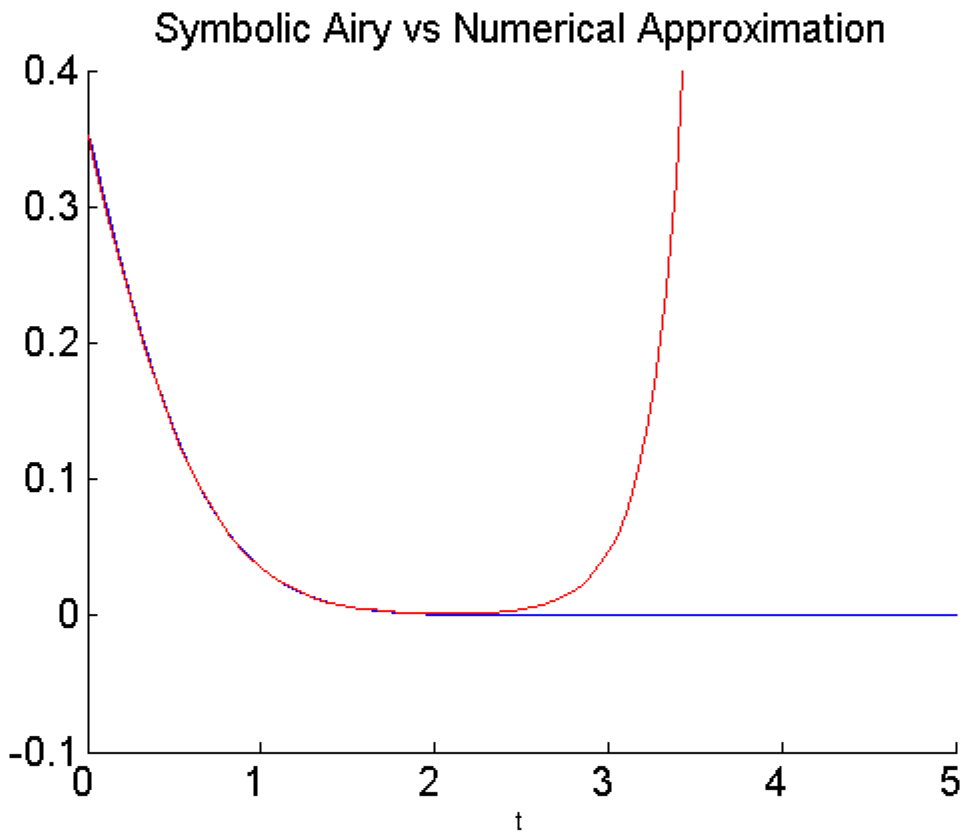
```
% Not likely; just the form of the equation y'' = ty and
% the derivative test suggests not. Let's see if we can
% justify that assertion. We know that airy(t) is the only
% solution of Airy's equation that decays at infinity.
% Let's solve the equation numerically and compare what
% we get (graphically) to the curve of airy(t):

figure; % set(gcf, 'Position', [1 1 1920 1420])
hold on
ezplot('airy(t)', [0 10])
yy0 = airy(0);
yyp0 = airy(1, 0);
[tt, yy] = ode45(rhs, [0 10], [yy0 yyp0]);
plot(tt, yy(:,1), 'r')
title('Symbolic Airy vs Numerical Approximation','FontSize',15)
set(gca,'XTickLabel',get(gca, 'XTickLabel'),'FontSize',15)

% Can you explain the graph?
```



This might help a little.

```
airy(0)
airy(1,0)
axis([0 10 -0.1 0.4])
```

```
        ans =

            0.3550


        ans =

           -0.2588
```



