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## Table of Contents

Transforms, especially the Laplace Transform .....	1
Right Triangles .....	1
Now for the second example. ....	5
The Laplace Transform .....	6

## Transforms, especially the Laplace Transform

```
% In this lecture we shall study the concept of a transform
% and then specialize to the Laplace Transform. Suppose you
% have some collection of objects on which there are well
% defined operations. For example:

% **The set of all right triangles with various geometric
%     and algebraic operations; or

% **The set of Riemann Integrable functions on the
%     interval [-pi,pi] with various analytic operations; or

% **The set of piecewise continuous functions on the
%     positive t-axis with no more than exponential growth and
%     the operation of differentiation.

% The last example as exactly the set of functions for which we can
% take a Laplace Transform. We'll come back to that, but first
% let's consider the first two examples to get a
% feel for what a transform can do to help us solve concrete
% problems.
```

## Right Triangles

```
% Given a right triangle, we shall associate with it a pair
% of real numbers in the set
%     {(x, y): 0 < x <= y}
% by assigning x to be the length of the shortest leg and
% y to be the length of the other leg.

% Now certain operations on the set of right triangles can
% be converted into operations on the pair of real numbers.
% For example, the operation of shrinking the size of each
% side of the triangle by the same amount, say lambda < 1,
% amounts to multiplying the coordinates (x, y) by lambda.
% Similarly with an expansion lambda > 1. One could also
% shrink or expand the two sides independently and that
% would correspond to multiplying the coordinates by
% distinct numbers. (One would have to restrict the
% multipliers to ensure that the multiple of x is less than
% or equal to the multiple of y.)

% Here is something more subtle. Given a
```

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```

% right triangle, how do you obtain a new triangle in which
% the length of the shortest leg remains the same but the
% size of the angle between the shortest leg and the
% hypotenuse is three quarters its original size. How much
% shorter does the longer leg have to be? Well if theta is
% the angle in question, then first of all it needs to be
% greater than pi/4 radians in order for x to be the
% shorter side. And if it remains the shorter side in the new
% triangle, then (3/4)theta must also be greater than pi/4.
% So it must be that theta > pi/3. Now we multiply y
% by lambda. Then the geometric problem is transformed
% into the algebraic problem:

```

```

%      theta = arctan(y/x).
% Find lambda < 1 so that
%      (3/4) arctan(y/x) = arctan(lambda*y/x).

```

```

% Not an easy problem by hand, but easy for Matlab.
clear all
close all

```

To illustrate, suppose  $x = 1$  and  $y = 3$ . Then

```

x = 1; y = 3; theta = atan(y/x)
lambda = x * tan((3/4)*theta)/y

```

```

% So you have to shorten the longer leg by a factor of 0.4533.
% in order to reduce the angle by 25%.

```

```

theta =
    1.2490

```

```

lambda =
    0.4533

```

Here's a harder problem. Find all the right triangles whose perimeter equals its area? After applying the transform, that is, converting the triangle into a pair of real numbers, the problem becomes: solve the equation  $(1/2)xy = x + y + \sqrt{x^2 + y^2}$ .

```

% That is some kind of quadratic curve.
% Let's see whether Matlab can simplify the equation
clear all
syms x y; simplify(((1/2)*x*y - (x+y))^2 - (x^2+y^2))

```

```

ans =
    -(x*y*(4*x + 4*y - x*y - 8))/4

```

Since  $x$  and  $y$  are both non-zero, the equation further simplifies to  $4*x + 4*y - x*y - 8 = 0$ .

---

```

% That is actually a hyperbola as you can see by making the
% substitution
% x = u + v, y = u - v.
syms u v; expr = 4*x + 4*y - x*y - 8; subs(expr, [x y], [u+v u-v])
expand(ans)

```

