II. Higher-Order Linear Ordinary Differential Equations
3. Supplement: Linear Algebraic Systems and Determinants

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3. Supplement: Linear Algebraic Systems and Determinants

This chapter is a supplement to Chapter 2 that reviews the material about linear algebraic systems that is used therein. In particular, it covers the relationship between linear algebraic systems and determinants that is used in Section 2.3. This material should be understood before covering that section.

3.1. Linear Algebraic Systems.

In each of the examples in Section 2.2 we were able to find a unique solution $c_1, c_2, \cdots, c_n$ to the linear algebraic system 2.5 which enabled us to solve the initial-value problem for any choice of initial data $y_0, y_1, \cdots, y_{n-1}$ by the method of superposition. In this supplement we will characterize when such a linear algebraic system has a unique solution.

A linear algebric system of n equations for n unknowns $x_1, x_2, \cdots, x_n$ has the general form

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1\,, \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2\,, \\ &\vdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n &= b_n\,. \end{aligned} \tag{3.1}$$

The n^2 numbers $\{a_{ij} : i, j = 1, 2, \cdots, n\}$ are called the *coefficients* of the system and the n numbers $b_1, b_2, \cdots, b_n$ are called the *forcing*. The system is called *homogeneous* if $b_1 = b_2 = \cdots = b_n = 0$, and is called *nonhomogenous* otherwise.

There are two questions regarding the existence of solutions that we want to address.

The **first question** is:

When does system (3.1) have a unique solution for every forcing $b_1, b_2, \cdots, b_n$?

The **second question** is:

When does system (3.1) with $b_1 = b_2 = \cdots = b_n = 0$ have a nonzero solution?

Here "nonzero solution" means a solution with $x_k \neq 0$ for some index k — i.e. a solution that is not the "trivial solution" $x_1 = x_2 = \cdots = x_n = 0$. Such a solution is also called a "nontrivial solution."

These questions are clearly related. Let us suppose that system (3.1) has two different solutions for some set of numbers $b_1, b_2, \cdots, b_n$. We denote one of these solutions by $x_1, x_2, \cdots, x_n$ and the other by $y_1, y_2, \cdots, y_n$. Set

$$z_1 = x_1 - y_1\,, \qquad z_2 = x_2 - y_2\,, \qquad \cdots, \qquad z_n = x_n - y_n\,.$$

Then we can show that $z_1, z_2, \cdots, z_n$ is a nonzero solution of

$$\begin{aligned} a_{11}z_1 + a_{12}z_2 + \cdots + a_{1n}z_n &= 0\,, \\ a_{21}z_1 + a_{22}z_2 + \cdots + a_{2n}z_n &= 0\,, \\ &\vdots \\ a_{n1}z_1 + a_{n2}z_2 + \cdots + a_{nn}z_n &= 0\,. \end{aligned} \tag{3.2}$$

But this is system (3.1) with $b_1 = b_2 = \cdots = b_n = 0$.

Conversely, if system (3.2) has a nonzero solution z_1, z_2, $\cdots$, z_n then no solution of (3.1) is unique for any forcing b_1, b_2, $\cdots$, b_n. To see this, suppose that y_1, y_2, $\cdots$, y_n is a solution of system (3.1) for some forcing b_1, b_2, $\cdots$, b_n:

$$\begin{aligned}
a_{11}y_1 + a_{12}y_2 + \cdots + a_{1n}y_n &= b_1 \,, \\
a_{21}y_1 + a_{22}y_2 + \cdots + a_{2n}y_n &= b_2 \,, \\
&\vdots \\
a_{n1}y_1 + a_{n2}y_2 + \cdots + a_{nn}y_n &= b_n \,.
\end{aligned}$$

Then for every $\alpha \neq 0$ we can construct another solution x_1, x_2, $\cdots$, x_n of system (3.1) with the same forcing b_1, b_2, $\cdots$, b_n by setting

$$x_1 = y_1 + \alpha z_1 \,, \qquad x_2 = y_2 + \alpha z_2 \,, \qquad \cdots \,, \qquad x_n = y_n + \alpha z_n \,.$$

Hence, the existence of a nonzero solution z_1, z_2, $\cdots$, z_n of (3.2) implies that for every forcing b_1, b_2, $\cdots$, b_n, system (3.1) either has no solution or has many solutions. Therefore it does not have a unique solution for any forcing.

3.2. **Determinants.** Answers to our questions can depend only on the coefficients $\{a_{ij} : i, j = 1, 2, \cdots, n\}$. It is helpful to write these coefficients as an $n{\times}n$ matrix $\mathbf{A}$:

$$\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix} .$$

The answers will be given in terms of a quantity $\det(\mathbf{A})$, called the *determinant* of $\mathbf{A}$. It is likely that you have seen determinants of 2×2 or 3×3 matrices in earlier courses. Here we review some methods for computing determinants that work well when either n is not too large or the matrix $\mathbf{A}$ has a special structure. These are most cases that you will face in this course.

3.2.1. *Special Formulas for Small Matrices.* Formulas for $\det(\mathbf{A})$ when $n = 1$, 2, and 3 are

$$\begin{aligned}
\det\begin{pmatrix} a_{11} \end{pmatrix} &= a_{11} \,, \\
\det\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} &= a_{11}a_{22} - a_{12}a_{21} \,, \\
\det\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} &= a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} \\
&\quad - a_{13}a_{22}a_{31} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} \,.
\end{aligned} \tag{3.3}$$

The best way to remember these formulas is visually.

The formula for the $2{\times}2$ determinant can be remembered as the product of the terms on the $\searrow$ diagonal minus the product of the terms on the $\swarrow$ diagonal. (Draw these two diagonal arrows on the above $2{\times}2$ matrix.)