The Nature of Special Functions

Finally, I would like to convince you that special functions like the Airy Function are really no more mysterious than many of our elementary functions -- like the sine, exponential or logarithm.

```
% In fact, how do you define sin(x)? In most caclulus
% books, the function sin(x) is defined by saying: let x
% measure an angle in radians, then draw a right triangle
% with that angle, whereupon sin(x) is the ratio of the
% length of the opposite side over the hypotenuse. If I
% ask you to tell me what sin(sqrt(3)) is, do you really
% have a good feeling for that value? So, many advanced
% calculus books attempt to put the definition of the
% sine function on a firmer analytic footing. They define
% the sine function as follows. First define the function
% ArcSin(t) by the integral formula:

%   ArcSin(t) = int_0^t 1/sqrt(1-s^2)) ds, -1 < t < 1.

% Then they engage in some calculus to establish that this
% is a differentiable function, monotone increasing,
%   InvSin(-1) = -pi/2,     InvSin(1) = pi/2,
% and the tangent line is vertical at the two endpoints.
% Finally, they define sin(x) to be the inverse function of
% ArcSin(t), which is then defined on [-pi/2, pi/2] and
% finally they extend it to the whole real line by invoking
```

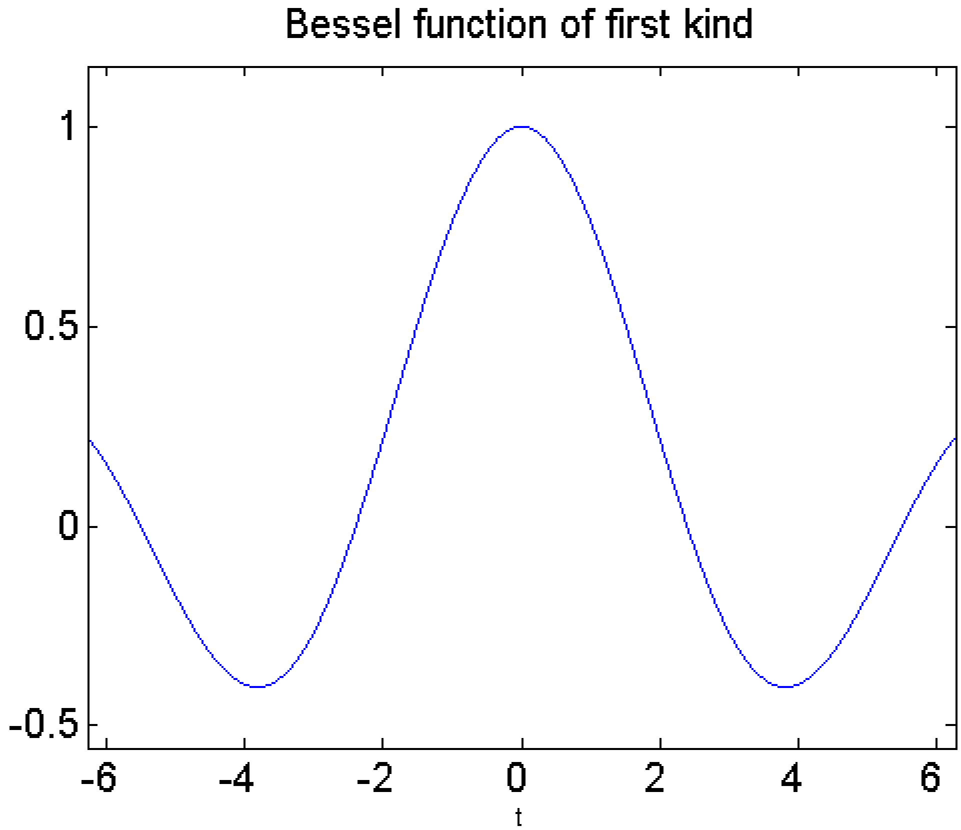
```
% periodicity.

% Now is that any simpler than the method we have used to
% define airy(t)? I won't go through the derivation
% but I will tell you that the basic Airy function can also
% be obtained by an integral. Here is the formula:

%   airy(t) = (1/pi) int_0^infty cos((1/3)s^3 + ts) ds.

% Not an easy integral, but we can deal with it numerically
% when necessary. This process is perhaps a little more
% complicated technically than:

%   ln(t) = int_0^t 1/s ds,
%   exp(t) = inverse function of log(t);

% but aren't we talking about more or less the same kind
% of object.

% The moral of the story: we can deal with airy(t) and most
% special functions in the same way that we deal with
% elementary functions:

% graphically, numerically, analytically and even
% symbolically on occasion.
% You should not be intimidated by special functions.

% There are more special functions than you can imagine:
% Bessel functions, Legendre functions, hypergeometric
% functions, the Riemann Zeta function and a score more.
% Many of these, but not all, are solutions of
% second order homogeneous non-constant coefficient
% equations. You may encounter some of them in your
% physics or engineering courses. Let's just see if Matlab
% can deal with the Bessel functions. Those are the
% solutions of the equation:

%       t^2 y''(t) + t y'(t) + (t^2 - n^2) y(t) = 0

% for different choices of an integer n.

dsolve('t^2*D2y + t*Dy +(t^2-n^2)*y=0')
```

```
ans =

C5*besselj(n, t) + C6*bessely(n, t)
```

