II. Higher-Order Linear Ordinary Differential Equations
1. Introduction to Higher-Order Linear Equations

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Contents

Return to Main Webpage

1. Introduction to Higher-Order Linear Equations

1.1. **Normal Forms and Solutions.**

Every n^{th}-order linear ordinary differential equation can be brought into the linear normal form

$$\frac{\mathrm{d}^n y}{\mathrm{d}t^n} + a_1(t)\frac{\mathrm{d}^{n-1} y}{\mathrm{d}t^{n-1}} + \cdots + a_{n-1}(t)\frac{\mathrm{d}y}{\mathrm{d}t} + a_n(t)y = f(t)\,. \tag{1.1}$$

Here $a_1(t), \cdots, a_n(t)$ are called *coefficients* while $f(t)$ is called the *forcing* or *driving*. When $f(t) = 0$ the equation is said to be *homogeneous*; otherwise it is said to be *nonhomogeneous* or *inhomogeneous*.

Definition 1.1. We say that $y = Y(t)$ is a *solution* of (1.1) over an interval (t_L, t_R) provided that:

- the function Y is n-times differentiable over (t_L, t_R), which simply means that $Y(t)$, $Y'(t)$, $\cdots$, $Y^{(n)}(t)$ are defined for every t in (t_L, t_R);
- the coefficients $a_1(t)$, $a_2(t)$, $\cdots$, $a_n(t)$, and the forcing $f(t)$ are defined for every t in (t_L, t_R);
- the equation

  $$Y^{(n)}(t) + a_1(t)Y^{(n-1)}(t) + \cdots + a_{n-1}(t)Y'(t) + a_n(t)Y(t) = f(t)$$

  is satisfied for every t in (t_L, t_R).

The first two bullets simply say that every term appearing in the equation is defined over the interval (t_L, t_R), while the third says the equation is satisfied at each time t in (t_L, t_R).

Remark. Recall from calculus that if a function has a derivative at a point then it must be continuous at that point. Our definition requires every solution $y = Y(t)$ of (1.1) to be n-times differentiable, which means that Y has derivatives Y', $\cdots$, $Y^{(n)}$ at every point in (t_L, t_R). Therefore Y, $\cdots$, $Y^{(n-1)}$ must be continuous at every point in (t_L, t_R). In other words, our definition requires every solution of (1.1) to have at least $n-1$ continuous derivatives over (t_L, t_R).

1.2. **Initial-Value Problems.**

An *initial-value problem* associated with (1.1) seeks a solution $y = Y(t)$ of (1.1) that also satisfies the *initial conditions*

$$Y(t_I) = y_0\,, \qquad Y'(t_I) = y_1\,, \qquad \cdots \qquad Y^{(n-1)}(t_I) = y_{n-1}\,, \tag{1.2}$$

for some *initial time* (or *initial point*) t_I and *initial data* (or *initial values*) $y_0, y_1, \cdots, y_{n-1}$. We will use the following basic existence and uniqueness theorem about initial-value problems, which we state without proof.

Theorem 1.1 (Basic Existence and Uniqueness Theorem). Suppose that the coefficients $a_1(t), a_2(t), \cdots, a_n(t)$ and the forcing $f(t)$ are all continuous functions over an interval (t_L, t_R). Then given any initial time t_I in (t_L, t_R) and any initial data $y_0, y_1, \cdots, y_{n-1}$ there exists a unique solution $y = Y(t)$ of the differential equation (1.1) that also satisfies the initial conditions (1.2). This solution has at least n continuous derivatives over (t_L, t_R).

Moreover, if the functions $a_1(t), a_2(t), \cdots, a_n(t)$, and $f(t)$ all have m continuous derivatives over (t_L, t_R) then this solution has at least $m+n$ continuous derivatives over (t_L, t_R).

Remark. The solutions asserted by this theorem have at least n continuous derivatives over (t_L, t_R), which is one more than was required by our definition of solution. This is because it assumes that the coefficients $a_1(t), a_2(t), \cdots, a_n(t)$, and forcing $f(t)$ are all continuous over (t_L, t_R), rather than assuming they are merely defined over (t_L, t_R) as was done in Definition 1.1 of what it means to be a solution. Moreover, it asserts that the more derivatives the coeffecients and forcing all have, the more deriviatives the solutions will have. In particular, it asserts that if the coeffecients and forcing all have derivatives of all orders then every solution does too.