The formula for the 3×3 determinant can be remembered by first augmenting the matrix by repeating the first two columns, thereby creating the 3×5 *augmented matrix*

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{11} & a_{12} \\ a_{21} & a_{22} & a_{23} & a_{21} & a_{22} \\ a_{31} & a_{32} & a_{33} & a_{31} & a_{32} \end{pmatrix}.$$

The formula is then the sum of the products of the terms on the $\searrow\searrow\searrow$ diagonals minus the sum of the products of the terms on the $\swarrow\swarrow\swarrow$ diagonals. (Draw these six diagonal arrows on the above 3×5 augmented matrix.)

3.2.2. *Laplace Expansions for General Matrices.* The determinant of the $n\times n$ matrix

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}$$

can be expanded in terms of the determinants of $(n-1)\times(n-1)$ submatrices of $\mathbf{A}$. Let $\mathbf{A}_{jk}$ denotes the $(n-1)\times(n-1)$ submatrix of $\mathbf{A}$ obtained by crossing out the j^{th} row and k^{th} column of $\mathbf{A}$. Then for any j we can expand $\det(\mathbf{A})$ about the j^{th} row of $\mathbf{A}$ as

$$\det(\mathbf{A}) = \sum_{k=1}^{n} (-1)^{j+k} a_{jk} \det(\mathbf{A}_{jk}), \tag{3.4}$$

while for any k we can expand $\det(\mathbf{A})$ about the k^{th} column of $\mathbf{A}$ as

$$\det(\mathbf{A}) = \sum_{j=1}^{n} (-1)^{j+k} a_{jk} \det(\mathbf{A}_{jk}). \tag{3.5}$$

These are the *Laplace expansions* for expanding a determinant. The computation can be reduced by expanding $\det(\mathbf{A})$ about a row or column with the most zero entries.

For $n = 2$ using formula (3.4) to expand about the first row ($j = 1$) gives

$$\det\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = a_{11}\det(a_{22}) - a_{12}\det(a_{21}) = a_{11}a_{22} - a_{12}a_{21}.$$

This is the formula for 2×2 determinants given in (3.3).

For $n = 3$ using formula (3.4) to expand about the first row ($j = 1$) gives

$$\begin{aligned} \det\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} &= a_{11}\det\begin{pmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{pmatrix} - a_{12}\det\begin{pmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{pmatrix} + a_{13}\det\begin{pmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{pmatrix} \\ &= a_{11}\big(a_{22}a_{33} - a_{23}a_{32}\big) - a_{12}\big(a_{21}a_{33} - a_{23}a_{31}\big) \\ &\qquad + a_{13}\big(a_{21}a_{32} - a_{22}a_{31}\big) \\ &= a_{11}a_{22}a_{33} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} + a_{12}a_{23}a_{31} \\ &\qquad + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31}. \end{aligned}$$

This is the formula for 3×3 determinants given in (3.3).

For $n = 4$ using formula (3.4) to expand about the first row ($j = 1$) gives

$$\det\begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{pmatrix} = a_{11}\det\begin{pmatrix} a_{22} & a_{23} & a_{24} \\ a_{32} & a_{33} & a_{34} \\ a_{42} & a_{43} & a_{44} \end{pmatrix} - a_{12}\det\begin{pmatrix} a_{21} & a_{23} & a_{24} \\ a_{31} & a_{33} & a_{34} \\ a_{41} & a_{43} & a_{44} \end{pmatrix}$$
$$+ a_{13}\det\begin{pmatrix} a_{21} & a_{22} & a_{24} \\ a_{31} & a_{32} & a_{34} \\ a_{41} & a_{42} & a_{44} \end{pmatrix} - a_{14}\det\begin{pmatrix} a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \\ a_{41} & a_{42} & a_{43} \end{pmatrix}.$$

Each of the 3×3 determinants above can be expanded further. The resulting formula for 4×4 determinants is a sum of 24 products.

Fully expanded, a similar formula for $n \times n$ determinants is a sum of $n!$ products. There is no simple "diagonal" picture that can be used to remember these formulas visually when $n > 3$. However the Laplace formulas (3.4) and (3.5) allow us to compute determinants without difficulty provided that either n is not too large or $\mathbf{A}$ has a simple structure.

Exercise. Prove the following evaluation of the determinant of a triangular $n \times n$ matrix

$$\det\begin{pmatrix} a_{11} & 0 & \cdots & 0 \\ a_{21} & a_{22} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ a_{n1} & \cdots & a_{n(n-1)} & a_{nn} \end{pmatrix} = a_{11}a_{22}\cdots a_{nn}\,.$$

Hint: Use the Laplace formula (3.4) and induction on n.

Remark. There are other ways to compute determinants that are much more efficient than the Laplace formulas (3.4) and (3.5) when n is not small and $\mathbf{A}$ does not have a simple structure. While these methods are not required for this course, they can be helpful. Therefore they are presented in Section 3.4.

3.3. **Existence of Solutions.** We now address the two questions posed in Section 3.1.

3.3.1. *Two Theorems.* Answers to the questions are given by the following theorems.

Theorem 3.1. System (3.1) has a unique solution for every forcing $b_1, b_2, \cdots, b_n$ if and only if $\det(A) \neq 0$.

Theorem 3.2. System (3.2) has a nonzero solution if and only if $\det(A) = 0$.

Remark. These theorems are used throughout the remainder of this course. Theorem 3.1 is used in Sections 2.3 – 2.5 to construct solutions of initial-value problems from general solutions of homogeneous linear differential equations. Theorem 3.2 is used in Section 2.6 to charaterize when solutions of homogeneous linear differential equations are linearly independent. They will be used again when we study first-order systems of differential equations.

