Summary of recipes for solving linear ODEs with constant coefficients with right-hand side in the characteristic form

We are considering ODEs of the form

$$y^{(n)} + a_1 y^{(n-1)} + \ldots + a_{n-1} y' + a_n y = f(t). \tag{1}$$

The polynomial

$$p(z) := z^n + a_1 z^{n-1} + \ldots a_{n-1} z + a_n \tag{2}$$

is the characteristics polynomial of Eq. (1). Eq. (1) can be written as

$$Ly := p(D)y = f(t), \quad \text{where} \quad D := \tfrac{d}{dt}. \tag{3}$$

The polynomial $p(z)$ has n complex roots according to the main theorem of algebra, hence it can be written as

$$p(z) = (z - z_1)^{m_1} (z - z_2)^{m_2} \ldots (z - z_k)^{m_k}, \quad \text{where} \quad m_1 + m_2 + \ldots + m_k = n. \tag{4}$$

We call the number m_i the multiplicity of the root z_i.

Suppose the right-hand side $f(t)$ of Eq. (1) is in the characteristic form, i.e.,

$$\begin{aligned} f(t) = {} & (F_0 t^d + F_1 t^{d-1} + \ldots + F_{d-1} t + F_d) e^{\mu t} \cos(\nu t) \\ & + (G_0 t^d + G_1 t^{d-1} + \ldots + G_{d-1} t + G_d) e^{\mu t} \sin(\nu t). \end{aligned} \tag{5}$$

The number d is called the degree of the right-hand side. The multiplicity of the root $z = \mu + i\nu$ of the characteristic polynomial $p(z)$ is called the multiplicity of the right-hand side. If $z = \mu + i\nu$ is not a root of $p(z)$ then the multiplicity os zero.

Method of undetermined coefficients

If the right-hand side is of the form of Eq. (5) then a particular solution should be sought of the form

$$\begin{aligned} y_p(t) = {} & (A_0 t^{d+m} + A_1 t^{d+m-1} + \ldots + A_{d-1} t^{m+1} + A_d t^m) e^{\mu t} \cos(\nu t) \\ & + (B_0 t^{d+m} + B_1 t^{d+m-1} + \ldots + B_{d-1} t^{m+1} + B_d t^m) e^{\mu t} \sin(\nu t), \end{aligned} \tag{6}$$

where m is the multiplicity of the root $z = \mu + i\nu$ of the polynomial $p(z)$. The particular cases of this rule are listed in Table 1.

$f(t)$	$y_p(t)$
Fe^{rt}	$At^m e^{rt}$, m is the multiplicity of $z = r$
$F\cos(\nu t)$ or $G\sin(\nu t)$	$t^m(A\cos(\nu t) + B\sin(\nu t))$, m is the multiplicity of $z = i\nu$
$Fe^{\mu t}\cos(\nu t)$ or $Ge^{\mu t}\sin(\nu t)$	$t^m e^{\mu t}(A\cos(\nu t) + B\sin(\nu t))$, m is the multiplicity of $z = \mu + i\nu$
$F_0t^d + \ldots + F_{d-1}t + F_d$	$A_0t^{d+m} + \ldots + A_{d-1}t^{m+1} + A_dt^m$, m is the multiplicity of $z = 0$
$(F_0t^d + \ldots + F_{d-1}t + F_d)e^{rt}$	$(A_0t^{d+m} + \ldots + A_{d-1}t^{m+1} + A_dt^m)e^{rt}$, m is the multiplicity of $z = r$
$(F_0t^d + \ldots + F_{d-1}t + F_d)\cos(\nu t)$ or $(G_0t^d + \ldots + G_{d-1}t + G_d)\sin(\nu t)$	$(A_0t^{d+m} + \ldots + A_{d-1}t^{m+1} + A_dt^m)\cos(\nu t) +$ $(B_0dt^{d+m} + \ldots + B_{d-1}t^{m+1} + B_dt^m)\sin(\nu t)$, m is the multiplicity of $z = i\nu$
$(F_0t^d + \ldots + F_{d-1}t + F_d)e^{\mu t}\cos(\nu t)$ or $(G_0t^d + \ldots + G_{d-1}t + G_d)e^{\mu t}\sin(\nu t)$	$(A_0t^{d+m} + \ldots + A_{d-1}t^{m+1} + A_dt^m)e^{\mu t}\cos(\nu t) +$ $(B_0dt^{d+m} + \ldots + B_{d-1}t^{m+1} + B_dt^m)e^{\mu t}\sin(\nu t)$, m is the multiplicity of $z = \mu + i\nu$