I leave it to you to explore other special functions in Matlab, but let us just draw the fundamental solutions of the Bessel equation for n = 0.

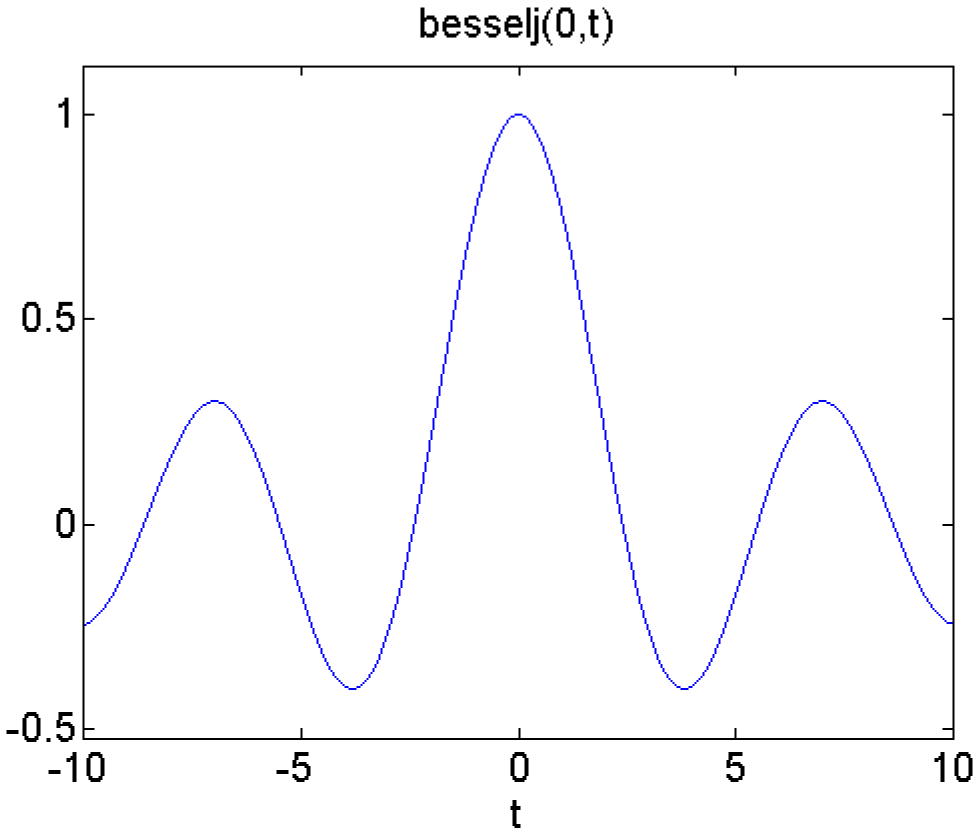
```
figure; % set(gcf, 'Position', [1 1 1920 1420])
ezplot('besselj(0,t)')
```

```
title('Bessel function of first kind','FontSize', 15)
set(gca,'XTickLabel',get(gca, 'XTickLabel'),'FontSize',15)
```