Half of Theorem 3.1 is implied by Theorem 3.2. Indeed, Theorem 3.2 implies that if $\det(A) = 0$ then system (3.2) has a nonzero solution. As we showed in Section 3.1, the existence of such a solution implies that system (3.1) does not have a unique solution for any forcing. Therefore we will establish Theorem 3.2 first, after which all that we need to do to establish Theorem 3.1 is show that if $\det(A) \neq 0$ then system (3.1) has a unique solution.

Proofs of Theorem 3.1 and Theorem 3.2 for general n are beyond the scope of this course, so are not given here. They are covered in sufficiently advanced linear algebra courses. However, we give proofs of these theorems for the cases $n = 1$ and $n = 2$ below. You should be able to follow these proofs. While you are not expected to replicate these proofs, you are expected to know both theorems.

3.3.2. *Proof of Theorem 3.2.* When $n = 1$ system (3.2) is simply the single equation

$$a_{11} z_1 = 0\,. \tag{3.6}$$

Clearly, if $\det(A) = a_{11} = 0$ then every z_1 satisfies (3.6). Conversely, if $\det(A) = a_{11} \neq 0$ then $z_1 = 0$ is the only solution of (3.6). Therefore Theorem 3.2 holds for $n = 1$.

When $n = 2$ system (3.2) is the two equations

$$\begin{aligned} a_{11} z_1 + a_{12} z_2 &= 0\,, \\ a_{21} z_1 + a_{22} z_2 &= 0\,. \end{aligned} \tag{3.7}$$

Clearly, if $\det(A) = a_{11}a_{22} - a_{12}a_{21} = 0$ then both

$$z_1 = a_{22}\,, \quad z_2 = -a_{21}\,, \qquad \text{and} \qquad z_1 = -a_{12}\,, \quad z_2 = a_{11}\,,$$

give solutions of (3.7), at least one of which is nonzero unless $a_{11} = a_{12} = a_{21} = a_{22} = 0$. However, when $a_{11} = a_{12} = a_{21} = a_{22} = 0$ then any values of z_1 and z_2 satisfy (3.7). Hence, system (3.7) has a nonzero solution in either case.

Conversely, assume that $\det(A) = a_{11}a_{22} - a_{12}a_{21} \neq 0$. Eliminate z_2 by multiplying the first equation in (3.7) by a_{22}, the second by a_{12}, and then subtracting the results to obtain

$$\big(a_{11}a_{22} - a_{12}a_{21}\big) z_1 = 0\,. \tag{3.8a}$$

Similarly, eliminate z_1 by multiplying the second equation in (3.7) by a_{11}, the second by a_{21}, and then subtracting the results to obtain

$$\big(a_{11}a_{22} - a_{12}a_{21}\big) z_2 = 0\,. \tag{3.8b}$$

It follows from (3.8) that $z_1 = z_2 = 0$ is the only solution of system (3.7) because $\det(A) = a_{11}a_{22} - a_{12}a_{21} \neq 0$. Therefore Theorem 3.2 holds for $n = 2$. □

3.3.3. *Proof of Theorem 3.1.* As remarked earlier, one direction of Theorem 3.1 follows from Theorem 3.2. Indeed, Theorem 3.2 implies that if $\det(A) = 0$ then system (3.2) has a nonzero solution. As we showed in Section 3.1, the existence of such a solution implies that system (3.1) does not have a unique solution for any forcing. Therefore all remains to be done in order to establish Theorem 3.1 is to show that if $\det(A) \neq 0$ then system (3.1) has a unique solution.

When $n = 1$ system (3.1) is simply the single equation

$$a_{11}x_1 = b_1\,.$$

If $\det(A) = a_{11} \neq 0$ then this clearly has the unique solution

$$x_1 = \frac{b_1}{a_{11}}\,.$$

Therefore Theorem 3.1 holds for $n = 1$.

When $n = 2$ system (3.1) is the two equations

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 &= b_1\,, \\ a_{21}x_1 + a_{22}x_2 &= b_2\,. \end{aligned} \tag{3.9}$$

First eliminate x_2 by multiplying the first equation in (3.9) by a_{22}, the second by a_{12}, and then subtracting the results to obtain

$$\big(a_{11}a_{22} - a_{12}a_{21}\big)x_1 = b_1a_{22} - a_{12}b_2\,. \tag{3.10a}$$

Similarly, eliminate x_1 by multiplying the second equation in (3.9) by a_{11}, the second by a_{21}, and then subtracting the results to obtain

$$\big(a_{11}a_{22} - a_{12}a_{21}\big)x_2 = a_{11}b_2 - b_1a_{21}\,. \tag{3.10b}$$

It is clear from (3.10) that if $\det(A) = a_{11}a_{22} - a_{12}a_{21} \neq 0$ then for any choice of b_1 and b_2 the system (3.9) has the unique solution

$$x_1 = \frac{b_1a_{22} - a_{12}b_2}{a_{11}a_{22} - a_{12}a_{21}}\,, \qquad x_2 = \frac{a_{11}b_2 - b_1a_{21}}{a_{11}a_{22} - a_{12}a_{21}}\,. \tag{3.11}$$

Therefore Theorem 3.1 holds for $n = 2$. □

Remark. Formula (3.11) for the solution of system (3.9) can be expressed in terms of ratios of determinants as

$$x_1 = \frac{\det\begin{pmatrix} b_1 & a_{12} \\ b_2 & a_{22} \end{pmatrix}}{\det\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}}\,, \qquad x_2 = \frac{\det\begin{pmatrix} a_{11} & b_1 \\ a_{21} & b_2 \end{pmatrix}}{\det\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}}\,.$$

