II. Higher-Order Linear Ordinary Differential Equations
5. Nonhomogeneous Equations: General Methods and Theory

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5. NONHOMOGENEOUS EQUATIONS: GENERAL METHODS AND THEORY

We are now ready to study nonhomogeneous linear equations. An n^{th}-order nonhomogeneous linear ordinary differential equation in the normal form can be expressed as

$$\mathrm{L}(t)y = f(t)\,, \tag{5.1}$$

where the linear differential operator $\mathrm{L}(t)$ has the form

$$\mathrm{L}(t) = \mathrm{D}^n + a_1(t)\mathrm{D}^{n-1} + \cdots + a_{n-1}(t)\mathrm{D} + a_n(t)\,, \qquad \text{where} \quad \mathrm{D} = \frac{d}{dt}\,. \tag{5.2}$$

We will assume throughout this chapter that the coefficients $a_1, a_2, \cdots, a_n$ and the forcing f are continuous over an interval (t_L, t_R), so that Therorem 1.1 can be applied.

Remark. Here we indicate that the differential operator above might depend upon t through its coefficients by writing $\mathrm{L}(t)$. In cases when the coefficients $a_1, a_2, \cdots, a_n$ are all constants we will write L. This was the case in the previous chapter where the operator notation was used to develop the Key Identity.

5.1. Particular and General Solutions.

Here we will use the linearity of $\mathrm{L}(t)$ to develop a general framework for solving nonhomogeneous linear equations in the form (5.1). Specifically, the operator $\mathrm{L}(t)$ has the property that if $Y_1(t)$, $Y_2(t)$, $\cdots$, $Y_m(t)$ are any n-times differentiable functions and c_1, c_2, $\cdots$, c_m are any constants then

$$\mathrm{L}(t)\big(c_1Y_1(t)+c_2Y_2(t)+\cdot+c_mY_m(t)\big) = c_1\mathrm{L}(t)Y_1(t)+c_2\mathrm{L}(t)Y_2(t)+\cdot+c_m\mathrm{L}(t)Y_m(t)\,. \tag{5.3}$$

This is the property of linearity that we introduced in Chapter 2.

Remark. The linearity of $\mathrm{L}(t)$ was used in Chapter 2 to developed the method of linear superposition for constructing general solutions of homogeneous linear equations.

We will develop a general framework for solving nonhomogeneous linear equations in the form (5.1) by exploiting the following facts.

Theorem 5.1. If $Y_1(t)$ and $Y_2(t)$ are solutions of (5.1) then $Z(t) = Y_1(t) - Y_2(t)$ is a solution of the associated homogeneous equation $\mathrm{L}(t)Z(t) = 0$.

Proof. Because $\mathrm{L}(t)Y_1(t) = f(t)$ and $\mathrm{L}(t)Y_2(t) = f(t)$, the linearity (5.3) of $\mathrm{L}(t)$ implies

$$\mathrm{L}(t)Z(t) = \mathrm{L}(t)\big(Y_1(t) - Y_2(t)\big) = \mathrm{L}(t)Y_1(t) - \mathrm{L}(t)Y_2(t) = f(t) - f(t) = 0\,.$$

Theorem 5.2. If $Y_P(t)$ is a solution of (5.1) and $Y_H(t)$ is a solution of the associated homogeneous equation $\mathrm{L}(t)Y_H(t) = 0$ then $Y(t) = Y_H(t) + Y_P(t)$ is also a solution of (5.1).

Proof. Because $\mathrm{L}(t)Y_H(t) = 0$ and $\mathrm{L}(t)Y_P(t) = f(t)$, the linearity (5.3) of $\mathrm{L}(t)$ implies

$$\mathrm{L}(t)Y(t) = \mathrm{L}(t)\big(Y_H(t) + Y_P(t)\big) = \mathrm{L}(t)Y_H(t) + \mathrm{L}(t)Y_P(t) = 0 + f(t) = f(t)\,.$$

Theorem 5.2 suggests that we can construct general solutions of the nonhomogeneous equation (5.1) as follows.

1. Find a general solution $Y_H(t)$ of the *associated homogeneous equation* $\mathrm{L}(t)y = 0$.
2. Find a *particular solution* $Y_P(t)$ of equation (5.1).
3. Then $Y_H(t) + Y_P(t)$ is a general solution of equation (5.1).