```

ans =

8*u - (u + v)*(u - v) - 8

```

```

ans =

- u^2 + 8*u + v^2 - 8

```

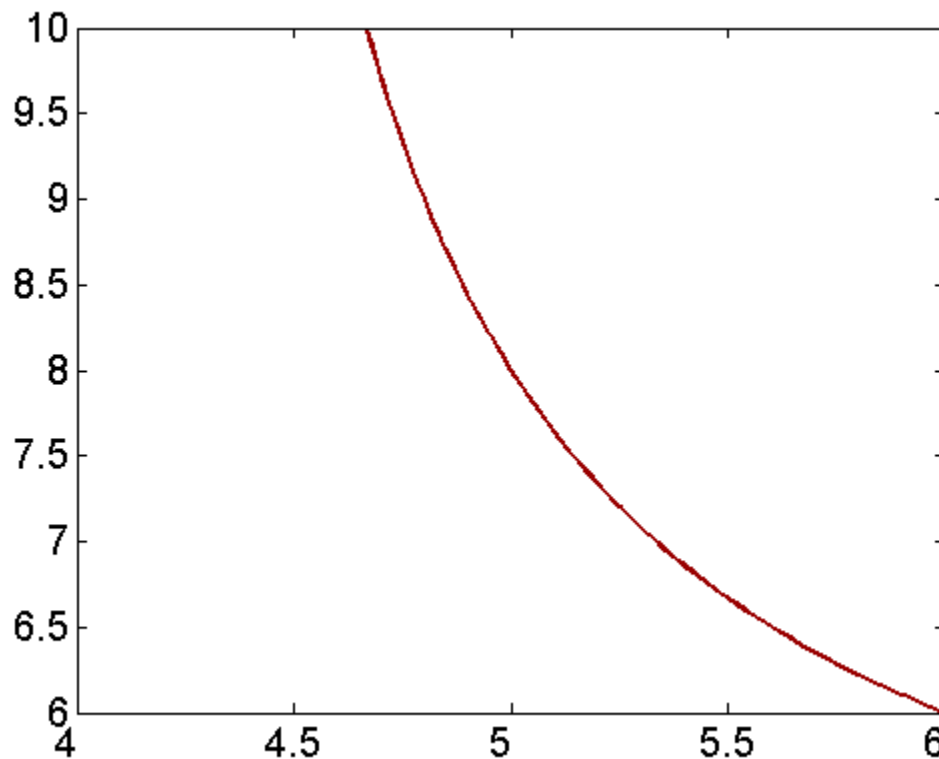
Also a little inspection reveals that  $x = 6, y = 8$  is a point on the curve. So now we can draw it.

```

figure; % set(gcf, 'Position', [1 1 1920 1420])
[X, Y] = meshgrid(4:0.1:8, 6:0.1:10);
contour(X, Y, 4.*X + 4.*Y - X.*Y - 8, [0 0],...
        'LineWidth',2,'Color',[.6 0 0])
set(gca, 'XTickLabel',get(gca, 'XTickLabel'),'FontSize',15)

% See the online help for an explanation of the
% option [0 0] in the contour command.