This is a special case of a more general formula called *Cramer's rule* that expresses the solution of the general $n\times n$ system (3.1) in terms of ratios of determinants. For example, the Cramer rule for $n = 3$ is

$$x_1 = \frac{\det\begin{pmatrix} b_1 & a_{12} & a_{13} \\ b_2 & a_{22} & a_{23} \\ b_3 & a_{32} & a_{33} \end{pmatrix}}{\det\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}}\,, \quad x_2 = \frac{\det\begin{pmatrix} a_{11} & b_1 & a_{13} \\ a_{21} & b_2 & a_{23} \\ a_{31} & b_3 & a_{33} \end{pmatrix}}{\det\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}}\,, \quad x_3 = \frac{\det\begin{pmatrix} a_{11} & a_{12} & b_1 \\ a_{21} & a_{22} & b_2 \\ a_{31} & a_{32} & b_3 \end{pmatrix}}{\det\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}}\,.$$

This is an inefficient way to find the solution of a linear algebraic system because each determinant has many terms in it that need to be computed. It is always faster to solve the linear system directly!

3.4. **Column and Row Operations.** The task of computing the determinant of a 3×3 or larger matrix can be eased by viewing it as an array of columns or rows.

3.4.1. *Column Operations.* A matrix $\mathbf{A}$ can be viewed as an array of columns as

$$\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix} = \begin{pmatrix} \mathbf{c}_1 & \mathbf{c}_2 & \cdots & \mathbf{c}_n \end{pmatrix} ,$$

where $\mathbf{c}_k$ is the k^{th} column of $\mathbf{A}$, which is given by

$$\mathbf{c}_k = \begin{pmatrix} a_{1k} \\ a_{2k} \\ \vdots \\ a_{nk} \end{pmatrix} , \qquad \text{for } k = 1, 2, \cdots, n .$$

We consider three types of *column operations* on the matrix.

- If $j < k$ then the j^{th} and k^{th} columns can be interchanged, so that the matrix

(3.12a)
$$\begin{pmatrix} \cdots & \mathbf{c}_j & \cdots & \mathbf{c}_k & \cdots \end{pmatrix} \quad \text{becomes} \quad \begin{pmatrix} \cdots & \mathbf{c}_k & \cdots & \mathbf{c}_j & \cdots \end{pmatrix} .$$

  This is the *column interchange* operation that we denote by $\mathbf{c}_j \leftrightarrow \mathbf{c}_k$.

- The j^{th} column can be multiplied by a nonzero scalar β, so that the matrix

(3.12b)
$$\begin{pmatrix} \cdots & \mathbf{c}_j & \cdots \end{pmatrix} \quad \text{becomes} \quad \begin{pmatrix} \cdots & \beta\mathbf{c}_j & \cdots \end{pmatrix} ,$$

  where the column $\beta\mathbf{c}_j$ is given by

$$\beta\mathbf{c}_j = \begin{pmatrix} \beta a_{1j} \\ \beta a_{2j} \\ \vdots \\ \beta a_{nj} \end{pmatrix} .$$

  This is the *column multiplication* operation that we denote by $\mathbf{c}_j \leftarrow \beta\mathbf{c}_j$.

- If $j \neq k$ then any multipule of the j^{th} column by a scalar γ can be added to the k^{th} column, so that when $j < k$ the matrix

(3.12c)
$$\begin{pmatrix} \cdots & \mathbf{c}_j & \cdots & \mathbf{c}_k & \cdots \end{pmatrix} \quad \text{becomes} \quad \begin{pmatrix} \cdots & \mathbf{c}_j & \cdots & \mathbf{c}_k + \gamma\mathbf{c}_j & \cdots \end{pmatrix} ,$$

  and when $k < j$ the matrix

(3.12d)
$$\begin{pmatrix} \cdots & \mathbf{c}_k & \cdots & \mathbf{c}_j & \cdots \end{pmatrix} \quad \text{becomes} \quad \begin{pmatrix} \cdots & \mathbf{c}_k + \gamma\mathbf{c}_j & \cdots & \mathbf{c}_j & \cdots \end{pmatrix} ,$$

  where in each case the column $\mathbf{c}_k + \gamma\mathbf{c}_j$ is given by

$$\mathbf{c}_k + \gamma\mathbf{c}_j = \begin{pmatrix} a_{1k} + \gamma a_{1j} \\ a_{2k} + \gamma a_{2j} \\ \vdots \\ a_{nk} + \gamma a_{nj} \end{pmatrix} .$$

  This is the *column addition* operation that we denote by $\mathbf{c}_k \leftarrow \mathbf{c}_k + \gamma\mathbf{c}_j$.

The useful fact is that determinants behave simply when a matrix is modified by any of these operations.

For the column interchange $\mathbf{c}_j \leftrightarrow \mathbf{c}_k$ we have

$$\det\begin{pmatrix}\cdots & \mathbf{c}_j & \cdots & \mathbf{c}_k & \cdots\end{pmatrix} = -\det\begin{pmatrix}\cdots & \mathbf{c}_k & \cdots & \mathbf{c}_j & \cdots\end{pmatrix} \quad \text{for } j < k\,. \tag{3.13a}$$

In other words, a column interchange changes the sign of the determinant.

For the column multiplication $\mathbf{c}_j \leftarrow \beta\mathbf{c}_j$ we have

$$\det\begin{pmatrix}\cdots & \mathbf{c}_j & \cdots\end{pmatrix} = \frac{1}{\beta}\det\begin{pmatrix}\cdots & \beta\mathbf{c}_j & \cdots\end{pmatrix}\,. \tag{3.13b}$$

In other words, a column multiplication changes the determinant by a factor of β.