Of course, step (1) reduces to finding a fundamental set of solutions of the associated homogeneous equation, $Y_1, Y_2, \cdots, Y_n$. Then

$$Y_H(t) = c_1 Y_1(t) + c_2 Y_2(t) + \cdots + c_n Y_n(t)\,.$$

If $\mathrm{L}(t)$ has constant coefficients (so that $\mathrm{L}(t) = \mathrm{L}$) then this can be done by the recipe of Chapter 4.

Example. We can check that $\frac{1}{4}t$ is a particular solution of

$$y'' + 4y = t\,.$$

This equation has constant coefficients. Its characteristic polynomial is $p(z) = z^2 + 4$, which has roots $\pm i2$. Therefore a general solution is

$$y = c_1 \cos(2t) + c_2 \sin(2t) + \tfrac{1}{4}t\,.$$

Example. We can check that $-\frac{1}{2}e^t$ is a particular solution of

$$v'' - v' - 2v = e^t\,.$$

This equation has constant coefficients. Its characteristic polynomial is $p(z) = z^2 - z - 2 = (z-2)(z+1)$, which has roots -1 and 2. Therefore a general solution is

$$v = c_1 e^{-t} + c_2 e^{2t} - \tfrac{1}{2}e^t\,.$$

These examples show that when $\mathrm{L}(t)$ has constant coefficients (so that $\mathrm{L}(t) = \mathrm{L}$), finding $Y_P(t)$ becomes the crux of matter. In Chapter 6 we will study methods for finding $Y_P(t)$ for equations with constant coefficients. If $\mathrm{L}(t)$ has variable coefficients then a fundamental set of solutions of the associated homogeneous equation will generally be given to you. In that case, finding $Y_P(t)$ again becomes the crux of matter. In Chapter 7 we will study methods for finding $Y_P(t)$ for equations with variable coefficients when a fundamental set of solutions of the associated homogeneous equation is known.

5.2. **Solutions of Initial-Value Problems.** An initial-value problem associated with an n^{th}-order nonhomogeneous linear equation has the form

$$\begin{gathered} \mathrm{D}^n y + a_1(t)\mathrm{D}^{n-1}y + \cdots + a_{n-1}(t)\mathrm{D}y + a_n(t)y = f(t)\,, \\ y(t_I) = y_0\,, \qquad y'(t_I) = y_1\,, \qquad \cdots \qquad y^{(n-1)}(t_I) = y_{n-1}\,, \end{gathered} \tag{5.4}$$

where t_I is the initial time and y_0, y_1, $\cdots$, y_{n-1} are the initial data.

Given a particular solution $Y_P(t)$ of the nonhomogeneous equation and a fundamental set of solutions of the associated homogeneous equation, Y_1, Y_2, $\cdots$, Y_n, we first construct a general solution of the nonhomogeneous equation as

$$Y(t) = c_1 Y_1(t) + c_2 Y_2(t) + \cdots + c_n Y_n(t) + Y_P(t)\,.$$

We then determine the values of the parameters $c_1, \cdots, c_n$ by requiring that this solution satisfy the initial conditions. This leads to a linear algebraic system of the form

$$\begin{aligned} c_1Y_1(t_I) &+ c_2Y_2(t_I) + \cdots + c_nY_n(t_I) = y_0 - Y_P(t_I)\,, \\ c_1Y_1'(t_I) &+ c_2Y_2'(t_I) + \cdots + c_nY_n'(t_I) = y_1 - Y_P'(t_I)\,, \\ &\vdots \\ c_1Y_1^{(n-1)}(t_I) &+ c_2Y_2^{(n-1)}(t_I) + \cdots + c_nY_n^{(n-1)}(t_I) = y_{n-1} - Y_P^{(n-1)}(t_I)\,. \end{aligned} \tag{5.5}$$

Because Y_1, Y_2, $\cdots$, Y_n, is a fundamental set of solutions of the associated homogeneous equation, their Wronskian $\mathrm{Wr}[Y_1, Y_2, \cdots, Y_n]$ is always nonzero. Therefore we can solve the linear algebraic system (5.5) for any initial time t_I and any initial data y_0, y_1, $\cdots$, y_{n-1}, and thereby always find the unique solution of the initial-value problem (5.4).

Remark. It is important to notice that the linear algebraic system (5.5) that determines the values of c_1, c_2, $\cdots$, c_n differs from the linear algebraic system 2.5 for solving homogeneous initial-value problems by the terms involving Y_P appearing on the right-hand side. This means that generally we will get the wrong answer if we apply the initial conditions to $Y_H(t)$, which is the general solution of the associated homogeneous equations, rather than applying it to $Y(t) = Y_H(t)+Y_P(t)$, which is the general solution to the nonhomogeneous equation.