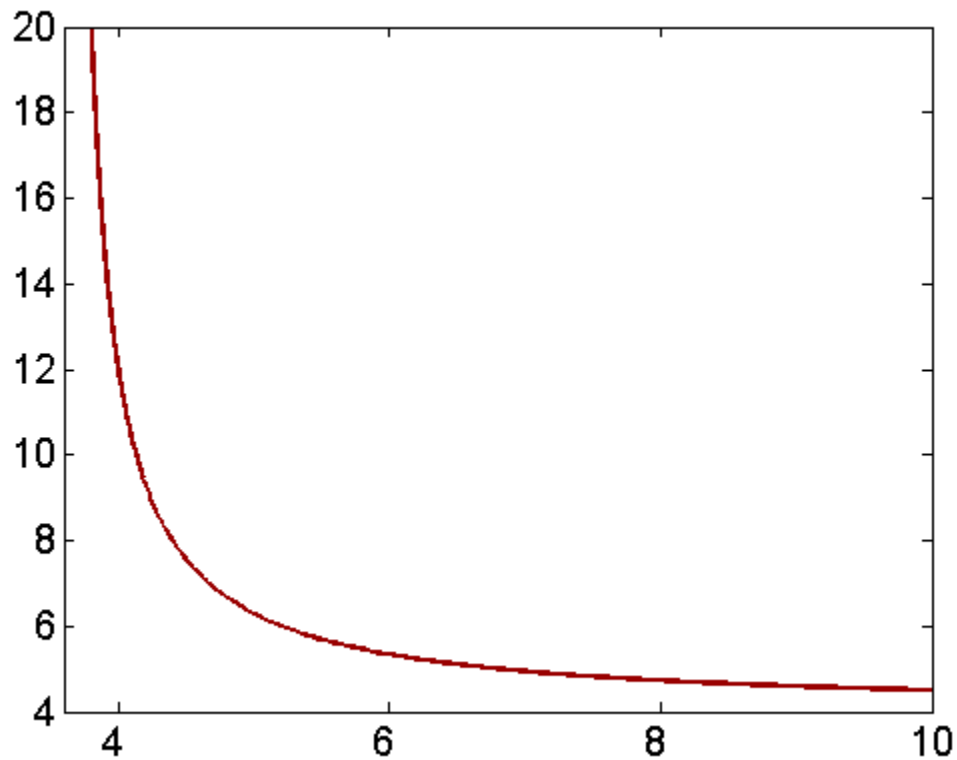
```



---

It's a very nice exercise to show algebraically that  $x = 4$  is an asymptote of the hyperbola. It should be clear then that  $y = 4$  is the other asymptote. Here is visual evidence:

```
figure; % set(gcf, 'Position', [1 1 1920 1420])
[X, Y] = meshgrid(4:0.1:20, 4:0.1:20);
contour(X, Y, 4.*X + 4.*Y - X.*Y - 8, [0 0],...
        'LineWidth',2,'Color',[.6 0 0])
set(gca, 'XTickLabel', get(gca, 'XTickLabel'), 'FontSize', 15)
```



Another nice exercise is to show that the only isosceles triangle which has equal perimeter and area corresponds to  $x = 4 + \sqrt{8}$ .

```
% Here is how to do it:
[a,b]=solve([x-y,expr])
% Why is only one answer legit?
```

$a =$

$$\frac{2 \cdot 2^{1/2} + 4}{4 - 2 \cdot 2^{1/2}}$$

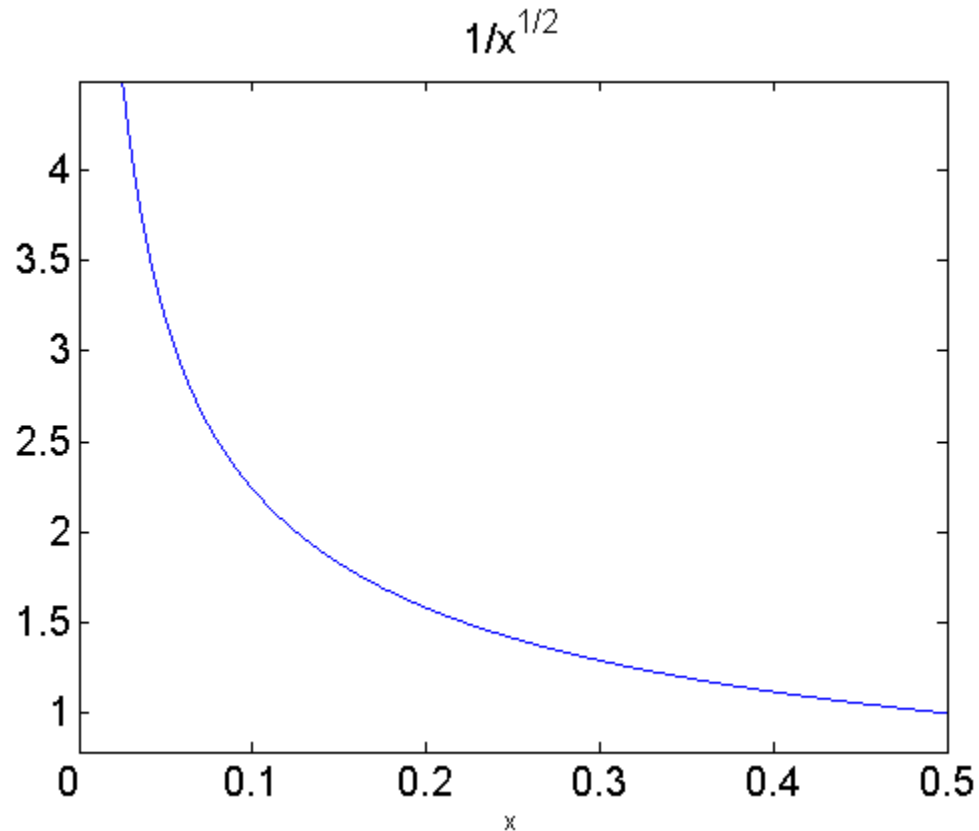
$b =$

$$\frac{2 \cdot 2^{1/2} + 4}{4 - 2 \cdot 2^{1/2}}$$

---

## Now for the second example.

