II. Higher-Order Linear Ordinary Differential Equations
2. Homogeneous Equations: General Methods and Theory

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2. Homogeneous Equations: General Methods and Theory

Before we examine the general case, we study the special case of homogeneous linear equations. These have the normal form

$$\frac{\mathrm{d}^n y}{\mathrm{d}t^n} + a_1(t)\frac{\mathrm{d}^{n-1}y}{\mathrm{d}t^{n-1}} + \cdots + a_{n-1}(t)\frac{\mathrm{d}y}{\mathrm{d}t} + a_n(t)y = 0\,. \tag{2.1}$$

We will assume throughout this chapter that the coefficients $a_1(t), a_2(t), \cdots, a_n(t)$ are continuous over an interval (t_L, t_R), so that Therorem 1.1 can be applied. We will learn how to construct any solution of equation (2.1) from certain sets of just n solutions. This construction depends heavily upon the linearity of equation (2.1). It reduces the problem of finding every solution of equation (2.1) to that of finding a so-called fundamental set of just n solutions.

2.1. Linear Differential Operators.

In order to see how the linearity of (2.1) enables the contruction of general solutions, we express it compactly as

$$\mathrm{L}(t)y = 0\,, \tag{2.2a}$$

where $\mathrm{L}(t)$ is the n^{th}-order differential *operator*

$$\mathrm{L}(t) = \mathrm{D}^n + a_1(t)\mathrm{D}^{n-1} + \cdots + a_{n-1}(t)\mathrm{D} + a_n(t)\,, \qquad \text{where} \quad \mathrm{D} = \frac{d}{dt}\,. \tag{2.2b}$$

The symbol $\mathrm{L}(t)y$ means the operator $\mathrm{L}(t)$ is applied to the n-times differentiable function y, which results in the linear combination of derivatives of y given by

$$\begin{aligned}\mathrm{L}(t)y &= \mathrm{D}^n y + a_1(t)\mathrm{D}^{n-1}y + \cdots + a_{n-1}(t)\mathrm{D}y + a_n(t)y \\ &= y^{(n)} + a_1(t)y^{(n-1)} + \cdots + a_{n-1}(t)y' + a_n(t)y\,.\end{aligned}$$

The word "operator" refers to the fact that the act of taking linear combinations of derivatives of a function is called an operation. The operator $\mathrm{L}(t)$ given by (2.2b) is simply the instructions for applying such an operation to any n-times differentiable function.

Remark. Here we indicate that the differential operator above might depend upon t through its coefficients by writing $\mathrm{L}(t)$. We will write L when we want to restrict to the case where the coefficients a_1, a_2, $\cdots$, a_n are all constants.

The differential operator $\mathrm{L}(t)$ defined by (2.2b) has an important property called *linearity.* (This is why we denote it with the letter L.) Here we explain this property.

Recall that the *general linear combination* of m functions $Y_1(t)$, $Y_2(t)$, $\cdots$, $Y_m(t)$ is the family

$$c_1Y_1(t) + c_2Y_2(t) + \cdots + c_mY_m(t)\,,$$

where c_1, c_2, $\cdots$, c_m are arbitrary constants that are called the *parameters* of the family.

We know from calculus that if $Y_1(t)$, $Y_2(t)$, $\cdots$, $Y_m(t)$ are differentiable then the derivative operator D has the property that

$$\mathrm{D}\big(c_1Y_1(t) + c_2Y_2(t) + \cdots + c_mY_m(t)\big) = c_1\mathrm{D}Y_1(t) + c_2\mathrm{D}Y_2(t) + \cdots + c_m\mathrm{D}Y_m(t)\,.$$

In other words, *the derivative of a linear combination of functions is the same linear combination of their derivatives.* Any such operator is said to be *linear.*

Other operators are linear. For example, if $Y_1(t)$, $Y_2(t)$, $\cdots$, $Y_m(t)$ are k-times differentiable then the k^{th} derivative operator D^k satisfies

$$\mathrm{D}^k\big(c_1Y_1(t)+c_2Y_2(t)+\cdots+c_mY_m(t)\big)=c_1\mathrm{D}^kY_1(t)+c_2\mathrm{D}^kY_2(t)+\cdots+c_m\mathrm{D}^kY_m(t)\,.$$

In other words, *the* k^{th} *derivative of a linear combination of functions is the same linear combination of their* k^{th} *derivatives.* Therefore the operator D^k is also linear. Clearly you are already familiar with linear operators, even if you have not called them that before.

If we multiply both sides of the last relation by the continuous function $a_{n-k}(t)$ then we obtain

$$\begin{aligned}&a_{n-k}(t)\mathrm{D}^k\big(c_1Y_1(t)+c_2Y_2(t)+\cdots+c_mY_m(t)\big)\\&\quad=c_1a_{n-k}(t)\mathrm{D}^kY_1(t)+c_2a_{n-k}(t)\mathrm{D}^kY_2(t)+\cdots+c_ma_{n-k}(t)\mathrm{D}^kY_m(t)\,.\end{aligned}$$

By summing the foregoing relations over k and using definition (2.2b) of $\mathrm{L}(t)$ we see that if $Y_1(t)$, $Y_2(t)$, $\cdots$, $Y_m(t)$ are n-times differentiable then the operator $\mathrm{L}(t)$ satisfies

$$\begin{aligned}&\mathrm{L}(t)\big(c_1Y_1(t)+c_2Y_2(t)+\cdots+c_mY_m(t)\big)\\&\quad=c_1\mathrm{L}(t)Y_1(t)+c_2\mathrm{L}(t)Y_2(t)+\cdots+c_m\mathrm{L}(t)Y_m(t)\,.\end{aligned}\tag{2.3}$$

In other words, $\mathrm{L}(t)$ *applied to a linear combination of functions is the same linear combination of* $\mathrm{L}(t)$ *applied to the functions.* Therefore the differential operator $\mathrm{L}(t)$ is a *linear operator.*

2.2. Method of Linear Superposition.

The linearity of $\mathrm{L}(t)$ can be used to construct general solutions of homogeneous linear equations. Indeed, we see from (2.3) that if each $Y_k(t)$ solves the homogeneous equation $\mathrm{L}(t)Y_k(t)=0$ then so does every linear combination of the $Y_k(t)$ — namely, for any values of the constants c_1, c_2, $\cdots$, c_m we have

$$\mathrm{L}(t)\big(c_1Y_1(t)+c_2Y_2(t)+\cdots+c_mY_m(t)\big)=0\,.$$

Therefore we have the following property of homogeneous equations.

Theorem 2.1 (Linear Superposition). If $Y_1(t)$ and $Y_2(t)$ are solutions of (2.1) then so is the linear combination

$$c_1Y_1(t)+c_2Y_2(t)\,,$$

for any values of the constants c_1 and c_2. More generally, if $Y_1(t)$, $Y_2(t)$, $\cdots$, $Y_m(t)$ are solutions of (2.1) then so is the linear combination

$$c_1Y_1(t)+c_2Y_2(t)+\cdots+c_mY_m(t)\,,$$

for any values of the constants c_1, c_2, $\cdots$, c_m.

Remark. This theorem states that any linear combination of solutions of (2.1) is also a solution of (2.1). It thereby provides a way to construct a whole family of solutions from a finite number of them.

2.2.1. *Application to Initial-Value Problems.* The initial-value problem associated with the n^{th}-order, homogeneous linear differential equation (2.2) is

$$\text{(2.4a)}\qquad \mathrm{L}(t)y = 0\,, \qquad y(t_I) = y_0\,, \quad y'(t_I) = y_1\,, \quad \cdots \quad y^{(n-1)}(t_I) = y_{n-1}\,,$$

where $\mathrm{L}(t)$ is the n^{th}-order differential *operator*

$$\text{(2.4b)}\qquad \mathrm{L}(t) = \mathrm{D}^n + a_1(t)\mathrm{D}^{n-1} + \cdots + a_{n-1}(t)\mathrm{D} + a_n(t)\,, \qquad \text{where} \quad \mathrm{D} = \frac{d}{dt}\,.$$

Now suppose that we know n "different" solutions, $Y_1(t)$, $Y_2(t)$, $\cdots$, $Y_n(t)$ of $\mathrm{L}(t)y = 0$. It is natural to ask if we can construct the solution of the initial-value problem (2.4a) as a linear combination of $Y_1(t)$, $Y_2(t)$, $\cdots$, $Y_n(t)$. Set

$$y(t) = c_1Y_1(t) + c_2Y_2(t) + \cdots + c_nY_n(t)\,.$$

By the superposition theorem this is a solution of $\mathrm{L}(t)y = 0$. We only have to check that values of c_1, c_2, $\cdots$, c_n can be found such that $y(t)$ satisfies the initial conditions in (2.4a) — namely, that

$$\text{(2.5)}\qquad \begin{aligned} y_0 &= y(t_I) = c_1Y_1(t_I) + c_2Y_2(t_I) + \cdots + c_nY_n(t_I)\,,\\ y_1 &= y'(t_I) = c_1Y_1'(t_I) + c_2Y_2'(t_I) + \cdots + c_nY_n'(t_I)\,,\\ &\ \ \vdots \qquad\qquad \vdots\\ y_{n-1} &= y^{(n-1)}(t_I) = c_1Y_1^{(n-1)}(t_I) + c_2Y_2^{(n-1)}(t_I) + \cdots + c_nY_n^{(n-1)}(t_I)\,. \end{aligned}$$

This is a system of n linear algebraic equations for the n unknowns c_1, c_2, $\cdots$, c_n. It seems likely that we can often solve this system. This is the so-called *method of linear superposition.* We illustrate it with the following examples.