For the column addition $\mathbf{c}_k \leftarrow \mathbf{c}_k + \gamma\mathbf{c}_j$ we have

$$\begin{aligned}\det\begin{pmatrix}\cdots & \mathbf{c}_j & \cdots & \mathbf{c}_k & \cdots\end{pmatrix} &= \det\begin{pmatrix}\cdots & \mathbf{c}_j & \cdots & \mathbf{c}_k + \gamma\mathbf{c}_j & \cdots\end{pmatrix} \quad \text{for } j < k\,,\\ \det\begin{pmatrix}\cdots & \mathbf{c}_k & \cdots & \mathbf{c}_j & \cdots\end{pmatrix} &= \det\begin{pmatrix}\cdots & \mathbf{c}_k + \gamma\mathbf{c}_j & \cdots & \mathbf{c}_j & \cdots\end{pmatrix} \quad \text{for } k < j\,,\end{aligned} \tag{3.13c}$$

In other words, a column addition does not change the determinant.

These rules are derived in linear algebra courses. They will not be derived here. Rather we will show how they can be used to compute determinants. The idea is to use them to reduce the matrix to the (easy) triangular case. For a lower trangular matrix we have

$$\det\begin{pmatrix} a_{11} & 0 & \cdots & 0 \\ a_{21} & a_{22} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ a_{n1} & \cdots & a_{n(n-1)} & a_{nn} \end{pmatrix} = a_{11}a_{22}\cdots a_{nn}\,,$$

while for an upper triangular matrix we have

$$\det\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ 0 & a_{22} & \ddots & a_{2n} \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & a_{nn} \end{pmatrix} = a_{11}a_{22}\cdots a_{nn}\,,$$

Example. Here we evaluate a 3×3 determinant using column operations.

$$\begin{aligned}\det\begin{pmatrix} 2 & 1 & 1 \\ 4 & 3 & 4 \\ 8 & 8 & 14 \end{pmatrix} &= 2\det\begin{pmatrix} 1 & 1 & 1 \\ 2 & 3 & 4 \\ 4 & 8 & 14 \end{pmatrix} && \boxed{\mathbf{c}_1 \leftarrow \tfrac{1}{2}\mathbf{c}_1} \\ &= 2\det\begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 2 \\ 4 & 4 & 10 \end{pmatrix} && \boxed{\begin{matrix}\mathbf{c}_2 \leftarrow \mathbf{c}_2 - \mathbf{c}_1 \\ \mathbf{c}_3 \leftarrow \mathbf{c}_3 - \mathbf{c}_1\end{matrix}} \\ &= 2\det\begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 4 & 4 & 2 \end{pmatrix} && \boxed{\mathbf{c}_3 \leftarrow \mathbf{c}_3 - \mathbf{c}_2} \\ &= 2 \cdot 2 = 4\,. \end{aligned}$$

The first step uses rule (3.13b) with $\beta = \frac{1}{2}$ to pull a factor of 2 from the first column. The second step uses rule (3.13c) with $\gamma = -1$ to make the top entries of the second and third columns zero. The third step uses rule (3.13c) with $\gamma = -2$ to make the middle entry of the third column zero. Finally, the determinant of the resulting lower triangular matrix was evaluated and answer was simplified.

Remark. Similar rules for row operations are given in the next subsection. Sometimes column operations are more useful for computing Wronskians, which are covered in Section 2.3.

Example. Compute the Wronskian $\mathrm{Wr}[U_1, U_2, U_3](t)$ for $U_1(t) = e^{2t}$, $U_2(t) = t\,e^{2t}$, and $U_3(t) = t^2 e^{2t}$. We have

$$\begin{aligned}
\mathrm{Wr}[U_1,U_2,U_3](t) &= \det\begin{pmatrix} U_1(t) & U_2(t) & U_3(t) \\ U_1'(t) & U_2'(t) & U_3'(t) \\ U_1''(t) & U_2''(t) & U_3''(t) \end{pmatrix} \\
&= \det\begin{pmatrix} e^{2t} & t\,e^{2t} & t^2e^{2t} \\ 2e^{2t} & e^{2t} + 2t\,e^{2t} & 2t\,e^{2t} + 2t^2e^{2t} \\ 4e^{2t} & 4e^{2t} + 4t\,e^{2t} & 2e^{2t} + 8t\,e^{2t} + 4t^2e^{2t} \end{pmatrix} \\
&= e^{2t} \cdot e^{2t} \cdot e^{2t} \det\begin{pmatrix} 1 & t & t^2 \\ 2 & 1+2t & 2t+2t^2 \\ 4 & 4+4t & 2+8t+4t^2 \end{pmatrix} \qquad \boxed{\begin{aligned} \mathbf{c}_1 &\leftarrow e^{-2t}\mathbf{c}_1 \\ \mathbf{c}_2 &\leftarrow e^{-2t}\mathbf{c}_2 \\ \mathbf{c}_3 &\leftarrow e^{-2t}\mathbf{c}_3 \end{aligned}} \\
&= e^{6t} \det\begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 2t \\ 4 & 4 & 2+8t \end{pmatrix} \qquad \boxed{\begin{aligned} \mathbf{c}_2 &\leftarrow \mathbf{c}_2 - t\mathbf{c}_1 \\ \mathbf{c}_3 &\leftarrow \mathbf{c}_3 - t^2\mathbf{c}_1 \end{aligned}} \\
&= e^{6t} \det\begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 4 & 4 & 2 \end{pmatrix} \qquad \boxed{\mathbf{c}_3 \leftarrow \mathbf{c}_3 - 2t\mathbf{c}_2} \\
&= 2e^{6t}\,.
\end{aligned}$$

3.4.2. *Row Operations.* A matrix $\mathbf{A}$ can be viewed as an array of rows as

$$\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix} = \begin{pmatrix} \mathbf{r}_1 \\ \mathbf{r}_2 \\ \vdots \\ \mathbf{r}_n \end{pmatrix},$$

where $\mathbf{r}_k$ is the k^{th} row of $\mathbf{A}$, which is given by

$$\mathbf{r}_k = \begin{pmatrix} a_{k1} & a_{k2} & \cdots & a_{kn} \end{pmatrix}, \qquad \text{for } k = 1, 2, \cdots, n\,.$$

We consider three types of *row operations* on the matrix.