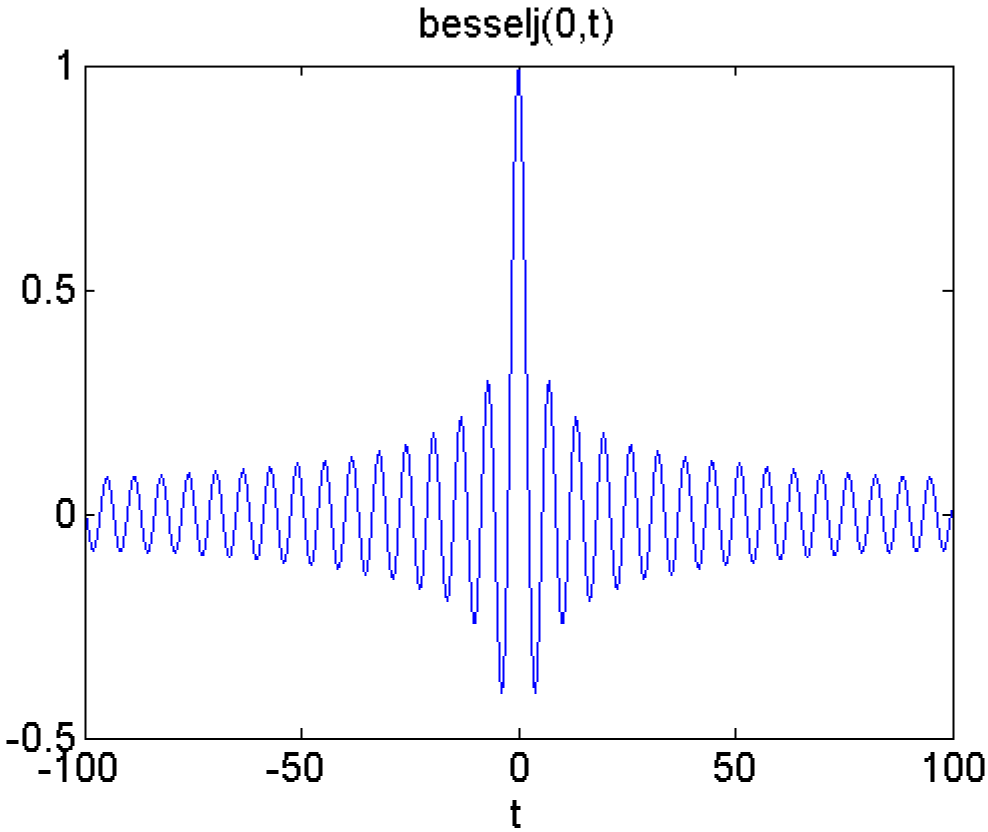
Looks like a sinusoidal; let's redraw on a bigger interval.

```
ezplot('besselj(0,t)', [-10 10])
```