Example. We can check that e^{2t} and e^{-t} are solutions of

$$y'' - y' - 2y = 0\,.$$

Find c_1 and c_2 such that $y(t) = c_1e^{2t} + c_2e^{-t}$ satifies the initial conditions

$$y(0) = 4\,, \qquad y'(0) = 2\,.$$

Solution. Because $y'(t) = c_12e^{2t} - c_2e^{-t}$, the initial conditions become

$$\begin{aligned} 4 &= y(0) = c_1e^0 + c_2e^0 = c_1 + c_2\,,\\ 2 &= y'(0) = 2c_1e^0 - c_2e^0 = 2c_1 - c_2\,. \end{aligned}$$

These can be solved to find $c_1 = c_2 = 2$. Hence, the solution of the initial-value problem is

$$y(t) = 2e^{2t} + 2e^{-t}\,.$$

Example. We can check that $\cos(2t)$ and $\sin(2t)$ are solutions of

$$h'' + 4h = 0\,.$$

Find c_1 and c_2 such that $h(t) = c_1\cos(2t) + c_2\sin(2t)$ satifies the initial conditions

$$h(0) = 3\,, \qquad h'(0) = -2\,.$$

Solution. Because $h'(t) = -2c_1 \sin(2t) + 2c_2 \cos(2t)$, the initial conditions become

$$3 = h(0) = c_1 \cos(0) + c_2 \sin(0) = c_1 \,,$$
$$-2 = h'(0) = -2c_1 \sin(0) + 2c_2 \cos(0) = 2c_2 \,.$$

These imply that $c_1 = 3$ and $c_2 = -1$. Hence, the solution of the initial-value problem is

$$h(t) = 3\cos(2t) - \sin(2t) \,.$$

Example. We can check that t and $t^2 - 1$ are solutions of

$$(1 + t^2)x'' - 2t\, x' + 2x = 0 \,.$$

Find c_1 and c_2 such that $x(t) = c_1 t + c_2(t^2 - 1)$ satifies the initial conditions

$$x(1) = 3 \,, \qquad x'(1) = 1 \,.$$

Solution. Because $x'(t) = c_1 + 2c_2 t$, the initial conditions become

$$3 = c_1 \,, \qquad 1 = c_1 + 2c_2 \,.$$

These can be solved to find

$$c_1 = 3 \,, \qquad c_2 = -1 \,.$$

Hence, the solution of the initial-value problem is given by

$$x(t) = 3t - (t^2 - 1) \,.$$

2.2.2. *General Initial Conditions.* We see that the method of linear superposition worked in each of the above examples. In each case we were able to find values of c_1 and c_2 such that the given initial conditions were satisfied. Were we just lucky or does the method work in general? In order to explore this question further, we will now consider initial-value problems in which the initial data y_0 and y_1 are left arbitrary rather than be given specific values. Such initial conditions are called *general initial conditions.*

Example. We can check that e^{2t} and e^{-t} are solutions of

$$y'' - y' - 2y = 0 \,.$$

Find c_1 and c_2 such that $y(t) = c_1 e^{2t} + c_2 e^{-t}$ satifies the general initial conditions

$$y(0) = y_0 \,, \qquad y'(0) = y_1 \,.$$

Solution. Because $y'(t) = c_1 2e^{2t} - c_2 e^{-t}$, the initial conditions become

$$y_0 = c_1 + c_2 \,, \qquad y_1 = 2c_1 - c_2 \,.$$

These can be solved to find

$$c_1 = \frac{y_0 + y_1}{3} \,, \qquad c_2 = \frac{2y_0 - y_1}{3} \,.$$

Hence, the solution of the initial-value problem for arbitrary y_0 and y_1 is given by

$$y(t) = \frac{y_0 + y_1}{3}\, e^{2t} + \frac{2y_0 - y_1}{3}\, e^{-t} \,.$$

Example. We can check that $\cos(2t)$ and $\sin(2t)$ are solutions of

$$h'' + 4h = 0 \,.$$

Find c_1 and c_2 such that $h(t) = c_1 \cos(2t) + c_2 \sin(2t)$ satifies the general initial conditions

$$h(0) = h_0\,, \qquad h'(0) = h_1\,.$$

Solution. Because $h'(t) = -2c_1 \sin(2t) + 2c_2 \cos(2t)$, the initial conditions become

$$h_0 = c_1\,, \qquad h_1 = 2c_2\,.$$

These can be easily solved to find

$$c_1 = h_0\,, \qquad c_2 = \frac{h_1}{2}\,.$$

Hence, the solution of the initial-value problem for arbitrary h_0 and h_1 is given by

$$h(t) = h_0 \cos(2t) + h_1 \frac{\sin(2t)}{2}\,.$$

Example. We can check that t and $t^2 - 1$ are solutions of

$$(1 + t^2)x'' - 2t\,x' + 2x = 0\,.$$

Find c_1 and c_2 such that $x(t) = c_1 t + c_2(t^2 - 1)$ satifies the general initial conditions

$$x(1) = x_0\,, \qquad x'(1) = x_1\,.$$

Solution. Because $x'(t) = c_1 + 2c_2 t$, the initial conditions become

$$x_0 = c_1\,, \qquad x_1 = c_1 + 2c_2\,.$$

These can be solved to find

$$c_1 = x_0\,, \qquad c_2 = \frac{x_1 - x_0}{2}\,.$$

Hence, the solution of the initial-value problem for arbitrary x_0 and x_1 is given by

$$x(t) = x_0 t + \frac{x_1 - x_0}{2}(t^2 - 1)\,.$$

The method can be applied to homogeneous equations of any order n, but the work needed to solve for the constants $c_1, \cdots, c_n$ is greater for larger n.

Example. We can check that e^{4t}, e^{3t}, and e^{-t} are solutions of

$$v''' - 6v'' + 5v' + 12v = 0\,.$$

Find c_1, c_2, and c_3 such that $v(t) = c_1 e^{4t} + c_2 e^{3t} + c_3 e^{-t}$ satifies the initial conditions

$$v(0) = v_0\,, \qquad v'(0) = v_1\,, \qquad v''(0) = v_2\,.$$

Solution. Because

$$\begin{aligned} v'(t) &= c_1 4e^{4t} + c_2 3e^{3t} - c_3 e^{-t}\,, \\ v''(t) &= c_1 16e^{4t} + c_2 9e^{3t} + c_3 e^{-t}\,, \end{aligned}$$

the initial conditions become

$$\begin{aligned} v_0 &= c_1 + c_2 + c_3\,, \\ v_1 &= 4c_1 + 3c_2 - c_3\,, \\ v_2 &= 16c_1 + 9c_2 + c_3\,. \end{aligned}$$

This is a system of three linear algebraic equations that needs to be solved. There are many methods that can be used to solve these equations. By using any of them we find that

$$c_1 = \frac{-3v_0 - 2v_1 + v_2}{5}\,, \qquad c_2 = \frac{4v_0 + 3v_1 - v_2}{4}\,, \qquad c_3 = \frac{12v_0 - 7v_1 + v_2}{20}\,.$$

Hence, the solution of the initial-value problem for arbitrary v_0, v_1, and v_2 is given by

$$v(t) = \frac{-3v_0 - 2v_1 + v_2}{5}\, e^{4t} + \frac{4v_0 + 3v_1 - v_2}{4}\, e^{3t} + \frac{12v_0 - 7v_1 + v_2}{20}\, e^{-t}\,.$$

Remark. In Figure 2.1 we compare the solutions e^{4t}, e^{3t}, e^{-t} to the solution $v(t)$ of the initial-value problem for $v_0 = 1$, $v_1 = 10$, and $v_2 = -10$, which is

$$v(t) = -\tfrac{33}{5}\, e^{4t} + 11\, e^{3t} - \tfrac{17}{5}\, e^{-t}\,.$$

For large t the dominant term in $v(t)$ is $-\frac{33}{5}\, e^{4t}$, which is why $Y(t)$ will decrease to $-\infty$ as $t \to \infty$.

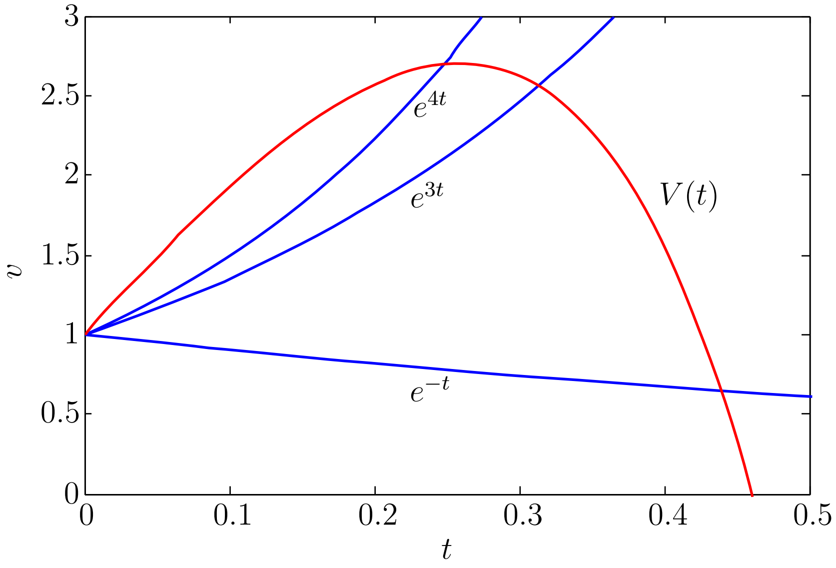

FIGURE 2.1. Plot comparing the solutions e^{4t}, e^{3t}, e^{-t} to the solution $V(t)$ of the IVP for $v_0 = 1$, $v_1 = 10$, and $v_2 = -10$.