- If $j < k$ then the j^{th} and k^{th} rows can be interchanged, so that the matrix

$$\text{(3.14a)} \qquad \begin{pmatrix} \vdots \\ \mathbf{r}_j \\ \vdots \\ \mathbf{r}_k \\ \vdots \end{pmatrix} \qquad \text{becomes} \qquad \begin{pmatrix} \vdots \\ \mathbf{r}_k \\ \vdots \\ \mathbf{r}_j \\ \vdots \end{pmatrix} .$$

  This is the *row interchange* operation that we denote by $\mathbf{r}_j \leftrightarrow \mathbf{r}_k$.

- The j^{th} row can be multiplied by a nonzero scalar β, so that the matrix

$$\text{(3.14b)} \qquad \begin{pmatrix} \vdots \\ \mathbf{r}_j \\ \vdots \end{pmatrix} \qquad \text{becomes} \qquad \begin{pmatrix} \vdots \\ \beta\mathbf{r}_j \\ \vdots \end{pmatrix} ,$$

  where the row $\beta\mathbf{r}_j$ is given by

$$\beta\mathbf{r}_j = \begin{pmatrix} \beta a_{j1} & \beta a_{j2} & \cdots & \beta a_{jn} \end{pmatrix} .$$

  This is the *row multiplication* operation that we denote by $\mathbf{r}_j \leftarrow \beta\mathbf{r}_j$.

- If $j \neq k$ then any multipule of the j^{th} row by a scalar γ can be added to the k^{th} row, so that when $j < k$ the matrix

$$\text{(3.14c)} \qquad \begin{pmatrix} \vdots \\ \mathbf{r}_j \\ \vdots \\ \mathbf{r}_k \\ \vdots \end{pmatrix} \qquad \text{becomes} \qquad \begin{pmatrix} \vdots \\ \mathbf{r}_j \\ \vdots \\ \mathbf{r}_k + \gamma\mathbf{r}_j \\ \vdots \end{pmatrix} ,$$

  and when $k < j$ the matrix

$$\text{(3.14d)} \qquad \begin{pmatrix} \vdots \\ \mathbf{r}_k \\ \vdots \\ \mathbf{r}_j \\ \vdots \end{pmatrix} \qquad \text{becomes} \qquad \begin{pmatrix} \vdots \\ \mathbf{r}_k + \gamma\mathbf{r}_j \\ \vdots \\ \mathbf{r}_j \\ \vdots \end{pmatrix} ,$$

  where in each case the row $\mathbf{r}_k + \gamma\mathbf{r}_j$ is given by

$$\mathbf{r}_k + \gamma\mathbf{r}_j = \begin{pmatrix} a_{k1} + \gamma a_{j1} & a_{k2} + \gamma a_{j2} & \cdots & a_{kn} + \gamma a_{jn} \end{pmatrix} .$$

  This is the *row addition* operation that we denote by $\mathbf{r}_k \leftarrow \mathbf{r}_k + \gamma\mathbf{r}_j$.

Determinants behave simply when a matrix is modified by any of these operations.

For the row interchange $\mathbf{r}_j \leftrightarrow \mathbf{r}_k$ we have

$$\det\begin{pmatrix} \vdots \\ \mathbf{r}_j \\ \vdots \\ \mathbf{r}_k \\ \vdots \end{pmatrix} = -\det\begin{pmatrix} \vdots \\ \mathbf{r}_k \\ \vdots \\ \mathbf{r}_j \\ \vdots \end{pmatrix} \quad \text{for } j < k\,. \tag{3.15a}$$

In other words, a row interchange changes the sign of the determinant.

For the row multiplication $\mathbf{r}_j \leftarrow \beta\mathbf{r}_j$ we have

$$\det\begin{pmatrix} \vdots \\ \mathbf{r}_j \\ \vdots \end{pmatrix} = \frac{1}{\beta}\det\begin{pmatrix} \vdots \\ \beta\mathbf{r}_j \\ \vdots \end{pmatrix}\,. \tag{3.15b}$$

In other words, a row multiplication changes the determinant by a factor of β.

For the row addition $\mathbf{r}_k \leftarrow \mathbf{r}_k + \gamma\mathbf{r}_j$ we have

$$\begin{aligned} \det\begin{pmatrix} \vdots \\ \mathbf{r}_j \\ \vdots \\ \mathbf{r}_k \\ \vdots \end{pmatrix} &= \det\begin{pmatrix} \vdots \\ \mathbf{r}_j \\ \vdots \\ \mathbf{r}_k + \gamma\mathbf{r}_j \\ \vdots \end{pmatrix} \quad \text{for } j < k\,, \\ \det\begin{pmatrix} \vdots \\ \mathbf{r}_k \\ \vdots \\ \mathbf{r}_j \\ \vdots \end{pmatrix} &= \det\begin{pmatrix} \vdots \\ \mathbf{r}_k + \gamma\mathbf{r}_j \\ \vdots \\ \mathbf{r}_j \\ \vdots \end{pmatrix} \quad \text{for } k < j\,, \end{aligned} \tag{3.15c}$$

In other words, a row addition does not change the determinant.

We now use these rules to compute the determinant of a 3×3 matrix.