Remark. For first-order linear equations ($n = 1$) this theorem was essentially proved when we showed that the unique solution of the initial-value problem

$$\frac{dy}{dt} + a(t)y = f(t)\,, \qquad Y(t_I) = y_0\,,$$

is given by the formula

$$Y(t) = \exp\left(-\int_{t_I}^{t} a(r)\,\mathrm{d}r\right)\left[y_0 + \int_{t_I}^{t} \exp\left(\int_{t_I}^{s} a(r)\,\mathrm{d}r\right)\, f(s)\,\mathrm{d}s\right]. \tag{1.3}$$

Because there is no such general formula for the solution of the initial-value problem when $n \geq 2$, the proof of this theorem for higher-order equations requires methods that are beyond the scope of this course.

Remark. In this chapter we will see that for special choices of coefficients we can construct explicit formulas for the solution of the initial-value problem when $n \geq 2$. Even in such cases we will use this theorem to assert the uniqueness of the solution.

Remark. This theorem states the "counting fact" that solutions of any n^{th}-order linear equation are uniquely specified by n additional pieces of information — specifically, the values of the solution Y and its first $n-1$ derivatives at an initial time t_I. It is natural to ask whether we have a similar result if we replace the n initial conditions (1.2) with any n conditions on Y. For example, can we use n conditions that specify the values of Y and some of its derivatives at more than one time t? Such a problem is a so-called boundary-value problem. In general solutions to such problems either may not exist or may not be unique. Therefore in this course we will focus on initial-value problems, which are simpler. Boundary-value problems are very important and are studied in more advanced courses.

The Basic Existence and Uniqueness Theorem (Theorem 1.1) can be used to argue that certain functions cannot be a solution of a given homogeneous linear ordinary differential equations of a given order. This is usually argued by contradition.

Example. Show that $\sin(t^3)$ cannot be a solution of any equation of the form

$$z''' + a_1(t)z'' + a_2(t)z' + a_3(t)z = 0\,,$$

where $a_1(t)$, $a_2(t)$, and $a_3(t)$, are continuous over an open interval containing 0.

Solution. Suppose otherwise — namely, suppose that $Z(t) = \sin(t^3)$ satisfies such an equation. Because

$$Z'(t) = 3t^2 \cos(t^3)\,, \qquad Z''(t) = 6t\cos(t^3) - 9t^4 \sin(t^3)\,.$$

we see that $Z(t)$ satisfies the equation and the initial conditions

$$Z(0) = Z'(0) = Z''(0) = 0\,.$$

However the Basic Existence and Uniqueness Theorem implies that $Z(t) = 0$ is the only solution of the equation that satisfies these initial conditions, which contradicts the fact that $Z(t) = \sin(t^3)$. □

1.3. **Intervals of Definition.** Most higher-order linear differental equations in the form (1.1) do not have recipes for finding analytic solutions. However numerical and graphical methods can be applied to all such equations. When applying these methods it is helpful to know the interval of definition of a solution beforehand. Fortunately, the Basic Existence and Uniqueness Theorem (Theorem 1.1) can be used to identify the interval of definition for solutions of (1.1) without finding a solution analytically. This is done very much like we identified the interval of definition for solutions of first-order linear equations. Specifically, if $Y(t)$ is the solution of the initial-value problem in the normal form (1.1-1.2) then its interval of definition will be (t_L, t_R) whenever:

- the initial time t_I is in (t_L, t_R),
- all the coefficients and the forcing are continuous over (t_L, t_R),
- either a coefficient or the forcing is undefined at each of $t = t_L$ and $t = t_R$.

The first two bullets along with the Basic Existence and Uniqueness Theorem imply that the interval of definition will be at least (t_L, t_R). The last bullet along with our definition of a solution imply that the interval of definition can be no bigger than (t_L, t_R) because the equation is undefined at $t = t_L$ and $t = t_R$. This argument works when $t_L = -\infty$ or $t_R = \infty$.

Remark. This does not mean that every solution of (1.1-1.2) will become undefined at either $t = t_L$ or $t = t_R$ when those endpoints are finite.

Remark. Remember to put the differential equation into normal form before checking where the coefficients and forcing are defined and continuous.

Example. Give the interval of definition for the solution of the initial-value problem

$$x''' + \frac{1}{t^2 - 9}\,x' + \frac{1}{t^2 - 4}\,x = \cos(t)\,, \qquad x(1) = 3\,, \quad x'(1) = 0\,, \quad x''(1) = 0\,.$$

Solution. The problem is linear and is already in normal form. Notice the following.

◇ The coefficient of x' is undefined at $t = \pm 3$ and is continuous elsewhere.
◇ The coefficient of x is undefined at $t = \pm 2$ and is continuous elsewhere.
◇ The forcing is continuous everywhere.