2.3. **Wronskians.** We now apply the material in Chapter 3 on linear algebraic systems to understand when and why the method of linear superposition works. Theorem 3.1 of Chapter 3 states that the linear algebraic system (2.5) will have a unique solution for any choice of initial data y_0, y_1, $\cdots$, y_{n-1} if and only if

$$\det\begin{pmatrix} Y_1(t_I) & Y_2(t_I) & \cdots & Y_n(t_I) \\ Y_1'(t_I) & Y_2'(t_I) & \cdots & Y_n'(t_I) \\ \vdots & \vdots & \ddots & \vdots \\ Y_1^{(n-1)}(t_I) & Y_2^{(n-1)}(t_I) & \cdots & Y_n^{(n-1)}(t_I) \end{pmatrix} \neq 0\,. \tag{2.6}$$

Therefore this condition must be satisfied for the method of linear superposition to work generally. In this section we explore this condition further. We begin with a definition.

Definition. Given any n functions $Y_1, Y_2, \cdots, Y_n$ that are $n-1$ times differentiable over an interval (t_L, t_R), a new function $\mathrm{Wr}[Y_1, Y_2, \cdots, Y_n]$, called the *Wronskian* of $Y_1, Y_2, \cdots, Y_n$, is defined for every t in (t_L, t_R) by

$$\mathrm{Wr}[Y_1, Y_2, \cdots, Y_n](t) = \det\begin{pmatrix} Y_1(t) & Y_2(t) & \cdots & Y_n(t) \\ Y_1'(t) & Y_2'(t) & \cdots & Y_n'(t) \\ \vdots & \vdots & \ddots & \vdots \\ Y_1^{(n-1)}(t) & Y_2^{(n-1)}(t) & \cdots & Y_n^{(n-1)}(t) \end{pmatrix}. \tag{2.7}$$

Condition (2.6) then can be recast as simply

$$\mathrm{Wr}[Y_1, Y_2, \cdots, Y_n](t_I) \neq 0\,. \tag{2.8}$$

Let us compute the Wronskians for each set of functions that appeared in the previous examples and check this condition.

Example. Let $Y_1(t) = e^{2t}$ and $Y_2(t) = e^{-t}$. Their Wronskian is

$$\begin{aligned} \mathrm{Wr}[Y_1, Y_2](t) &= \det\begin{pmatrix} Y_1(t) & Y_2(t) \\ Y_1'(t) & Y_2'(t) \end{pmatrix} = \det\begin{pmatrix} e^{2t} & e^{-t} \\ 2e^{2t} & -e^{-t} \end{pmatrix} \\ &= -e^{2t}e^{-t} - 2e^{2t}e^{-t} = -3e^{t}\,. \end{aligned}$$

Because $\mathrm{Wr}[Y_1, Y_2](t) \neq 0$ for every t, condition (2.8) holds for any inital time t_I.

Example. Let $H_1(t) = \cos(2t)$ and $H_2(t) = \sin(2t)$. Their Wronskian is

$$\begin{aligned} \mathrm{Wr}[H_1, H_2](t) &= \det\begin{pmatrix} H_1(t) & H_2(t) \\ H_1'(t) & H_2'(t) \end{pmatrix} = \det\begin{pmatrix} \cos(2t) & \sin(2t) \\ -2\sin(2t) & 2\cos(2t) \end{pmatrix} \\ &= 2\cos(2t)^2 + 2\sin(2t)^2 = 2\,. \end{aligned}$$

Because $\mathrm{Wr}[H_1, H_2](t) \neq 0$ for every t, condition (2.8) holds for any inital time t_I.

Example. Let $X_1(t) = t$ and $X_2(t) = t^2 - 1$. Their Wronskian is

$$\mathrm{Wr}[X_1, X_2](t) = \det\begin{pmatrix} X_1(t) & X_2(t) \\ X_1'(t) & X_2'(t) \end{pmatrix} = \det\begin{pmatrix} t & t^2-1 \\ 1 & 2t \end{pmatrix} = 2t^2 - (t^2 - 1) = t^2 + 1\,.$$

Because $\mathrm{Wr}[X_1, X_2](t) \neq 0$ for every t, condition (2.8) holds for any inital time t_I.

Example. Let $V_1(t) = e^{4t}$, $V_2(t) = e^{3t}$, and $V_3(t) = e^{-t}$. Their Wronskian is

$$\begin{aligned}
\mathrm{Wr}[V_1, V_2, V_3](t) = \det\begin{pmatrix} V_1(t) & V_2(t) & V_3(t) \\ V_1'(t) & V_2'(t) & V_3'(t) \\ V_1''(t) & V_2''(t) & V_3''(t) \end{pmatrix} &= \det\begin{pmatrix} e^{4t} & e^{3t} & e^{-t} \\ 4e^{4t} & 3e^{3t} & -e^{-t} \\ 16e^{4t} & 9e^{3t} & e^{-t} \end{pmatrix} \\
&= 3e^{4t}e^{3t}e^{-t} - 16e^{4t}e^{3t}e^{-t} + 36e^{4t}e^{3t}e^{-t} \\
&\quad - 48e^{4t}e^{3t}e^{-t} + 9e^{4t}e^{3t}e^{-t} - 4e^{4t}e^{3t}e^{-t} \\
&= (3 - 16 + 36 - 48 + 9 - 4)e^{6t} = -20e^{6t}\,.
\end{aligned}$$

Because $\mathrm{Wr}[V_1, V_2, V_3](t) \neq 0$ for every t, condition (2.8) holds for any inital time t_I.

Because the Wronskians in the above examples were nonzero for every t, condition (2.8) holds for any inital time t_I. It is natural to ask if this reflects a general fact. The next result will allow us to show that this is indeed the case.

Theorem 2.2 (Abel's Wronskian Theorem). If $Y_1, Y_2, \cdots, Y_n$ are n solutions of the n^{th}-order, homogeneous equation

$$y^{(n)} + a_1(t)y^{(n-1)} + \cdots + a_{n-1}(t)y' + a_n(t)y = 0\,, \tag{2.9}$$

then $w = \mathrm{Wr}[Y_1, Y_2, \cdots, Y_n](t)$ solves the first-order, homogeneous linear equation

$$w' + a_1(t)w = 0\,. \tag{2.10}$$

Proof for the Second-Order Case. We will not give a proof of Abel's Wronskian Theorem in its general setting because it requires using properties of $n\times n$ determinants that are not covered in this course. However we will give a proof for the second-order case, which is the case encountered most often in this course.

Let Y_1 and Y_2 be two solutions of the second-order homogeneous linear equation

$$y'' + a_1(t)y' + a_2(t)y = 0\,.$$

Their Wronskian is given by

$$\mathrm{Wr}[Y_1, Y_2](t) = \det\begin{pmatrix} Y_1(t) & Y_2(t) \\ Y_1'(t) & Y_2'(t) \end{pmatrix} = Y_1(t)Y_2'(t) - Y_1'(t)Y_2(t)\,.$$

Differentiating this formula and then using the differential equation to eliminate $Y_1''(t)$ and $Y_2''(t)$ yields

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{Wr}[Y_1, Y_2](t) &= \frac{\mathrm{d}}{\mathrm{d}t}\big(Y_1(t)Y_2'(t) - Y_1'(t)Y_2(t)\big) \\
&= Y_1'(t)Y_2'(t) + Y_1(t)Y_2''(t) - Y_1''(t)Y_2(t) - Y_1'(t)Y_2'(t) \\
&= Y_1(t)Y_2''(t) - Y_1''(t)Y_2(t) \\
&= Y_1(t)\big(-a_1(t)Y_2'(t) - a_2(t)Y_2(t)\big) - \big(-a_1(t)Y_1'(t) - a_2(t)Y_1(t)\big)Y_2(t) \\
&= -a_1(t)\big(Y_1(t)Y_2'(t) - Y_1'(t)Y_2(t)\big) - a_2(t)\big(Y_1(t)Y_2(t) - Y_1(t)Y_2(t)\big) \\
&= -a_1(t)\mathrm{Wr}[Y_1, Y_2](t)\,,
\end{aligned}$$

which is equivalent to the first-order equation (2.10) asserted by the theorem. □

Exercise. Give a proof of Abel's Wronskian Theorem for the third-order case along the lines of the one given above for the second-order case.

Remark. Equation (2.10) is easy to remember. Its coefficient $a_1(t)$ is the coefficient of the $(n-1)^{\text{th}}$-order derivative of y in the n^{th}-order equation (2.9) satisfied by y.