Example. Here we evaluate a 3×3 determinant using column operations.

$$\begin{aligned} \det\begin{pmatrix} 2 & 1 & 1 \\ 4 & 3 & 4 \\ 8 & 8 & 14 \end{pmatrix} &= \det\begin{pmatrix} 2 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 4 & 10 \end{pmatrix} && \boxed{\begin{matrix} \mathbf{r}_2 \leftarrow \mathbf{r}_2 - 2\mathbf{r}_1 \\ \mathbf{r}_3 \leftarrow \mathbf{r}_3 - 4\mathbf{r}_1 \end{matrix}} \\ &= \det\begin{pmatrix} 2 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 2 \end{pmatrix} && \boxed{\mathbf{r}_3 \leftarrow \mathbf{r}_3 - 4\mathbf{r}_2} \\ &= 2 \cdot 2 = 4\,. \end{aligned}$$

The first step uses rule (3.15c) with $\gamma = -2$ and $\gamma = -4$ to make the first entries of the second and third row zero. The second step uses rule (3.15c) with $\gamma = -4$ to make the middle entry of the third row zero. Finally, the determinant of the resulting upper triangular matrix was evaluated and answer was simplified.

The Wronskian computed earlier with column operations can be computed just as easily with row operations.

Example. Compute the Wronskian $\mathrm{Wr}[U_1, U_2, U_3](t)$ for $U_1(t) = e^{2t}$, $U_2(t) = t\,e^{2t}$, and $U_3(t) = t^2 e^{2t}$. We have

$$\begin{aligned}
\mathrm{Wr}[U_1, U_2, U_3](t) &= \det\begin{pmatrix} U_1(t) & U_2(t) & U_3(t) \\ U_1'(t) & U_2'(t) & U_3'(t) \\ U_1''(t) & U_2''(t) & U_3''(t) \end{pmatrix} \\
&= \det\begin{pmatrix} e^{2t} & t\,e^{2t} & t^2 e^{2t} \\ 2e^{2t} & e^{2t} + 2t\,e^{2t} & 2t\,e^{2t} + 2t^2 e^{2t} \\ 4e^{2t} & 4e^{2t} + 4t\,e^{2t} & 2e^{2t} + 8t\,e^{2t} + 4t^2 e^{2t} \end{pmatrix} \\
&= e^{2t} \cdot e^{2t} \cdot e^{2t} \det\begin{pmatrix} 1 & t & t^2 \\ 2 & 1 + 2t & 2t + 2t^2 \\ 4 & 4 + 4t & 2 + 8t + 4t^2 \end{pmatrix} && \boxed{\begin{array}{l} \mathbf{r}_1 \leftarrow e^{-2t}\mathbf{r}_1 \\ \mathbf{r}_2 \leftarrow e^{-2t}\mathbf{r}_2 \\ \mathbf{r}_3 \leftarrow e^{-2t}\mathbf{r}_3 \end{array}} \\
&= e^{6t} \det\begin{pmatrix} 1 & t & t^2 \\ 0 & 1 & 2t \\ 0 & 4 & 2 + 8t \end{pmatrix} && \boxed{\begin{array}{l} \mathbf{r}_2 \leftarrow \mathbf{r}_2 - 2\mathbf{r}_1 \\ \mathbf{r}_3 \leftarrow \mathbf{r}_3 - 4\mathbf{r}_1 \end{array}} \\
&= e^{6t} \det\begin{pmatrix} 1 & t & t^2 \\ 0 & 1 & 2t \\ 0 & 0 & 2 \end{pmatrix} && \boxed{\mathbf{r}_3 \leftarrow \mathbf{r}_3 - 4\mathbf{c}_2} \\
&= 2e^{6t}\,.
\end{aligned}$$

Exercises on Linear Algebraic Systems and Determinants

(1) Calculate $\det(\mathbf{A})$ for the following matrices.

(a) $\mathbf{A} = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$

(b) $\mathbf{A} = \begin{pmatrix} 5 & 7 \\ -3 & 2 \end{pmatrix}$

(c) $\mathbf{A} = \begin{pmatrix} 3 & -1 & 2 \\ 2 & 1 & 1 \\ 4 & -5 & 3 \end{pmatrix}$

(d) $\mathbf{A} = \begin{pmatrix} 1 & 1 & \pi \\ 2 & -1 & 1 \\ 1 & -\pi & 6 \end{pmatrix}$

Short Answer
Solution

(2) Calculate $\det(B)$ for the following matrices.

(a) $\mathbf{B} = \begin{pmatrix} 2 & 3 & 4 \\ 0 & -5 & 7 \\ 0 & 0 & 19 \end{pmatrix}$

(b) $\mathbf{B} = \begin{pmatrix} 1 & 4 & 0 \\ 0 & 0 & 2 \\ 4 & -2 & 5 \end{pmatrix}$

(c) $\mathbf{B} = \begin{pmatrix} 3 & 3 & 3 \\ -1 & -1 & -1 \\ 10 & 4 & 1 \end{pmatrix}$

(d) $\mathbf{B} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$

Short Answer
Solution

(3) Decide whether the following systems have nonzero solutions.

(a) $\begin{cases} 2x + 4y = 0 \\ 3x + 2y = 0 \end{cases}$

(b) $\begin{cases} 10x + 5y + 7z = 0 \\ 6x + y + 9z = 0 \\ 2x + 2y - z = 0 \end{cases}$

(c) $\begin{cases} x_1 + 2x_2 = 0 \\ x_2 + 2x_3 = 0 \\ x_3 + 2x_1 = 0 \end{cases}$

Short Answer
Solution

(4) Decide whether the following systems have unique solutions.