Therefore the interval of definition is $(-2, 2)$ because:

- the initial time $t = 1$ is in $(-2, 2)$;
- all the coefficients and the forcing are continuous over $(-2, 2)$;
- the coefficient of x is undefined at $t = -2$ and $t = 2$. □

Example. Give the interval of definition for the solution of the initial-value problem

$$t^2 u''' + 2u = e^t \,, \qquad u(-1) = 3 \,, \quad u'(-1) = 0 \,, \quad u''(-1) = 0 \,.$$

Solution. The problem is linear. Its normal form is

$$u''' + \frac{2}{t^2}\, u = \frac{e^t}{t^2} \,, \qquad u(-1) = 3 \,, \quad u'(-1) = 0 \,, \quad u''(-1) = 0 \,.$$

Notice the following.

- ◇ The coefficient of u is undefined at $t = 0$ and is continuous elsewhere.
- ◇ The forcing is undefined at $t = 0$ and is continuous elsewhere.

Therefore the interval of definition is $(-\infty, 0)$ because:

- the initial time $t = -1$ is in $(-\infty, 0)$;
- all the coefficients and the forcing are continuous over $(-\infty, 0)$;
- the coefficient of u and the forcing are undefined at $t = 0$. □

Remark. The next two examples have the same differential equation but different initial times. This leads to different answers.

Example. Give the interval of definition for the solution of the initial-value problem

$$\frac{\mathrm{d}^4 v}{\mathrm{d}t^4} + \frac{1}{t-4}\frac{\mathrm{d}v}{\mathrm{d}t} = \frac{e^t}{2+t} \,, \qquad v(0) = v'(0) = v''(0) = v'''(0) = 0 \,.$$

Solution. The problem is linear and is already in normal form. Notice the following.

- ◇ The coefficient of v' is undefined at $t = 4$ and is continuous elsewhere.
- ◇ The forcing is undefined at $t = -2$ and is continuous elsewhere.

Therefore the interval of definition is $(-2, 4)$ because:

- the initial time $t = 0$ is in $(-2, 4)$;
- all the coefficients and the forcing are continuous over $(-2, 4)$;
- the forcing is undefined at $t = -2$;
- the coefficient of v' is undefined at $t = 4$. □

Example. Give the interval of definition for the solution of the initial-value problem

$$\frac{\mathrm{d}^4 v}{\mathrm{d}t^4} + \frac{1}{t-4}\frac{\mathrm{d}v}{\mathrm{d}t} = \frac{e^t}{2+t} \,, \qquad v(6) = v'(6) = v''(6) = v'''(6) = 0 \,.$$

Solution. The problem is linear and is already in normal form. Notice the following.

- ◇ The coefficient of v' is undefined at $t = 4$ and is continuous elsewhere.
- ◇ The forcing is undefined at $t = -2$ and is continuous elsewhere.

Therefore the interval of definition is $(4, \infty)$ because:

- the initial time $t = 6$ is in $(4, \infty)$;
- all the coefficients and the forcing are continuous over $(4, \infty)$;
- the coefficient of v' is undefined at $t = 4$. □

1.4. **Overview of Higher-Order Linear Equations.** Linear equations were the simplest class of first-order equations that we have studied. In Part I, Chapter 2 we studied:

1. a recipe for general solutions of homogeneous linear equations that required finding one primitive,
2. a recipe for general solutions of nonhomogeneous linear equations that required finding two primitives,
3. an existence and uniqueness theory for initial-value problems that laid the groundwork for the graphical and numerical methods we studied later.

Linear equations are also the simplest class of higher-order equations. Their analogous existence and uniqueness theory for initial-value problems was just given. However, there are no perfect analogs of the general solution recipes that we found for first-order linear equations. In the subsequent chapters we develop analytic methods for constructing general solutions of special higher-order linear equations. We use of some of the graphical methods that were introduced for first-order equations.

We begin by studying an **analytic method** that allows us to construct a general solution of an n^{th}-order *homogeneous* linear equation from a sufficiently nice set of n solutions, called a *fundamental set of solutions.*

We need some facts about *linear algebraic systems* and *determinants* to characterize when a set of n solutions is fundamental.

We then give a recipe to construct a fundamental set of solutions for higher-order homogeneous linear equations with *constant coefficients.*

Next, we study an **analytic method** that allows us to construct a general solution of an n^{th}-order *nonhomogeneous* linear equation from a *particular solution* of that equation and a general solution of the *associated homogeneous equation.*

We then study three **analytic methods** for constructing a particular solution for higher-order nonhomogeneous linear equations with *constant coefficients.* The first two methods yield an explicit solution but require the forcing to have a very special form. The third method can be applied to a general forcing but requires finding n primitives for an n^{th}-order equation.