Example. Consider the homogeneous linear equation

$$x''' + \frac{\cos(3t)}{t+2}\, x' + \frac{e^t}{t-5}\, x = 0\,.$$

Let $X_1(t)$, $X_2(t)$, and $X_3(t)$ be solutions of this equation over the interval $(-2,5)$. Because this equation is third-order and the coefficient of x'' is zero, the Abel Theorem says that $w = \mathrm{Wr}[X_1, X_2, X_3](t)$ satisfies

$$w' + 0\, w = 0\,.$$

In other words, $\mathrm{Wr}[X_1, X_2, X_3](t)$ is constant over $(-2,5)$.

Example. Consider the homogeneous linear equation

$$v'''' - 3v''' + t^3 v'' + e^t v' - \sin(5t)v = 0\,.$$

Let $V_1(t)$, $V_2(t)$, $V_3(t)$, and $V_4(t)$ be solutions of this equation over $(-\infty,\infty)$. Because this is a four-order equation and the coefficient of v''' is -3, the Abel Theorem says that $w = \mathrm{Wr}[V_1, V_2, V_3, V_4](t)$ satisfies

$$w' - 3\, w = 0\,.$$

Therefore $\mathrm{Wr}[X_1, X_2, X_3](t) = ce^{3t}$ over $(-\infty,\infty)$ for some number c.

Now let $Y_1(t)$, $Y_2(t)$, $\cdots$, $Y_n(t)$ be solutions of the n^{th}-order homogeneous linear equation (2.9) over an interval $(t_{\text{L}}, t_{\text{R}})$ and let $\mathrm{Wr}[Y_1, Y_2, \cdots, Y_n](t)$ be their Wronskian. If we know $\mathrm{Wr}[Y_1, Y_2, \cdots, Y_n](t_I)$ at some time t_I in the interval $(t_{\text{L}}, t_{\text{R}})$ then we can use the Abel Theorem to determine $\mathrm{Wr}[Y_1, Y_2, \cdots, Y_n](t)$ for every t in $(t_{\text{L}}, t_{\text{R}})$. Indeed, the Abel Theorem implies that $w = \mathrm{Wr}[Y_1, Y_2, \cdots, Y_n](t)$ satisfies the initial-value problem

$$w' + a_1(t)w = 0\,, \qquad w(t_I) = \mathrm{Wr}[Y_1, Y_2, \cdots, Y_n](t_I)\,.$$

By solving this initial-value problem we obtain the relation

$$\mathrm{Wr}[Y_1, Y_2, \cdots, Y_n](t) = \mathrm{Wr}[Y_1, Y_2, \cdots, Y_n](t_I)\, \exp\!\left(-\int_{t_I}^{t} a_1(r)\, \mathrm{d}r\right). \tag{2.11}$$

An important consequence of this relation is that the Wronskian of n solutions of (2.9) is either *always zero* or *never zero*.

Theorem 2.3. If $Y_1, Y_2, \cdots, Y_n$ are solutions of (2.9) over an interval (t_L, t_R) then their Wronskian $\mathrm{Wr}[Y_1, Y_2, \cdots, Y_n]$ is either *zero everywhere* in (t_L, t_R) or *zero nowhere* in (t_L, t_R).

Proof. Suppose that $\mathrm{Wr}[Y_1, Y_2, \cdots, Y_n](t_I) = 0$ for some t_I in (t_L, t_R). Then relation (2.11) immediately implies that $\mathrm{Wr}[Y_1, Y_2, \cdots, Y_n](t) = 0$ everywhere in (t_L, t_R). On the other hand, suppose that $\mathrm{Wr}[Y_1, Y_2, \cdots, Y_n](t_I) \neq 0$ for some t_I in (t_L, t_R). Then because the exponential factor in relation (2.11) is always positive, that relation implies that $\mathrm{Wr}[Y_1, Y_2, \cdots, Y_n](t) \neq 0$ everywhere in (t_L, t_R). $\square$

2.4. **Fundamental Sets of Solutions and General Solutions.** Theorem 2.3 shows us that either condition (2.8) holds *everywhere* $Y_1, Y_2, \cdots, Y_n$ are defined, or it holds *nowhere*. This means that when the Wronskian $\mathrm{Wr}[Y_1, Y_2, \cdots, Y_n]$ is nonzero we can always find the unique solution of any initial-value problem for any initial time t_I and any initial data y_0, y_1, $\cdots$, y_n by the method of linear superposition. This fact motivates the following definition.

> **Definition.** A set of n solutions of an n^{th}-order homogeneous linear ordinary differential equation is said to be *fundamental* if its Wronskian is nonzero.

The importance of this concept is evident in the following.

> **Theorem 2.4.** Let Y_1, Y_2, $\cdots$, Y_n be a fundamental set of solutions of equation (2.9) over the interval (t_L, t_R). Then every solution of (2.9) over the interval (t_L, t_R) can be expressed as a unique linear combination of Y_1, Y_2, $\cdots$, Y_n that can be determined by the method of linear superposition.

Proof. Let $Y(t)$ be any solution of (2.9) over (t_L, t_R). Consider the n-parameter family

$$y(t) = c_1 Y_1(t) + c_2 Y_2(t) + \cdots + c_n Y_n(t)\,. \tag{2.12}$$

We claim that there is a unique choice of c_1, c_2, $\cdots$, c_n such that $Y(t) = y(t)$. Let t_I be any time in (t_L, t_R). We will pick c_1, c_2, $\cdots$, c_n so that

$$Y(t_I) = y(t_I)\,, \qquad Y'(t_I) = y'(t_I)\,, \qquad \cdots \qquad , Y^{(n-1)}(t_I) = y^{(n-1)}(t_I)\,.$$

By (2.12) this means that c_1, c_2, $\cdots$, c_n satisfy the linear algebraic system

$$\begin{array}{rcccccccc}
Y(t_I) = & y(t_I) & = & c_1 Y_1(t_I) & + & c_2 Y_2(t_I) & + \cdots + & c_n Y_n(t_I)\,, \\
Y'(t_I) = & y'(t_I) & = & c_1 Y_1'(t_I) & + & c_2 Y_2'(t_I) & + \cdots + & c_n Y_n'(t_I)\,, \\
\vdots & \vdots & & & & & & \\
Y^{n-1}(t_I) = & y^{(n-1)}(t_I) & = & c_1 Y_1^{(n-1)}(t_I) & + & c_2 Y_2^{(n-1)}(t_I) & + \cdots + & c_n Y_n^{(n-1)}(t_I)\,.
\end{array}$$

We know that $\mathrm{Wr}[Y_1, Y_2, \cdots, Y_n](t_I) \neq 0$ because Y_1, Y_2, $\cdots$, Y_n is a fundamental set of solutions. Hence, this linear algebraic system has a unique solution c_1, c_2, $\cdots$, c_n. Because $Y(t)$ and $y(t)$ for these values of c_1, c_2, $\cdots$, c_n are solutions of (2.9) with the same initial values at t_I, we conclude that $Y(t) = y(t)$ by the uniqueness assertion of Theorem 1.1. □

Theorem 2.4 motivates the following definition.

> **Definition.** If Y_1, Y_2, $\cdots$, Y_n is a fundamental set of solutions of an n^{th}-order homogeneous linear ordinary differential equation then the n-parameter family (2.12) is called a *general solution* of the equation.

Example. In Section 2.2 we saw that $Y_1(t) = e^{2t}$ and $Y_2(t) = e^{-t}$ are solutions of

$$y'' - y' - 2y = 0\,.$$

Earlier in this section we saw that their Wronskian is

$$\mathrm{Wr}[Y_1, Y_2](t) = -3e^{t} \neq 0\,.$$

Therefore they are a fundamental set of solutions and a general solution is

$$y = c_1 e^{2t} + c_2 e^{-t}\,.$$

Example. In Section 2.2 we saw that $H_1(t) = \cos(2t)$ and $H_2(t) = \sin(2t)$ are solutions of

$$h'' + 4h = 0\,.$$

Earlier in this section we saw that their Wronskian is

$$\mathrm{Wr}[H_1, H_2](t) = 2 \neq 0\,.$$

Therefore they are a fundamental set of solutions and a general solution is

$$h = c_1 \cos(2t) + c_2 \sin(2t)\,.$$

Example. In Section 2.2 we saw that $X_1(t) = t$ and $X_2(t) = t^2 - 1$ are solutions of

$$(1 + t^2)x'' - 2t\,x' + 2x = 0\,.$$

Earlier in this section we saw that their Wronskian is

$$\mathrm{Wr}[X_1, X_2](t) = t^2 + 1 \neq 0\,.$$

Therefore they are a fundamental set of solutions and a general solution is

$$x = c_1 t + c_2(t^2 - 1)\,.$$

Example. In Section 2.2 we saw that $V_1(t) = e^{4t}$, $V_2(t) = e^{3t}$, and $V_3(t) = e^{-t}$ are solutions of

$$v''' - 6v'' + 5v' + 12v = 0\,.$$

Earlier in this section we saw that their Wronskian is

$$\mathrm{Wr}[V_1, V_2, V_3](t) = -20e^{6t} \neq 0\,.$$

Therefore they are a fundamental set of solutions and a general solution is

$$v = c_1 e^{4t} + c_2 e^{3t} + c_3 e^{-t}\,.$$

Remark. Notice that all of the Wronskians computed in the four examples above are consistent with what we would expect from the Abel Theorem. In the first example the equation was

$$y'' - y' - 2y = 0\,,$$

so the Abel Theorem says that $w(t) = \mathrm{Wr}[Y_1, Y_2](t)$ should satisfy

$$w' - w = 0\,.$$

We obtained $\mathrm{Wr}[Y_1, Y_2](t) = -3e^t$, which is a solution of this equation. The subsequent examples can be checked similarly. In this manner the Abel Theorem provides a check on the computation of Wronskians.