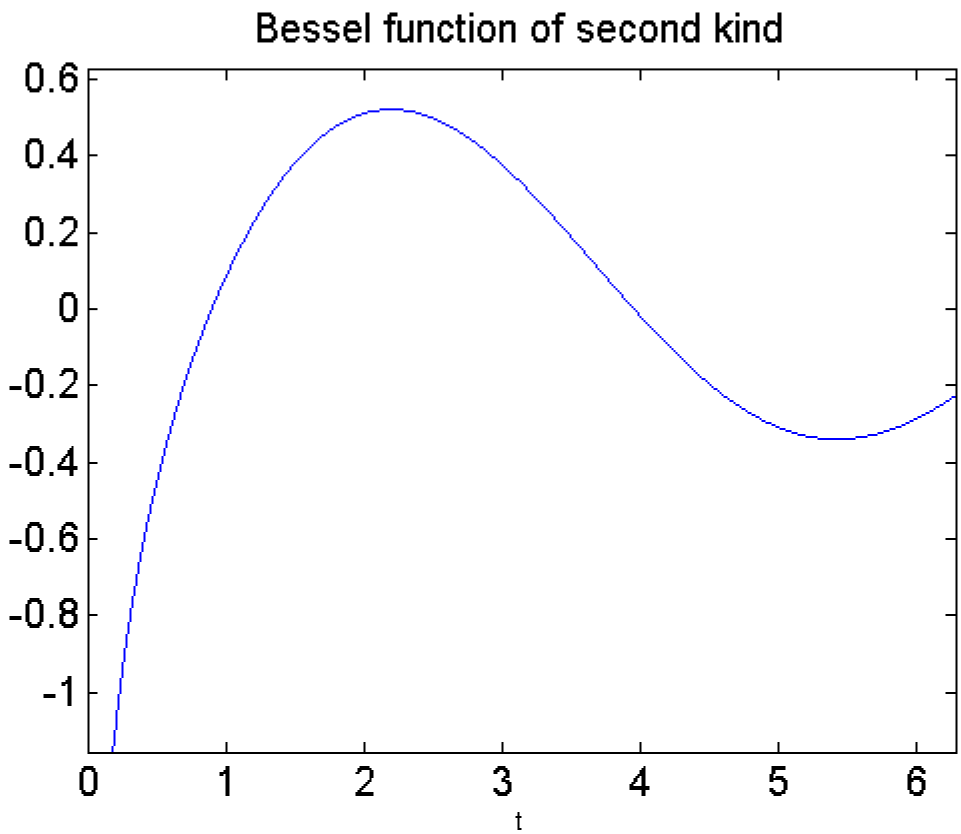
and even bigger interval

```
ezplot('besselj(0,t)', [-100 100 -0.5 1])
```



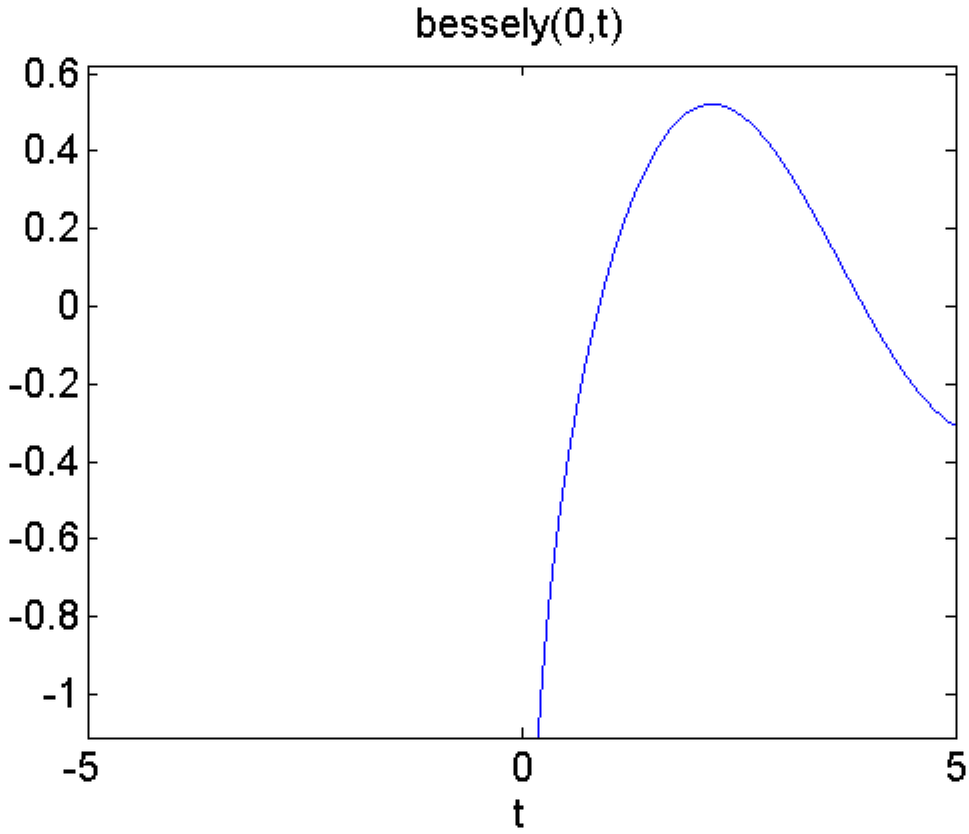
Damped oscillation clearly. Actually, the oscillatory behavior is not surprising. For t large, the middle term in the equation

```
%   t y'' + y' + t y =0

% is negligible and so the sinusoidal behavior is evident.
% Also, it looks like the solution curve has the same
% behavior in both directions. In fact bessel functions
% of order zero are even. That's a good exercise: show
% that if y(t) satisfies the Bessel equation (with n  = 0),
% then so does y(-t).

% Now for the other solution:
figure; % set(gcf, 'Position', [1 1 1920 1420])
ezplot('bessely(0,t)')
title('Bessel function of second kind','FontSize', 15)
set(gca,'XTickLabel',get(gca, 'XTickLabel'),'FontSize',15)
```