```
% We consider the functions on [-pi, pi] that are Riemann
% integrable. You will have to look back at your calculus
% book if you don't remember upper sums and lower sums, etc.
% Certainly all continuous functions are
% integrable, but there are others, like
%     f(x) = 1/sqrt(|x|).
clear all
close all
syms x
figure; % set(gcf, 'Position', [1 1 1920 1420])
ezplot(1/sqrt(x), [0 1])
set(get(gca, 'Title'), 'FontSize', 15)
set(gca, 'XTickLabel', get(gca, 'XTickLabel'), 'FontSize', 15)
```



Clearly the y-axis is a vertical asymptote for the curve; Still, there is a finite area under the curve:

```
int(1/sqrt(x), 0, pi)
```

```
ans =
```

```
2*pi^(1/2)
```

---

Now for the transform. To every such function we associate or transform it into a sequence of complex numbers via the following formula:

```
%  $a_n = (1/2\pi) \int_{-\pi}^{\pi} e^{-i n t} f(t) dt, -\infty < n < \infty$ 
```

```
% This transform is bound up with the theory of
% Fourier Series that you will certainly meet in later Math,
% Physics or Engineering courses. There are many interesting
% properties that relate the function  $f(t)$  to the
% sequence  $\{a_n\}$ , which comprises in fact the
% Fourier Coefficients of  $f$ .
```

```
% The associated Fourier Series is nothing more than
%  $\sum_{-\infty}^{\infty} a_n e^{i n t}$ .
```

```
% We can use this observation and the transform to solve the
% following problem. Which functions are square integrable?
% That is, when is it true that
%  $\int_{-\pi}^{\pi} |f(t)|^2 dt < \infty$ .
```

```
% It's not true always as you can see from the example
%  $f(t) = 1/\sqrt{|t|}$ .
```

```
int(1/x, 0, pi)
```

```
ans =
```

```
Inf
```

We compute as follows  $\int_{-\pi}^{\pi} f(t)^2 dt = \int_{-\pi}^{\pi} f(t)f(t) dt$  \_\_\_\_\_

```
%=  $\int_{-\pi}^{\pi} [\sum_{-\infty}^{\infty} a_n e^{i n t} \sum_{-\infty}^{\infty} a_m e^{i m t}] dt$ 
```

```
%
% But  $(1/2\pi) \int_{-\pi}^{\pi} e^{i j t} dt = 0$  unless  $j = 0$ ,
% in which case it is 1.
```

```
%
% So the above reduces to
```

```
%  $\sum_{-\infty}^{\infty} |a_n|^2$ .
```

```
% This solves the problem. An integrable function is
% square integrable precisely when its Fourier coefficients
% comprise a square summable sequence.
```

## The Laplace Transform

Finally we look at the collection of functions that we have identified for which we can take a Laplace Transform, namely:

```
% The set of piecewise continuous functions on the positive
```

---

```

% t-axis with no more than exponential growth.
% The Laplace Transform of such a function f(t) is then
% defined as