2.5. **Natural Fundamental Sets of Solutions.** In the last section we saw how to use the Wronskian to identify a fundamental set of solutions Y_1, Y_1, $\cdots$, Y_n, to an n^{th}-order homogeneous linear ordinary differential equation in the normal form

$$y^{(n)} + a_1(t)y^{(n-1)} + \cdots + a_{n-1}(t)y' + a_n(t)y = 0\,, \tag{2.13}$$

Given any initial time t_I, in Section 2.2 we saw how to use any such fundamental set to construct solutions of (2.13) that satisfy the general initial conditions

$$y(t_I) = y_0\,, \quad y'(t_I) = y_1\,, \quad y''(t_I) = y_2\,, \quad \cdots \quad y^{(n-1)}(t_I) = y_{n-1}\,, \tag{2.14}$$

where the initial-values y_0, y_1, $\cdots$, y_{n-1} are arbitrary rather than specific numbers. These solutions always have the form

$$y = N_0(t)y_0 + N_1(t)y_1 + N_2(t)y_2 + \cdots + N_{n-1}(t)y_{n-1}\,. \tag{2.15}$$

Therefore the functions $N_0(t)$, $N_1(t)$, $\cdots$, $N_{n-1}(t)$, are solutions of (2.13) such that for each $j = 0, 1, \cdots, n-1$ the function $N_j(t)$ satisfies the initial conditions

$$N_j^{(k)}(t_I) = \delta_{jk} \quad \text{for every } k = 0, 1, \cdots, n-1\,. \tag{2.16}$$

Here δ_{jk} denotes the Kronecker delta, which is defined by

$$\delta_{jk} = \begin{cases} 1 & \text{if } j = k\,, \\ 0 & \text{otherwise}\,. \end{cases}$$

The Wronskian of $N_0(t)$, $N_1(t)$, $\cdots$, $N_{n-1}(t)$ can be evaluated at the initial time t_I using the initial conditions (2.16). We find that

$$\begin{aligned} \mathrm{Wr}[N_0, N_1, \cdots, N_{n-1}](t_I) &= \det \begin{pmatrix} N_0(t_I) & N_1(t_I) & \cdots & N_{n-1}(t_I) \\ N_0'(t_I) & N_1'(t_I) & \cdots & N_{n-1}'(t_I) \\ \vdots & \vdots & \ddots & \vdots \\ N_0^{(n-1)}(t_I) & N_1^{(n-1)}(t_I) & \cdots & N_{n-1}^{(n-1)}(t_I) \end{pmatrix} \\ &= \det \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & 1 \end{pmatrix} = 1\,. \end{aligned}$$

Because this is nonzero, $N_0(t)$, $N_1(t)$, $\cdots$, $N_{n-1}(t)$ is a fundamental set of solutions to equation (2.13). It is called the *natural fundamental set* associated with the initial time t_I because the solution of the initial-problem satifying the general initial conditions (2.14) has the simple form (2.15).

Remark. If the coefficients a_1, a_2, $\cdots$, a_n in the differential equation (2.13) are all continuous over an interval (t_L, t_R) that contains the initial time t_I then our Basic Existence and Uniqueness Theorem (Theorem 1.1) insures that for each $j = 0, 1, \cdots, n-1$ the solution $N_j(t)$ is uniquely defined over the interval (t_L, t_R) by the initial conditions (2.16). It can then be checked that the solution of the initial-problem satifying the general initial conditions (2.14) is given by (2.15).

Remark. The general solution constructed from the natural fundamental set $N_0(t)$, $N_1(t)$, $\cdots$, $N_{n-1}(t)$ is given by (2.15), where initial values y_0, y_1, $\cdots$, y_{n-1} are the n parameters of the general solution.

A fast way to find the natrual fundamental set associated with a given initial time t_I is to simply solve the initial-problem satisfying the general initial conditions (2.14) and then put its solution into the form (2.15).

Example. Find the natural fundamental set of solutions associated with the initial time 0 for the differential equation

$$y'' - y' - 2y = 0\,.$$

Solution. In Section 2.2 we used the fundamental set of solutions e^{2t} and e^{-t} to construct solutions that satisfy the general initial conditions

$$y(0) = y_0\,, \qquad y'(0) = y_1\,.$$

We found that

$$y(t) = \frac{y_0 + y_1}{3}\, e^{2t} + \frac{2y_0 - y_1}{3}\, e^{-t}\,.$$

By grouping the terms that multiply y_0 and those that multiply y_1 we obtain

$$y(t) = \frac{e^{2t} + 2e^{-t}}{3}\, y_0 + \frac{e^{2t} - e^{-t}}{3}\, y_1\,.$$

We can read off from this that the natural fundamental set of solutions is

$$N_0(t) = \frac{e^{2t} + 2e^{-t}}{3}\,, \qquad N_1(t) = \frac{e^{2t} - e^{-t}}{3}\,.$$

Remark. Notice that because

$$N_0'(t) = \frac{2e^{2t} - 2e^{-t}}{3}\,, \qquad N_1'(t) = \frac{2e^{2t} + e^{-t}}{3}\,,$$

we see that $N_0(t)$ and $N_1(t)$ satisfy the inital conditions (2.16) — namely, that

$$N_0(0) = 1\,, \quad N_0'(0) = 0\,, \qquad \text{and} \qquad N_1(0) = 0\,, \quad N_1'(0) = 1\,.$$

You should check that the inital conditions (2.16) are satisfied in the next example too.

Example. Find the natural fundamental set of solutions associated with the initial time 0 for the differential equation

$$h'' + 4h = 0\,.$$

Solution. In Section 2.2 we used the fundamental set of solutions $\cos(2t)$ and $\sin(2t)$ to construct solutions that satisfy the general initial conditions

$$h(0) = h_0\,, \qquad h'(0) = h_1\,.$$

We found that

$$h(t) = \cos(2t) h_0 + \frac{\sin(2t)}{2} h_1\,.$$

We can read off from this that the natural fundamental set of solutions is

$$N_0(t) = \cos(2t)\,, \qquad N_1(t) = \frac{\sin(2t)}{2}\,.$$

Example. Find the natural fundamental set of solutions associated with the initial time 1 for the differential equation

$$(1+t^2)x'' - 2t\,x' + 2x = 0\,.$$

Solution. In Section 2.2 we used the fundamental set of solutions t and t^2-1 to construct solutions that satisfy the general initial conditions

$$x(1) = x_0\,, \qquad x'(1) = x_1\,.$$

We found that

$$x(t) = x_0 t + \frac{x_1 - x_0}{2}(t^2-1)\,.$$

By grouping the terms that multiply x_0 and those that multiply x_1 we obtain

$$x(t) = \frac{2t - t^2 + 1}{2}\,x_0 + \frac{t^2-1}{2}\,x_1\,.$$

We can read off from this that the natural fundamental set of solutions is

$$N_0(t) = \frac{2t - t^2 + 1}{2}\,, \qquad N_1(t) = \frac{t^2-1}{2}\,.$$

Example. Find the natural fundamental set of solutions associated with the initial time 0 for the differential equation

$$v''' - 6v'' + 5v' + 12v = 0\,.$$

Solution. In Section 2.2 we used the fundamental set of solutions e^{4t}, e^{3t}, and e^{-t} to construct solutions that satisfy the general initial conditions

$$v(0) = v_0\,, \qquad v'(0) = v_1\,, \qquad v''(0) = v_2\,.$$

We found that

$$v(t) = \frac{-3v_0 - 2v_1 + v_2}{5}\,e^{4t} + \frac{4v_0 + 3v_1 - v_2}{4}\,e^{3t} + \frac{12v_0 - 7v_1 + v_2}{20}\,e^{-t}\,.$$

By grouping the terms multiplying v_0, v_1, and v_2 we obtain

$$v(t) = \frac{-12e^{4t} + 20e^{3t} + 12e^{-t}}{20}\,v_0 + \frac{-8e^{4t} + 15e^{3t} - 7e^{-t}}{20}\,v_1 + \frac{4e^{4t} - 5e^{3t} + e^{-t}}{20}\,v_2\,.$$

We can read off from this that the natural fundamental set of solutions is

$$\begin{aligned} N_0(t) &= \frac{-12e^{4t} + 20e^{3t} + 12e^{-t}}{20}\,, \\ N_1(t) &= \frac{-8e^{4t} + 15e^{3t} - 7e^{-t}}{20}\,, \\ N_2(t) &= \frac{4e^{4t} - 5e^{3t} + e^{-t}}{20}\,. \end{aligned}$$

2.6. **Linear Independence of Solutions.** In Section 2.4 we defined fundamental sets of solutions of a homogeneous linear ordinary differential equation in terms of nonzero Wronskians. Here we develop another characterization of fundamental sets of solutions based on the following notions.