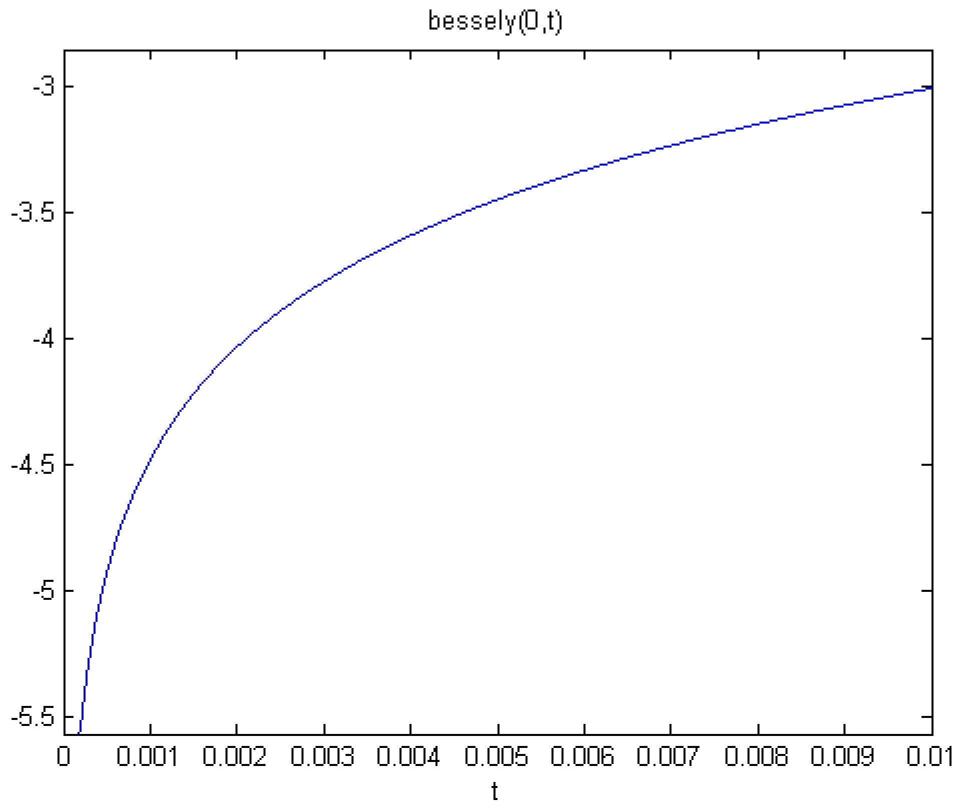
that's interesting; why no negative values?