%       $L\{f\}(s) = F(s) = \int_0^{\infty} e^{-st} f(t) dt.$ 

% The improper integral converges because of the assumptions
% on the growth of f. F(s) is a continuous function that
% decays to zero as  $s \rightarrow \infty$ . The key property of the
% Laplace Transform that makes it useful in the study
% of differential equations is that it converts the operation
% of differentiation into multiplication by the Laplace
% Transform variable; thus analytic problems in differential
% equations are converted into algebraic problems. The
% formulaic expression of the conversion is
% computed in the text as:

%       $L\{f'\}(s) = s L\{f\}(s) - f(0)$ 
%       $L\{f''\}(s) = s^2 L\{f\}(s) - s f(0) - f'(0),$ 

% perfect for initial value problems; and especially suited
% to problems involving discontinuous functions for which
% Laplace transforms can be computed.

```

A nice scheme for using Matlab to implement solutions via the Laplace Transform is presented in DEwM. Let's use it to solve the following explicit problem:

```

%       $y''(t) + 2 y'(t) - 2 y(t) = g(t) + \delta_2(t),$ 
%       $y(0) = 1, y'(0) = -1,$ 

% where  $\delta_2$  is the Dirac generalized function
% concentrated at  $t = 2$ , and  $g(t)$  is given as follows:

%  $g(t) = 0, t < 0; t^2, 0 \leq t < 1; 2, 1 \leq t.$ 

% Note that  $g$  is discontinuous at  $t = 1$ , but the full right side
% of the equation will also be discontinuous at  $t = 2$ , because
% of the Dirac delta function concentrated there.

```

```

clear all
close all

```

Here is the scheme presented in DEwM:

```

syms s t Y
g = ['heaviside(t)*t^2 + heaviside(t - 1)*(2 - t^2)'];
rhs = strcat(g, '+ dirac(t-2)');
eqn = sym(['D(D(y))(t) + 2*D(y)(t) - 2*y(t) = ' rhs]);
lteqn = laplace(eqn, t, s);
neweqn = subs(lteqn, {'laplace(y(t),t,s)', 'y(0)', 'D(y)(0)'}, {Y,1,-1});
ytrans = simplify(solve(neweqn,Y));
y = ilaplace(ytrans, s, t);
figure; % set(gcf, 'Position', [1 1 1920 1420])
ezplot(y, [0 3])

```

```

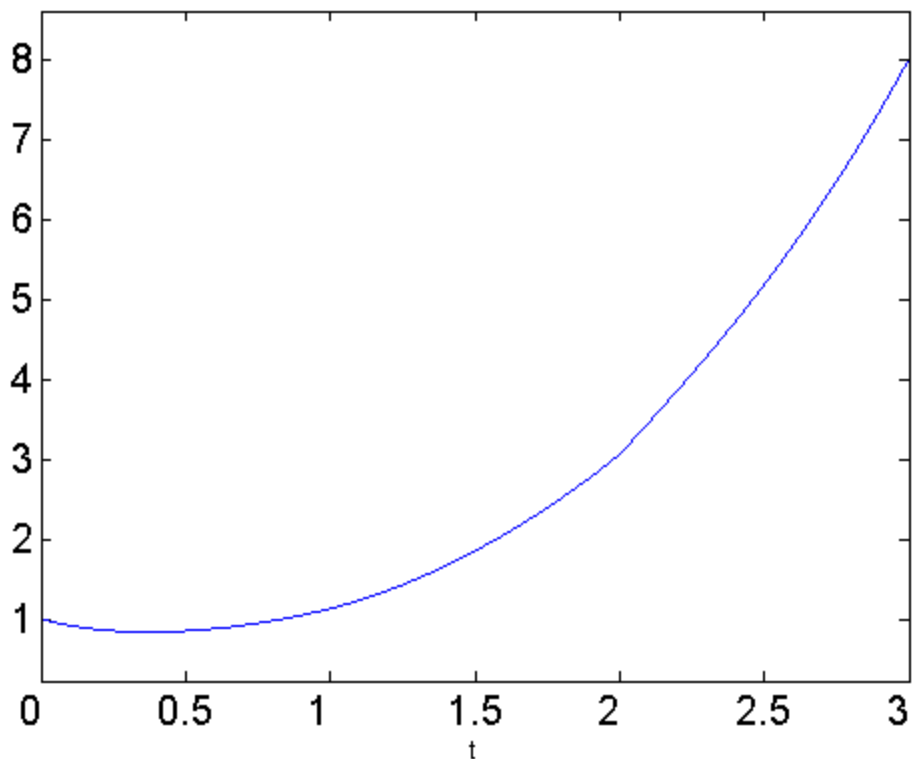
set(get(gca, 'Title'),'FontSize', 15)
set(gca, 'XTickLabel', get(gca, 'XTickLabel'), 'FontSize', 15)