> **Definition.** Functions Y_1, Y_2, $\cdots$, Y_m defined over an interval (t_L, t_R) are said to be *linearly dependent* if there exists constants c_1, c_2, $\cdots$, c_m, not all zero, such that
>
> $$0 = c_1 Y_1(t) + c_2 Y_2(t) + \cdots + c_m Y_m(t) \quad \text{for every } t \text{ in } (t_L, t_R)\,. \tag{2.17}$$
>
> Otherwise they are said to be *linearly independent.*

If Y_1, Y_2, $\cdots$, Y_m are linearly dependent then for any c_k that is nonzero, we can solve (2.17) for $Y_k(t)$ as a linear combination of the other functions. For example, if $c_1 \neq 0$ then

$$Y_1(t) = -\frac{c_2}{c_1}\,Y_2(t) - \cdots - \frac{c_m}{c_1}\,Y_m(t) \quad \text{for every } t \text{ in } (t_L, t_R)\,.$$

Because there is at least one nonzero c_k, this can always be done for some Y_k.

Example. The functions $\cos(2t)$, $\cos(t)^2$ and 1 are linearly dependent over $(-\infty, \infty)$ because

$$\cos(2t) = \cos(t)^2 - \sin(t)^2 = 2\cos(t)^2 - 1\,.$$

Remark. If one of the functions Y_1, Y_2, $\cdots$, Y_m is identically zero over (t_L, t_R) then the set is linearly dependent. For example, suppose that $Y_1(t) = 0$ for every t in (t_L, t_R). Then (2.17) holds with $c_1 = 1$ and $c_2 = \cdots = c_m = 0$.

Remark. Two functions Y_1 and Y_2, neither of which is identically zero, are linearly dependent if and only if they are proportional to each other.

Example. The functions t and t^2 are linearly independent over $(0, 1)$ because they are not proportional to each other. It is clear graphically that there is no constant k such that $t^2 = kt$ for every t in $(0, 1)$ because the parabola $y = t^2$ is not a line. Hence, these functions are not linearly dependent.

A good way to generally approach establishing linear independence is the following. A set functions Y_1, Y_2, $\cdots$, Y_m defined over an interval (t_L, t_R) is linearly independent if and only if the linear relation (2.17) can only hold when $c_1 = c_2 = \cdots = c_m = 0$. When a set of functions is linearly independent there are many ways to show this.

Example. Show that the functions 1, t and t^2 are linearly independent over $(-\infty, \infty)$.

Solution. Begin by supposing the linear relation

$$0 = c_1 + c_2 t + c_3 t^2 \qquad \text{for every } t \text{ in } (-\infty, \infty)\,.$$

If we set $t = 0$, $t = 1$, and $t = -1$ into this relation, we obtain the linear algebraic system

$$\begin{aligned} 0 &= c_1\,, \\ 0 &= c_1 + c_2 + c_3\,, \\ 0 &= c_1 - c_2 + c_3\,. \end{aligned}$$

This can be easily solved to show that $c_1 = c_2 = c_3 = 0$, whereby we conclude that 1, t and t^2 are linearly independent.

Remark. A similar argument works if we had chosen to evaluate the linear relations at any other three distinct points, say $t = 2$, $t = 4$, and $t = 6$. We chose to use $t = 0$, $t = 1$, and $t = -1$ because they led to a simple linear algebraic system.

Alternative Solution. Another approach to the above example is to differentiate the linear relation twice with respect to t, thereby obtaining

$$\begin{aligned} 0 &= c_1 + c_2 t + c_3 t^2 \,, \\ 0 &= c_2 + 2c_3 t \,, \qquad\qquad \text{for every } t \text{ in } (-\infty, \infty) \,. \\ 0 &= 2c_3 \,, \end{aligned}$$

If we set $t = 0$ into these equations we immediately see that $c_1 = c_2 = c_3 = 0$, whereby we conclude that 1, t and t^2 are linearly independent.

The approach taken in our alternative solution generalizes as follows.

Theorem 2.5. If Y_1, Y_2, $\cdots$, Y_m is a set of $m - 1$ times differentiable functions over an interval (t_L, t_R) such that $\mathrm{Wr}[Y_1, Y_2, \cdots, Y_m](t_I) \neq 0$ for some t_I in (t_L, t_R) then they are linearly independent.

Proof. We show this by supposing the linear relation

$$0 = c_1 Y_1(t) + c_2 Y_2(t) + \cdots + c_m Y_m(t) \quad \text{for every } t \text{ in } (t_L, t_R) \,.$$

If we differentiate this relation $m - 1$ times with respect to t and evaluate the resulting relationships at $t = t_I$, we obtain the linear algebraic system

$$\begin{aligned} 0 &= \ c_1 Y_1(t_I) \ + \ c_2 Y_2(t_I) \ + \cdots + \ c_m Y_m(t_I) \,, \\ 0 &= \ c_1 Y_1'(t_I) \ + \ c_2 Y_2'(t_I) \ + \cdots + \ c_m Y_m'(t_I) \,, \\ &\ \vdots \\ 0 &= c_1 Y_1^{(m-1)}(t_I) + c_2 Y_2^{(m-1)}(t_I) + \cdots + c_m Y_m^{(m-1)}(t_I) \,. \end{aligned}$$

Because $\mathrm{Wr}[Y_1, Y_2, \cdots, Y_m](t_I) \neq 0$, it follows from Theorem 3.1 of Chapter 3 that $c_1 = c_2 = \cdots = c_m = 0$ is the only solution to this system, from which we conclude the functions Y_1, Y_2, $\cdots$, Y_m are linearly independent. ☐

It is natural to ask if linear independence implies having a Wronskian that is nonzero somewhere (or what is the same, if having a Wronskian that is zero everywhere implies linear dependence.) The following example shows that this is not the case.

Example. Consider the functions $Y_1(t) = t^2$ and $Y_2(t) = |t|t$ over $(-\infty, \infty)$. Because $Y_1'(t) = 2t$ and $Y_2'(t) = 2|t|$ over $(-\infty, \infty)$, we have

$$\begin{aligned} \mathrm{Wr}[Y_1, Y_2](t) &= \det\begin{pmatrix} Y_1(t) & Y_2(t) \\ Y_1'(t) & Y_2'(t) \end{pmatrix} = \det\begin{pmatrix} t^2 & |t|t \\ 2t & 2|t| \end{pmatrix} \\ &= 2|t|t^2 - 2|t|t^2 = 0 \qquad \text{for every } t \text{ in } (-\infty, \infty) \,. \end{aligned}$$

Therefore the Wronskian of Y_1 and Y_2 is *zero everywhere*. However, Y_1 and Y_2 are *linearly independent*! This can be seen from the fact that Y_1 and Y_2 are *not proportional*, which is evident from their graphs. Alternatively, we could argue they are linearly independent by first supposing the linear relation

$$0 = c_1 t^2 + c_2 |t|t \qquad \text{for every } t \text{ in } (-\infty, \infty) \,.$$

If we set $t = 1$ and $t = -1$ into this relation, we obtain the linear algebraic system

$$0 = c_1 + c_2\,, \qquad 0 = c_1 - c_2\,.$$

This can be easily solved to show that $c_1 = c_2 = 0$, whereby we conclude that Y_1 and Y_2 are linearly independent.

The above example shows that a set of linearly independent functions can have a Wronskian that is zero everywhere. However, the following theorem shows this cannot happen for sets of n solutions of an n^{th}-order homogeneous equation.

Theorem 2.6. If $Y_1, Y_2, \cdots, Y_n$ are solutions of (2.9) over an interval (t_L, t_R) then the following properties are equivalent:

(i) $\mathrm{Wr}[Y_1, Y_2, \cdots, Y_n]$ is nonzero everywhere in (t_L, t_R),
(ii) $\mathrm{Wr}[Y_1, Y_2, \cdots, Y_n]$ is nonzero somewhere in (t_L, t_R),
(iii) $Y_1, Y_2, \cdots, Y_n$ are linearly independent.

Remark. This is the same as saying that following properties are equivalent:

(i′) $\mathrm{Wr}[Y_1, Y_2, \cdots, Y_n]$ is zero somewhere in (t_L, t_R),
(ii′) $\mathrm{Wr}[Y_1, Y_2, \cdots, Y_n]$ is zero everywhere in (t_L, t_R),
(iii′) $Y_1, Y_2, \cdots, Y_n$ are linearly dependent.

The above properties are simply the negations of (i), (ii), and (iii) respectively.

Remark. This theorem shows that properties (i), (ii), and (iii) are all equivalent to $Y_1, Y_2, \cdots, Y_n$ being a fundamental set of solutions to (2.9). The equivalence of (i) and (ii) was established by Theorem 2.3. Below we give an alternative proof of this fact.

Proof. It is clear that (i) implies (ii). The fact that (ii) implies (iii) is just Theorem 2.5. Neither of these implications requires the hypothesis that $Y_1, Y_2, \cdots, Y_n$ are solutions of (2.9). All that remains to be proved is that (iii) implies (i). We will do this by contradiction.