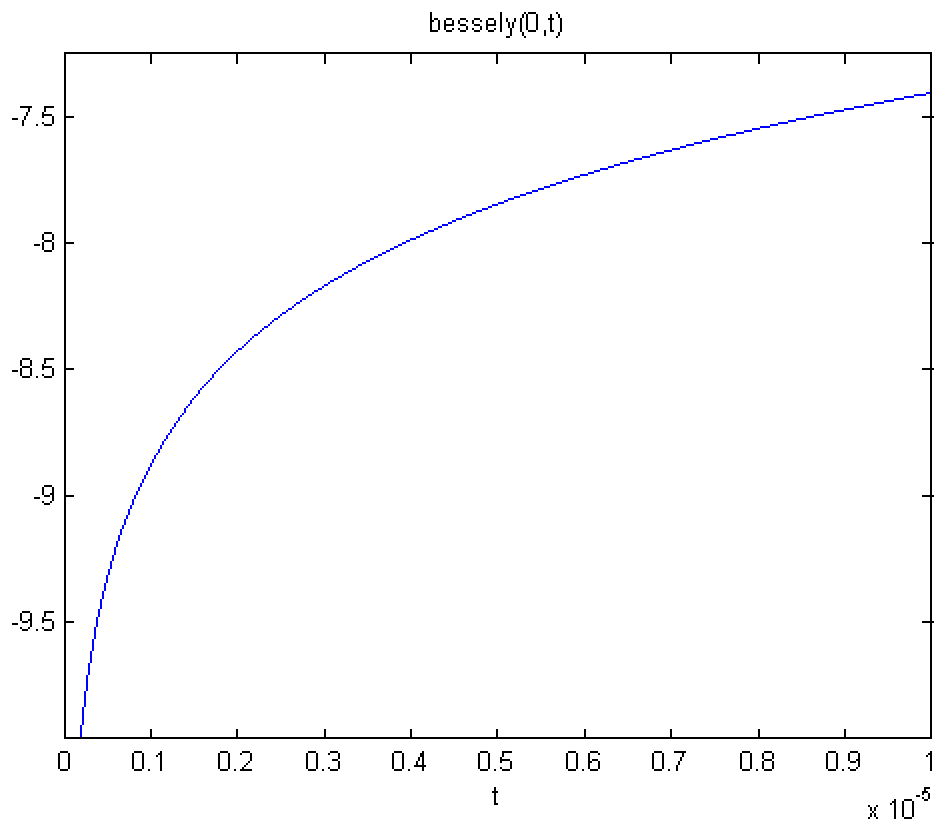
```
ezplot('bessely(0,t)', [-5 5])
```



There are no negative values because bessely(0,t) has a logarithmic singularity at t = 0. So Matlab discards the negative values, which are just a mirror reflection of the positive values. Let's see the singularity more clearly.

```
figure; ezplot('bessely(0,t)', [0 0.01])
figure; ezplot('bessely(0,t)', [0 0.00001])
```


bessely(0,t)
-3
-3.5
-4
-4.5
-5
-5.5
0
0.001
0.002
0.003
0.004
0.005
0.006
0.007
0.008
0.009
0.01
t


bessely(0,t)
-7.5
-8
-8.5
-9
-9.5
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
t
x 10-5

In fact lim bessely(0,t) as t-> 0+ is -Inf; although the divergence is very slow (logarithmic).

Let's look at one more -- the hypergeometric equation:

```
dsolve('t*(1 - t)*D2y + (c-(1 + a + b)*t)*Dy - a*b*y=0')
pretty(ans)
% solves the hypergeometric equation. I leave further
% experimentation with special functions as solutions of
% second order homogeneous equations to you; there are
% lots of examples in Boyce & DiPrima. In fact, most of
% the examples appear in the chapter on series solutions.
% Since that topic is not covered in our syllabus, I will
% have to leave it to you to look at on your own or to
% encounter in other math or engineering courses.
```

```
ans =

C8*hypergeom([a/2 + b/2 - ((a + b + 1)^2 - 2*b - 4*a*b - 2*a - 1)^(1/2)/2,


                  / -- a   b       a   b      --          \       1 - c
  C8 hypergeom| |  - + - - #1, - + - + #1  |, [c], t | + C9 t
                  \ -- 2   2       2   2      --          /

                  / -- a   b               a   b               --           \
     hypergeom| |  - + - - c - #1 + 1, - + - - c + #1 + 1  |, [2 - c], t |
                  \ -- 2   2               2   2               --           /

  where

                         2                           1/2
        ((a + b + 1)  - 2 b - 4 a b - 2 a - 1)
     #1 == -----------------------------------------
                                 2
```

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