Next, we study two **analytic methods** for constructing a particular solution for higher-order nonhomogeneous linear equations with *variable coefficients* from a fundamental set of solutions to the associated homogeneous equation. These methods can be applied to a general forcing but require finding n primitives for an n^{th}-order equation. (For $n = 1$ these methods reduce to the recipe for solving nonhomogeneous linear first-order equations that we studied in Part I, Chapter 2.)

We then **apply** some of the foregoing analytical and graphical methods to study mathematical models of spring-mass systems. The new ingredient here is learning the connection between the physical system and a second-order differential equation.

Finally, we study an **analytic method** called the *Laplace transform method* that allows efficient solution of initial-value problems for higher-order nonhomogeneous linear equations with *constant coefficients* and a forcing with jumps.

Exercises on Higher-Order Linear Equations

(1) Put the differential equation
$$\frac{t}{e^t+1}\, y'''(t) + 3ty'(t) = \cos(t)$$
into normal form. What is its order?

Solution

For #2 – #7, state the largest interval on which a solution to the given initial value problem is determined by the conditions given.

(2) $y'' + 5y' + 3y = e^t$, $y(0) = 0$, $y'(0) = 1$.

Short Answer
Solution

(3) $w\ddot{z} + 5\dot{z} + 3z = e^w$, $z(1) = 0$, $\dot{z}(1) = 1$.

Short Answer
Solution

(4) $y''' + \frac{1}{t}\, y'' + \tan(t)y' + y = \frac{t}{t^2-4}$, $y(3) = 0$, $y'(3) = -1$, $y''(3) = 2\pi$.

Short Answer
Solution

(5) $(x^2 - 1)\ddot{y} + \frac{y}{x-3} = e^x \cos(x)$, $y(2) = \dot{y}(2) = \pi$.

Short Answer
Solution

(6) $y'' + a(t)y' + b(t) = 0$, $y(3) = y'(3) = 0$, where
$$a(t) = \begin{cases} 1 & t \le 2 \\ -1 & t > 2 \end{cases}$$
and
$$b(t) = \begin{cases} 0 & t < 0 \\ 1 & 0 \le t \le 5 \\ 2 & t > 5. \end{cases}$$

Short Answer
Solution

(7) $z'' + c(w)z = \sec(w)$, $z(1) = 1$, $z'(1) = -3$, where
$$c(w) = \begin{cases} w & , w < 0 \\ w^2 & , w > 0. \end{cases}$$

Short Answer
Solution

(8) (a) Can x^3 be the solution to a differential equation of the form $w''+aw'+bw = 0$, where a and b are real constants? Explain why or why not.
(b) What is special about the exponent 3 in part (a)? Suggest a generalization, and explain why it's correct.

Solution

(9) Show that $f(t) = e^t + e^{2t} + e^{3t}$ cannot be the solution of any differential equation of the form $\mathrm{D}^2 y + a\mathrm{D}y + by = c$, where a, b, and c are constants. [*Hint.* Show that if you plug in f into the equation, then the left side $(f'' + af' + bf)$ must be unbounded for large t, regardless of the choice of a or b.]

Solution

(10) (a) Check that, for any real constants a and b, the function $ae^t + be^{-t}$ is a solution to the differential equation $\mathrm{D}^2 y - y = 0$.
(b) Try to find values of a and b so that the function $ae^t + be^{-t}$ satisfies the initial conditions $y(0) = y'(0) = 0$, or prove it can't be done.
(c) Try to find values of a and b so that the function $ae^t + be^{-t}$ satisfies the initial conditions $y(0) = 1$, $y'(0) = -1$, or prove it can't be done. [*Hint.* Later we will see that in fact there's a way to satisfy *any* initial conditions by appropriately choosing values of a and b. So it can be done.]

Solution

(11) Put the differential equation

$$(u^2 + 4)x'''' + 2ux' + 2x = \sin(u)$$

into normal form. What is its order? Is it homogeneous or not?

Solution

(12) Put the differential equation

$$y''' + \frac{3}{x}y' + \frac{\cos(5x)}{4+x}y = \frac{e^x}{x-3}$$

into normal form. What is its order? Is it homogeneous or not?

Solution

(13) Find the largest interval $a < x < b$ on which the initial-value problem

$$\cos(x)y'' + \sin(x)y = x,\ \ y(1) = -1,\ \ y'(1) = 1.$$

can be guaranteed to have a solution.

Short Answer
Solution

(14) State the largest interval on which a solution to the given initial value problem is determined by the conditions given:

$(u^2 - 9)x'' + 3x = \log(|30 - 4u|)$, $x(5) = -3$, $x'(5) = 2$.

Short Answer
Solution

Navigation to Other Chapters

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