Suppose that $Y_1, Y_2, \cdots, Y_n$ are linearly independent and $\mathrm{Wr}[Y_1, Y_2, \cdots, Y_n](t_I) = 0$ for some t_I in (t_L, t_R). Because

$$\det\begin{pmatrix} Y_1(t_I) & Y_2(t_I) & \cdots & Y_n(t_I) \\ Y_1'(t_I) & Y_2'(t_I) & \cdots & Y_n'(t_I) \\ \vdots & \vdots & \ddots & \vdots \\ Y_1^{(n-1)}(t_I) & Y_2^{(n-1)}(t_I) & \cdots & Y_n^{(n-1)}(t_I) \end{pmatrix} = \mathrm{Wr}[Y_1, Y_2, \cdots, Y_n](t_I) = 0\,,$$

Theorem 3.2 of Chapter 3 implies that the linear algebraic system

$$\begin{aligned} 0 &= c_1 Y_1(t_I) + c_2 Y_2(t_I) + \cdots + c_n Y_n(t_I)\,, \\ 0 &= c_1 Y_1'(t_I) + c_2 Y_2'(t_I) + \cdots + c_n Y_n'(t_I)\,, \\ &\;\vdots \\ 0 &= c_1 Y_1^{(n-1)}(t_I) + c_2 Y_2^{(n-1)}(t_I) + \cdots + c_n Y_n^{(n-1)}(t_I)\,, \end{aligned} \tag{2.18}$$

has a nonzero solution c_1, c_2, $\cdots$, c_n. Now define

$$y(t) = c_1 Y_1(t) + c_2 Y_2(t) + \cdots + c_n Y_n(t)\,. \tag{2.19}$$

Because $Y_1, Y_2, \cdots, Y_n$ are solutions of (2.9), Theorem 2.1 (Superposition) implies that $y(t)$ is also a solution of (2.9). By (2.19) and (2.18) we see that $y(t)$ satisfies the initial conditions

$$y(t_I) = 0\,, \qquad y'(t_I) = 0\,, \qquad \cdots \qquad y^{(n-1)}(t_I) = 0\,.$$

The uniqueness assertion of Theorem 1.1 then implies that $y(t) = 0$ for every t in (t_L, t_R). Hence, by (2.19) we have

$$0 = c_1 Y_1(t) + c_2 Y_2(t) + \cdots + c_n Y_n(t)\,,$$

where the c_1, c_2, $\cdots$, c_n are not all zero. But this implies that $Y_1, Y_2, \cdots, Y_n$ are linearly dependent, which is a contradiction. Therefore we conclude that $\mathrm{Wr}[Y_1, Y_2, \cdots, Y_n]$ is nonzero everywhere in (t_L, t_R), thereby showing that (iii) implies (i). ☐

2.7. **Method of Order Reduction.** While there is no general recipe for constructing any solution to the n^{th}-order linear differential equation (2.9) when $n > 1$, if we are able to find one solution then we can reduce the problem of finding other solutions to that of solving an $(n-1)^{\text{th}}$-order linear differential equation. This method is called *order reduction*. It is particularly useful when $n = 2$ because in that case the second-order equation is reduced to a first-order equation, and there is a general recipe for constructing solutions of first-order linear equations.

Suppose we know that $Y(t)$ is a solution of (2.9). If we set $y = Y(t)u$ and compute its first n derivatives we obtain

$$\begin{aligned} y' &= Y'(t)u + Y(t)u'\,, \\ y'' &= Y''(t)u + 2Y'(t)u' + Y(t)u''\,, \\ &\vdots \\ y^{(n)} &= Y^{(n)}(t)u + nY^{(n-1)}(t)u' + \cdots + nY'(t)u^{(n-1)} + Y(t)u^{(n)}\,. \end{aligned}$$

When these expressions are substituted into (2.9) all the terms involving u will drop out because $Y(t)$ is a solution of (2.9). The result will be an linear differential equation of order $n-1$ for u'.

We will illustrate this method by using it to construct a general solution of some second-order equations when one solution of that equation is given to us.

Example. Given that $Y(t) = e^{2t}$ is one solution, find a general solution of

$$y'' - y' - 2y = 0\,.$$

Solution. Set $y = e^{2t}u$, so that

$$\begin{aligned} y' &= 2e^{2t}u + e^{2t}u'\,, \\ y'' &= 4e^{2t}u + 4e^{2t}u' + e^{2t}u''\,. \end{aligned}$$

Upon substituting these expressions into the equation we obtain

$$\begin{aligned}0 &= y'' - y' - 2y\\ &= \big(4e^{2t}u + 4e^{2t}u' + e^{2t}u''\big) - \big(2e^{2t}u + e^{2t}u'\big) - 2e^{2t}u\\ &= e^{2t}u'' + \big(4e^{2t} - e^{2t}\big)u' + \big(4e^{2t} - 2e^{2t} - 2e^{2t}\big)u\\ &= e^{2t}u'' + 3e^{2t}u' = e^{2t}\big(u'' + 3u'\big)\,,\end{aligned}$$

which is equivalent to

$$u'' + 3u' = 0\,.$$

Because u does not appear in this equation, it is equivalent to the first-order equation

$$w' + 3w = 0\,, \qquad \text{where } w = u'\,.$$

A general solution of this first-order equation is $w = c_1 e^{-3t}$, whereby we can integrate to obtain $u = -\frac{1}{3}c_1 e^{-3t} + c_2$. We thereby obtain the general solution

$$y = e^{2t}u = e^{2t}\big(-\tfrac{1}{3}c_1 e^{-3t} + c_2\big) = -\tfrac{1}{3}c_1 e^{-t} + c_2 e^{2t}\,.$$

Remark. In particular, we see that e^{-t} is also a solution of the equation. It is easy to check that e^{-t} and e^{2t} are linearly independent, and are thereby a fundamental set of solutions of the equation.

Example. Given that $X(t) = t$ is one solution, find a general solution of

$$(1 + t^2)x'' - 2t\,x' + 2x = 0\,.$$

Solution. Set $x = tu$, so that

$$\begin{aligned}x' &= u + tu'\,,\\ x'' &= 2u' + tu''\,.\end{aligned}$$

Upon substituting these expressions into the equation we obtain

$$\begin{aligned}0 &= (1 + t^2)x'' - 2t\,x' + 2x\\ &= (1 + t^2)\big(2u' + tu''\big) - 2t\big(u + tu'\big) + 2tu\\ &= (1 + t^2)tu'' + \big(2(1 + t^2) - 2t^2\big)u'\,,\end{aligned}$$

which is equivalent to

$$(1 + t^2)tu'' + 2u' = 0\,.$$

Because u does not appear in this equation, it is equivalent to the first-order equation

$$w' + \frac{2}{(1 + t^2)t}\,w = 0\,, \qquad \text{where } w = u'\,.$$

By the partial fraction identity

$$\frac{2}{(1 + t^2)t} = \frac{2}{t} - \frac{2t}{1 + t^2}\,,$$

we see that

$$\int \frac{2}{(1 + t^2)t}\,\mathrm{d}t = \int \frac{2}{t} - \frac{2t}{1 + t^2}\,\mathrm{d}t = \log(t^2) - \log(1 + t^2) = \log\!\left(\frac{t^2}{1 + t^2}\right).$$

The first-order equation for w thereby has the integrating factor form

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{t^2}{1+t^2}\,w\right) = 0\,.$$

A general solution of the first-order equation for w is

$$w = c_1\frac{1+t^2}{t^2} = c_1\left(1+\frac{1}{t^2}\right),$$

which we can integrate to find

$$u = \int w\,dt = c_1\left(t-\frac{1}{t}\right) + c_2\,.$$

We thereby obtain the general solution

$$x = tu = c_1(t^2-1) + c_2 t\,.$$

Remark. In particular, we see that t^2-1 is also a solution of the equation. It is easy to check that t^2-1 and t are linearly independent and are thereby a fundamental set of solutions of the equation.

EXERCISES ON HOMOGENEOUS EQUATIONS

Find the general solution to the following differential equations.

(1) Consider the second-order differential equation $x'' = x$. The four functions $\cosh(u)$, $\sinh(u)$, e^u, and e^{-u} are all solutions to it. Is that weird? Why (or why not)?

Solution

(2) Consider the fourth-order differential equation $y^{(4)} = y$. Check that the functions $y_1(t) = \cos(t)$ and $y_2(t) = \sin(t)$ both are solutions to it. There should be two other fundamentally different solutions; can you find them by inspection?

Solution

(3) Check that e^{3z} and e^{4z} are solutions to $w'' - 7w' + 12w = 0$. Then find c_1 and c_2 so that $W(z) = c_1e^{3z} + c_2e^{4z}$ is a solution to the differential equation satisfying the initial conditions $W(0) = 2$, $W'(0) = 0$.

Solution

(4) Check that $e^x \sin(x)$ and $e^x \cos(x)$ are solutions to $y'' - 2y' + 2y = 0$. Then find c_1 and c_2 so that $Y(x) = c_1e^x \sin(x) + c_2e^x \cos(x)$ is a solution to the differential equation satisfying the initial conditions $Y(0) = 3$, $Y'(0) = -2$.

Short Answer
Solution

(5) (a) Check that e^x and $x^2 + 2x + 2$ are solutions to the equation

$$xy'' - (x + 2)y' + 2y = 0.$$

(b) Find c_1 and c_2 so that $Y(x) = c_1e^x + c_2(x^2 + 2x + 2)$ is a solution to the differential equation satisfying the initial conditions $Y(1) = e$, $Y'(1) = e+2$.

(c) On what interval(s) is the function Y you found in part (b) a solution to the equation?

Short Answer
Solution

(6) Check that w, w^3, and w^5 are solutions to the third-order differential equation

$$w^3z''' - 6w^2z'' + 15wz' - 15z = 0 \ .$$

Then find constants c_1, c_2, and c_3 so that $Z(w) = c_1w + c_2w^3 + c_3w^5$ are solutions to the equation on $(0, \infty)$ satisfying $Z(2) = 0$, $Z'(2) = 0$, and $Z''(2) = 1$.