Table 1: Forms of the right-hand sides and the corresponding particular solutions for the method of undetermined coefficients

Using the Key Identity

The Key Identity for Eq. (2) and its first three derivatives with respect to z are given by

$$\begin{aligned}
&L(e^{zt}) \equiv p(D)e^{zt} = p(z)e^{zt},\\
&L(te^{zt}) = (p(z)t + p'(z))e^{zt},\\
&L(t^2e^{zt}) = (p(z)t^2 + 2p'(z)t + p''(z))e^{zt},\\
&L(t^3e^{zt}) = (p(z)t^3 + 3p'(z)t^2 + 3p''(z)t + p'''(z))e^{zt}.
\end{aligned}$$

In order to find a particular solution for the right-hand side with degree d and multiplicity m we will need to write $d+1$ equations that are the derivatives of the Key Identity of orders m, $m+1$, ..., $m+d$. To apply this approach to $f(t)$ containing trigonometric functions we need to take real and imaginary parts of complex functions. By Euler's formula,

$$e^{i\nu t} = \cos(\nu t) + i\sin(\nu t).$$

If $z = u + iv$ is a complex number then u and v are called its real and imaginary parts respectvely and denoted by

$$u \equiv \mathrm{Re}(z), \quad v \equiv \mathrm{Im}(z).$$

Therefore,

$$\cos(\nu t) = \mathrm{Re}\left(e^{i\nu t}\right) \quad \text{and} \quad \sin(\nu t) = \mathrm{Im}\left(e^{i\nu t}\right).$$

The Key Identity approach for degree $d = 0$ is summarized in Table 2.

The Key Identity approach for degree $d = 1$ and multiplicity $m = 0$ is summarized in Table 3.

The Key Identity approach for degree $d = 1$ and multiplicity $m = 0$ is summarized in Table 4.

$f(t)$	$y_p(t)$
Fe^{rt}	$F\dfrac{t^m e^{rt}}{p^{(m)}(r)}$, m is the multiplicity of $z = r$
$F\cos(\nu t)$	$\mathrm{Re}\left(F\dfrac{t^m e^{i\nu t}}{p^{(m)}(i\nu)}\right)$, m is the multiplicity of $z = i\nu$
$G\sin(\nu t)$	$\mathrm{Im}\left(G\dfrac{t^m e^{i\nu t}}{p^{(m)}(i\nu)}\right)$, m is the multiplicity of $z = i\nu$
$Fe^{\mu t}\cos(\nu t)$	$\mathrm{Re}\left(Fe^{\mu t}\dfrac{t^m e^{i\nu t}}{p^{(m)}(\mu + i\nu)}\right)$, m is the multiplicity of $z = \mu + i\nu$
$Ge^{\mu t}\sin(\nu t)$	$\mathrm{Im}\left(Ge^{\mu t}\dfrac{t^m e^{i\nu t}}{p^{(m)}(\mu + i\nu)}\right)$, m is the multiplicity of $z = \mu + i\nu$

Table 2: The Key identity approach for $d = 0$.