% Mention that the "simplify" command is missing in the book.
% Please use it when employing this template for solving eqns
% by the Laplace transform with Matlab.

% The graph illustrates a point made in DEwM. The solution
% function will be differentiable at t = 1, but only
% continuous at t = 2. That might be a little hard to see
% from the graph, so

```

$$- 1) ((\exp(1 - t) (\cosh(3^{1/2} (t - 1)) + (3^{1/2} \sinh(3^{1/2} (t - 1))))/3))/2 .$$

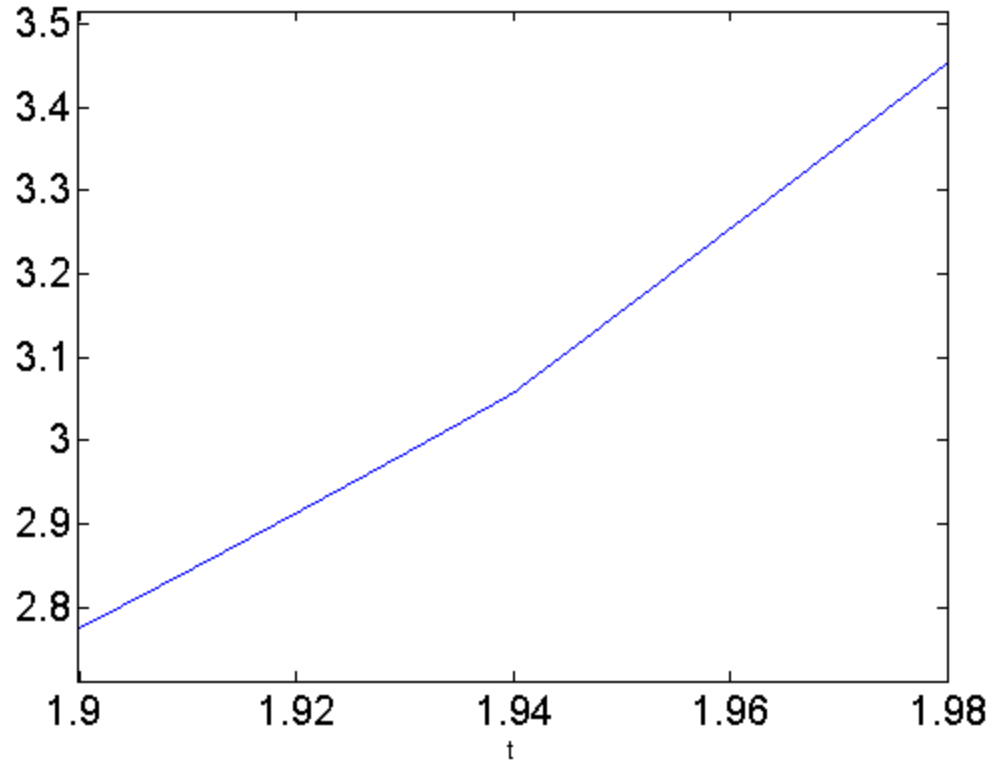