Short Answer
Solution

(7) One fundamental set of solutions for the differential equation $y'' - 2y' - 8y = 0$ is $\{e^{-2w}, e^{4w}\}$. Find a solution $Y = c_1e^{-2w} + c_2e^{4w}$ which satisfies the general initial conditions $Y(0) = y_0$, $Y'(0) = y_1$.

Short Answer
Solution

(8) One fundamental set of solutions for the differential equation $y'' + 9y = 0$ is $\{\cos(3t), \sin(3t)\}$. Find a solution $Y = c_1 \cos(3t) + c_2 \sin(3t)$ that satisfies the general initial conditions $Y(\frac{\pi}{2}) = y_0$, $Y'(\frac{\pi}{2}) = y_1$.

Short Answer
Solution

(9) One fundamental set of solutions for the differential equation $u^2x'' + 4ux' = 0$ is $\{1, u^{-3}\}$. Find a solution $X(u) = c_1 + c_2u^{-3}$ that satisfies the general initial conditions $X(1) = x_0$, $X'(1) = x_1$.

Short Answer
Solution

(10) (a) One fundamental set of solutions for the differential equation $y'' - \frac{1}{x}y' = 0$ is $\{1, x^2\}$. Find a solution $Y(x) = c_1 + c_2x^2$ that satisfies the general initial conditions $Y(-2) = y_0$, $Y'(-2) = y_1$.
(b) What is the interval of definition for your solution in (a)?
(c) What happens if you try to solve for the initial conditions $y(0) = 1$, $y'(0) = 2$?

Short Answer
Solution

(11) Compute the Wronskian of the functions $W_1(z) = z$ and $W_2(z) = \cos(z)$. Is this defined for all points z?

Solution

(12) Compute the Wronskian of the functions $X_1(u) = \cosh(u)$, $X_2(u) = \sinh(u)$, and $X_3(u) = e^u$. Notice this also explains what was going on in Exercise #1.

Solution

(13) (Continuation of Exercise #3) Show that $\{e^{3t}, e^{4t}\}$ is a fundamental set of solutions for the second-order differential equation $y'' - 7y' + 12y = 0$.

Solution

(14) (Continuation of Exercise #4) Show that $\{e^x \sin(x), e^x \cos(x)\}$ is a fundamental set of solutions for the second-order differential equation $y'' - 2y' + 2y = 0$.

Solution

(15) (Continuation of Exercise #6) Show that $\{x, x^3, x^5\}$ is a fundamental set of solutions for the third-order differential equation

$$x^3y''' - 6x^2y'' + 15xy' - 15y = 0.$$

Solution

(16) Show that the functions e^t, e^{2t}, and e^{3t} are linearly independent on $(-\infty, \infty)$.

Solution

(17) Show that $\log(z)$, $\log(5z)$, and 1 are linearly dependent over $(0, \infty)$.

Solution

(18) (Continuation of Exercise #3) Find a natural fundamental set for the differential equation $y'' - 7y' + 12y = 0$ associated with the initial time 0.

Short Answer
Solution

(19) (Continuation of Exercise #9) Find a natural fundamental set for the differential equation $w^2z'' + 4wz' = 0$ associated with the initial time 1.

Short Answer
Solution

(20) (Continuation of Exercise #8) Find a natural fundamental set for the differential equation $y'' + 9y = 0$ associated with the initial time $\frac{\pi}{2}$.

Short Answer
Solution

(21) Check that e^z is a solution to $w'' - 2w' + w = 0$, and then use reduction of order to find a fundamental set.

Short Answer
Solution

(22) It's clear that $Y_1(t) = 1$ solves the differential equation $ty'' + y' = 0$. Find another solution by reduction of order.

Short Answer
Solution

(23) If n is a positive integer, then $Y_1(t) = 1$ is also a solution to $y'' - \frac{n}{x}y' = 0$. Complete a fundamental set for this equation. [*Hint.* Your answer will have an n in it somewhere.]

Short Answer
Solution

(24) Check that $Z_1(u) = e^{2u}\cos(u)$ is a solution to $D^2z - 4Dz + 5z = 0$. Then use reduction of order to complete a fundamental set for it.

Short Answer
Solution

(25) Check that $Z_1(w) = w$ is a solution to $(w-1)(w-2)\ddot{z} - w\ddot{z} + z = 0$. Then use reduction of order to find a second linearly independent solution.

Short Answer
Solution

(26) As suggested in the text, give a proof of Abel's theorem for the third-order case. That is to say, suppose $Y_1(t)$, $Y_2(t)$, and $Y_3(t)$ are three solutions to the differential equation

$$y'''(t) + a_1(t)y''(t) + a_2(t)y'(t) + a_3(t)y(t) = 0,$$

then show that their Wronskian $W(t) = W[Y_1, Y_2, Y_3](t)$ satisfies the first-order differential equation

$$W' + a_1(t)W = 0.$$

Solution

(27) Suppose we have a second-order homogeneous differential equation and we happen to know one of the solutions. Then the method of reduction of order will always give us a first-order differential equation whose solution is a linearly independent solution to the equation. In the problems above, the first-order differential equation is solvable, but this doesn't happen in general—often we

wind up with an integral that we can't solve. This is not to say all is lost; having an integral (or even just a differential equation in the first place) opens up the possibility of determining values of the function by use of numerical methods, after all.

Consider the differential equation

$$y'' - 2(2x^2 + 1)y = 0.$$

One of the homogeneous solutions is $Y_1(x) = e^{x^2}$. Check this. Then use the method of reduction of order to come up with an expression for another solution to the homogeneous equation, linearly independent from Y_1. [You should guess from the paragraph preceeding this that you probably will get stuck at an integral.]

Short Answer
Solution

(28) Show that $\left\{x^{-1}, x^{\frac{3}{2}}\right\}$ is a fundamental set of solutions for the second-order differential equation $2x^2y'' + xy' - 3y = 0$. Make sure to check to see if the Wronskian of x_1 and x_2 is defined everywhere.

(Note: you will need first, to verify that each of the two functions $\left\{x^{-1}, x^{\frac{3}{2}}\right\}$ satisfies the differential equation and second, to show that they form a fundamental set of solutions.)

Solution

(29) Without solving, determine the Wronskian of two solutions evaluated at $t = 1$ for the following differential equation:

$$t^5\ddot{y} - 2t^2\dot{y} - t^7y = 0\ , y(1) = 5\ , \dot{y} = 10.$$

What is the interval of definition of this particular solution to the initial-value problem?

Short Answer
Solution

(30) Without solving, determine the Wronskian of two solutions evaluated at $u = 4$ for the following differential equation:

$$2u^2y'' + uy' - 3y = 0.$$

Is the Wronskian defined for all u?

Short Answer
Solution

(31) The following problem is an application of the Method of Linear Superposition, **Theorem 2.1**. Assume that $\cos(x)$ and x are both solutions of the equation $p(\mathrm{D})w = q(x)$, for a certain polynomial $p(x)$ and a certain function $q(x)$.

(a) Write down a nonzero solution of the equation $p(\mathrm{D})w = 0$.

(b) Write down a solution $w(x)$ of $p(\mathrm{D})w = q(x)$ such that $w(0) = 2$.

Remark We haven't discussed how to solve nonhomogeneous second-order linear differential equations yet, but we don't need to know that yet!

Short Answer
Solution

(32) Prove the following statement: “If $f(t)$ and $g(t)$ are linearly independent solutions of a second-order linear homogeneous differential equation on an interval I, then f and g cannot have a maximum at the same location in I.”

(**Hint** : This is a more theoretical argument than what you’ve seen before. Think of an argument by contradiction. Your proof should include complete sentences.)

Solution

(33) Verify that $y_1(x) = 1$ and $y_2(x) = x^{\frac{1}{2}}$ are solutions to the differential equation $yy'' + (y')^2 = 0$ for $x > 0$. Then show that $c_1 + c_2 x^{\frac{1}{2}}$ is not in general a solution of this equation. Can you explain why this result doesn’t contradict the Method of Linear Superposition in **Theorem 2.1**?

Solution

(34) **More exploration of the Wronskian**

If the functions w_1 and w_2 are linearly independent solutions of $w'' + p(u)w' + q(u)w = 0$, determine what the necessary and sufficient conditions are such that the functions $w_3 = \alpha w_1 + \beta w_2$ and $w_4 = \gamma w_1 + \epsilon w_2$ also form a linearly independent set of solutions.

Short Answer
Solution

(35) **A clever use of Abel’s Theorem** :

(a) Consider the differential equation $z'' + 2az' + a^2 z = 0$. Show that one of the solutions of the equation is e^{-au}.

(b)Use Abel’s Formula to show that the Wronskian of any two solutions of the given equaton is

$$W(u) = z_1(u)z_2'(u) - z_1'(u)z_2(u) = ce^{-2au},$$

where c is a constant.

(c) Consider $z_1(u) = e^{-au}$ from part (a) and use the result in (b) to obtain a differential equation satisfied by the second solution $z_2(u)$. Then solve this equation to show that $z_2(u) = ue^{-au}$.

Solution

(36) Use the method of order reduction to find a second solution of the differential equation $z^2\ddot{w} + 2z\dot{w} - 2w = 0$, $z > 0$, $w_1(z) = z$.

Short Answer
Solution

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