$f(t)$	$y_p(t)$
Fte^{rt}. r is not a root of $p(z)$, i.e., $m = 0$, $d = 1$.	Write 2 equations $L(e^{rt}) = p(r)e^{rt},$ $L(te^{rt}) = p(r)te^{rt} + p'(r)e^{rt}$ Express te^{rt} in terms of $L(e^{rt})$, $L(te^{rt})$, $p(r)$, and $p'(r)$, and multiply by F. Obtain $Fte^{rt} = L(\ldots).$ The expression in the parentheses is $y_p(t)$.
$Ft\cos(\nu t)$ or $Ft\sin(\nu t)$, where $i\nu$ is not a root of $p(z)$, i.e., $m = 0$, $d = 1$.	Write 2 equations $L(e^{i\nu t}) = p(i\nu)e^{i\nu t},$ $L(te^{i\nu t}) = p(i\nu)te^{i\nu t} + p'(i\nu)e^{i\nu t}.$ Express $te^{i\nu t}$ in terms of $L(e^{i\nu t})$, $L(te^{i\nu t})$, $p(i\nu)$, and $p'(i\nu)$, and multiply by F. Obtain $Fte^{i\nu t} = L(\ldots).$ Then for $f(t) = Ft\cos(\nu t)$ $y_p(t)$ is the real part of the expression in the parentheses, while for $f(t) = Ft\sin(\nu t)$ y_p is the imaginary part of the expression in the parentheses.
$Fte^{\mu t}\cos(\nu t)$ or $Fte^{\mu t}\cos(\nu t)$, where $\mu + i\nu$ is not a root of $p(z)$, i.e., $m = 0$, $d = 1$.	Write 2 equations $L(e^{(\mu+i\nu)t}) = p(\mu + i\nu)e^{(\mu+i\nu)t},$ $L(te^{(\mu+i\nu)t}) = p(\mu + i\nu)te^{(\mu+i\nu)t} + p'(\mu + i\nu)e^{(\mu+i\nu)t}.$ Express $te^{(\mu+i\nu)t}$ in terms of $L(e^{(\mu+i\nu)t})$, $L(te^{(\mu+i\nu)t})$, $p(\mu + i\nu)$, and $p'(\mu + i\nu)$, and multiply by F. Obtain $Fte^{(\mu+i\nu)t} = L(\ldots).$ Then for $f(t) = Fte^{\mu t}\cos(\nu t)$ $y_p(t)$ is the real part of the expression in the parentheses, while for $f(t) = Fte^{\mu t}\sin(\nu t)$ y_p is the imaginary part of the expression in the parentheses.

Table 3: The Key identity approach for $d = 1$ and $m = 0$.

$f(t)$	$y_p(t)$
Fte^{rt}, r is a root of $p(z)$ of multiplicity 1, i.e., $m = 1$, $d = 1$.	Note that $p(r) = 0$. Write 2 equations $L(te^{rt}) = p'(r)e^{rt}$, $L(t^2e^{rt}) = 2p'rte^{rt} + p''re^{rt}$. Express te^{rt} in terms of $L(te^{(\mu+i\nu)t})$, $L(t^2e^{rt})$, $p'(r)$, and $p''(r)$, and multiply by F. Obtain $Fte^{rt} = L(\ldots)$. Then $y_p(t)$ is the expression in the parentheses.
$Ft\cos(\nu t)$ or $Ft\sin(\nu t)$, where $i\nu$ is a root of $p(z)$ of multiplicity 1, i.e., $m = 1$, $d = 1$.	Note that $p(i\nu) = 0$. Write 2 equations $L(te^{(i\nu)t}) = p'(i\nu)e^{(i\nu)t}$, $L(t^2e^{(i\nu)t}) = 2p'(i\nu)te^{(i\nu)t} + p''(i\nu)e^{(i\nu)t}$. Express $te^{(i\nu)t}$ in terms of $L(te^{(i\nu)t})$, $L(t^2e^{(i\nu)t})$, $p'(i\nu)$, and $p''(i\nu)$, and multiply by F. Obtain $Fte^{(i\nu)t} = L(\ldots)$. Then for $f(t) = Ft\cos(\nu t)$ $y_p(t)$ is the real part of the expression in the parentheses, while for $f(t) = Ft\sin(\nu t)$ y_p is the imaginary part of the expression in the parentheses.
$Fte^{\mu t}\cos(\nu t)$ or $Fte^{\mu t}\sin(\nu t)$, where $\mu + i\nu$ is a root of $p(z)$ of multiplicity 1, i.e., $m = 1$, $d = 1$.	Note that $p(\mu + i\nu) = 0$. Write 2 equations $L(te^{(\mu+i\nu)t}) = p'(\mu + i\nu)e^{(\mu+i\nu)t}$, $L(t^2e^{(\mu+i\nu)t}) = 2p'(\mu + i\nu)te^{(\mu+i\nu)t} + p''(\mu + i\nu)e^{(\mu+i\nu)t}$. Express $te^{(\mu+i\nu)t}$ in terms of $L(te^{(\mu+i\nu)t})$, $L(t^2e^{(\mu+i\nu)t})$, $p'(\mu + i\nu)$, and $p''(\mu + i\nu)$, and multiply by F. Obtain $Fte^{(\mu+i\nu)t} = L(\ldots)$. Then for $f(t) = Fte^{\mu t}\cos(\nu t)$ $y_p(t)$ is the real part of the expression in the parentheses, while for $f(t) = Fte^{\mu t}\sin(\nu t)$ $y_p(t)$ is the imaginary part of the expression in the parentheses.

Table 4: The Key identity approach for $d = 1$ and $m = 1$.