(a) $\begin{cases} 4x + 10y = 6 \\ -2x - 5y = -3 \end{cases}$

(b) $\begin{cases} x + y = 0 \\ y + z = 0 \\ x + y + z = 1 \end{cases}$

(c) $\begin{cases} 2x_1 - 3x_2 + 5x_3 = 8 \\ 3x_1 - 3x_2 + 9x_3 = 12 \\ x_1 - 2x_2 + 2x_3 = 6 \end{cases}$

Short Answer
Solution

(5) As suggested in the text, show that the determinant of a lower triangular matrix is the product of its diagonal entries. Is the same true of upper triangular matrices?

Solution

(6) Suppose that the system of equations

$$\begin{cases} a_{11}z_1 + a_{12}z_2 + \cdots + a_{1n}z_n = 0 \\ a_{21}z_1 + a_{22}z_2 + \cdots + a_{2n}z_n = 0 \\ \vdots \\ a_{n1}z_1 + a_{n2}z_2 + \cdots + a_{nn}z_n = 0 \end{cases}$$

has a nonzero solution $(z_1, \cdots, z_n)$. Let $(b_1, \cdots, b_n)$ be n real numbers (this will be our forcing terms), and suppose that the list of numbers $(y_1, \cdots, y_n)$ satisfies the equations

$$(\star) \qquad \begin{cases} a_{11}y_1 + a_{12}y_2 + \cdots + a_{1n}y_n = b_1 \\ a_{21}y_1 + a_{22}y_2 + \cdots + a_{2n}y_n = b_2 \\ \vdots \\ a_{n1}y_1 + a_{n2}y_2 + \cdots + a_{nn}y_n = b_n \end{cases}$$

The goal of this problem is to show that the solution $(y_1, \cdots, y_n)$ is necessarily not unique. To do this, we'll actually produce infinitely many other solutions of $(\star)$. Let α be any real number, and let $x_i = y_i + \alpha z_i$. Show that $(x_1, \cdots, x_n)$ also satisfies $(\star)$. [This was a step in the proof of Theorems 2.16 and 2.17.]

Solution

(7) Let $\mathbf{A}$ be an $n \times n$ matrix, say

$$\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix},$$

and suppose that $\mathbf{A}$ has a column of zeros. For concreteness let it be the first column: $a_{11} = a_{21} = \cdots = a_{n1} = 0$. Show that the system

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = 0 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = 0 \\ \qquad\qquad\qquad\qquad\qquad \vdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n = 0 \end{cases}$$

has a nonzero solution by explicitly giving one. [*Note.* This implies by Theorem A.2 that $\det(A) = 0$.]

Solution

(8) Let $\mathbf{A}$ be an $n \times n$ matrix as in the previous problem, say

$$\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix},$$

and this time suppose that $\mathbf{A}$ has a *row* of zeros. For concreteness let it be the first row: $a_{11} = a_{12} = \cdots = a_{1n} = 0$. Explain why $\det(\mathbf{A}) = 0$.

[*Note.* This time the solution to the system of linear equations is not straightforward to write down, but we know from this problem that one is out there somewhere.]

Solution

(9) One might naïvely hope that the determinant function is linear, i.e., $\det(\mathbf{A} + \mathbf{B}) = \det(\mathbf{A}) + \det(\mathbf{B})$. For 2×2 matrices, this is the hope that

$$\det\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} + \det\begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} \stackrel{?}{=} \det\begin{pmatrix} a_{11} + b_{11} & a_{12} + b_{12} \\ a_{21} + b_{21} & a_{22} + b_{22} \end{pmatrix}.$$

Unfortunately, this is flagrantly false. Find two 2×2 matrices $\mathbf{A}$ and $\mathbf{B}$ for which $\det(\mathbf{A} + \mathbf{B}) \neq \det(\mathbf{A}) + \det(\mathbf{B})$.

Solution

(10) One might also naïvely hope that you can pull scalars through a determinant in the sense that $\det(k\mathbf{A}) = k\det(A)$. For 2×2 matrices, this is the hope that

$$\det\begin{pmatrix} ka_{11} & ka_{12} \\ ka_{21} & ka_{22} \end{pmatrix} \stackrel{?}{=} k\det\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}.$$

This too is false, but this time something similar is true. What is the relationship between $\det(k\mathbf{A})$ and $\det(\mathbf{A})$ if $\mathbf{A}$ is a 2×2 matrix? (What if $\mathbf{A}$ is an $n \times n$ matrix?)

Solution

(11) Calculate the determinant for the following matrices:

(a) $\mathbf{X} = \begin{pmatrix} x & y \\ 0 & t \end{pmatrix}$,

(b) $\mathbf{Y} = \begin{pmatrix} \alpha & 1 \\ 2 & 5 \end{pmatrix}$, where α, x, y, and t are scalar parameters. Under what conditions is the determinant nonzero for each matrix?

Short Answer
Solution

(12) Calculate the determinant for the following matrices:

(a) $\mathbf{M} = \begin{pmatrix} 0 & 2 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & \alpha \end{pmatrix}$,

(b) $\mathbf{N} = \begin{pmatrix} 2 & 2 & \frac{5}{7} \\ a & b & c \\ 4 & 4 & \frac{10}{7} \end{pmatrix}$,

(c) $\mathbf{P} = \begin{pmatrix} 1 & 2 & 3 \\ 0 & \beta & 0 \\ 1 & 0 & 5 \end{pmatrix}$,

where a, b, c, e, f, α, and β are scalar parameters.

Under what conditions is the determinant in (c) nonzero?

Short Answer
Solution

(13) Decide under which conditions the following systems have unique solutions (here, a, b, and α are scalars, while x and y are the unknowns).

(a)

$$a \cdot x + b \cdot y = 0,$$
$$x + y = 1.$$

(b)

$$x + y = 0,$$
$$y + z = 0,$$
$$\alpha \cdot x + y + z = 0.$$

Short Answer
Solution

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