```

figure; % set(gcf, 'Position', [1 1 1920 1420])
ezplot(y, [1.9 2.1])
set(get(gca, 'Title'),'FontSize', 15)
set(gca, 'XTickLabel', get(gca, 'XTickLabel'), 'FontSize', 15)

```

$$- 1) ((\exp(1 - t) (\cosh(3^{1/2} (t - 1))) + (3^{1/2} \sinh(3^{1/2} (t - 1))))/3))/2 .$$



Finally, the formula for the solution has been computed, although not displayed. Let's take a look.

y

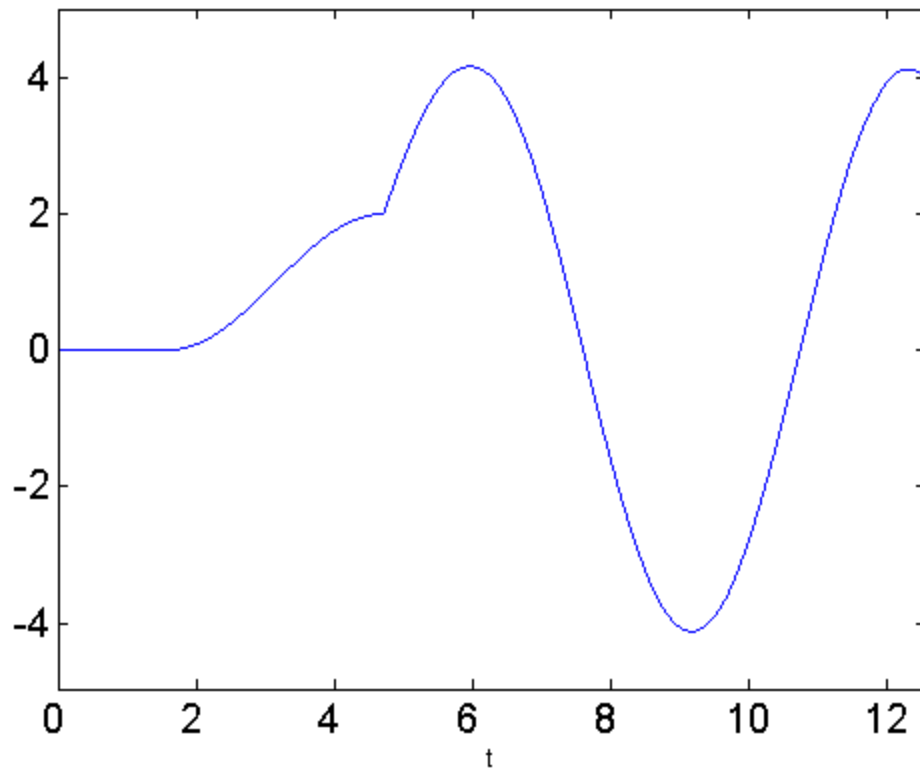
y =

$$\text{heaviside}(t - 1) * ((\exp(1 - t) * (\cosh(3^{1/2} * (t - 1))) + (3^{1/2} * \sinh(3^{1/2} * (t - 1))))/3))$$

Here's another example:  $y''(t) + y(t) = u_{\pi/2}(t) + 3*\delta_{3\pi/2}(t) - u_{2\pi}(t)$ ,  $y(0) = 0$ ,  $y'(0) = 0$ .

```
syms s t Y
g = ['heaviside(t-pi/2) - heaviside(t - 2*pi)'];
rhs = strcat(g, '+ 3*dirac(t-3*pi/2)');
eqn = sym(['D(D(y))(t) + y(t) = ' rhs]);
lteqn = laplace(eqn, t, s);
neweqn = subs(lteqn, {'laplace(y(t),t,s)', 'y(0)', 'D(y)(0)'}, {Y,0,0});
ytrans = simplify(solve(neweqn,Y));
y = ilaplace(ytrans, s, t);
figure; % set(gcf, 'Position', [1 1 1920 1420])
ezplot(y, [0 4*pi])
set(get(gca, 'Title'), 'FontSize', 15)
set(gca, 'XTickLabel', get(gca, 'XTickLabel'), 'FontSize', 15)
```

$\text{heaviside}(t - (3\pi)/2) \sin(t - (3\pi)/2) + \dots - \text{heaviside}(t - \pi/2) (\cos(t - \pi$



Again, let's look at the solution formula.

y

y =

$$3 \cdot \text{heaviside}(t - (3\pi)/2) \cdot \sin(t - (3\pi)/2) + \text{heaviside}(t - 2\pi) \cdot (\cos(t - \pi))$$

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