Example. Solve the initial-value problem

$$y'' + 4y = t\,, \qquad y(0) = 3\,, \quad y'(0) = -1\,.$$

Solution. We saw earlier that a general solution of the nonhomogeneous equation is

$$y(t) = c_1\cos(2t) + c_2\sin(2t) + \tfrac{1}{4}t\,.$$

Because

$$y'(t) = -2c_1\sin(2t) + 2c_2\cos(2t) + \tfrac{1}{4}\,,$$

the initial conditions yield

$$y(0) = c_1 = 3\,, \qquad y'(0) = 2c_2 + \tfrac{1}{4} = -1\,.$$

These can be solved to find $c_1 = 3$ and $c_2 = -\frac{5}{8}$. Therefore the solution of the initial-value problem is

$$y(t) = 3\cos(2t) - \tfrac{5}{8}\sin(2t) + \tfrac{1}{4}t\,.$$

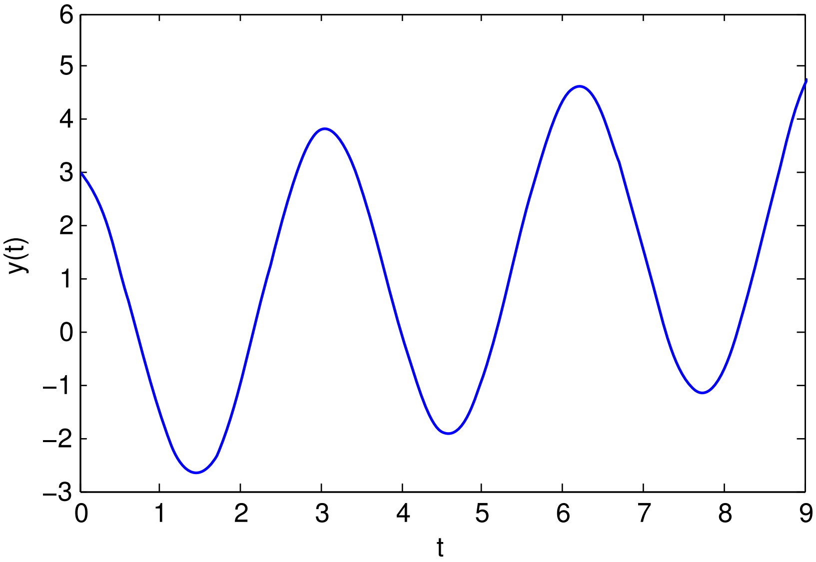

FIGURE 5.1. Solution to the initial-value problem $y'' + 4y = t$, $y(0) = 3$, $y'(0) = -1$.

Example. Solve the initial-value problem

$$v'' - v' - 2v = e^t\,, \qquad v(0) = -2\,, \quad v'(0) = 5\,.$$

Solution. We saw earlier that a general solution of the nonhomogeneous equation is

$$v(t) = c_1 e^{-t} + c_2 e^{2t} - \tfrac{1}{2} e^t\,.$$

Because

$$v'(t) = -c_1 e^{-t} + 2c_2 e^{2t} - \tfrac{1}{2} e^t\,,$$

the initial conditions yield

$$v(0) = c_1 + c_2 - \tfrac{1}{2} = -2\,, \qquad v'(0) = -c_1 + 2c_2 - \tfrac{1}{2} = 5\,,$$

which is equivalent to the system

$$c_1 + c_2 = -\tfrac{3}{2}\,, \qquad -c_1 + 2c_2 = \tfrac{11}{2}\,.$$

This can be solved to find $c_1 = -\frac{17}{6}$ and $c_2 = \frac{4}{3}$. Therefore the solution of the initial-value problem is

$$v(t) = -\tfrac{17}{6} e^{-t} + \tfrac{4}{3} e^{2t} - \tfrac{1}{2} e^t\,.$$

Remark. It is worth repeating that when solving an initial-value problem for a nonhomogeneous equation, the initial conditions should be applied to a general solution of the nonhomogeneous equation! The solution $Y_H(t) + Y_P(t)$ *will not* satisfy the initial conditions when the solution $Y_H(t)$ of the associated homogeneous equation does *unless* the particular solution $Y_P(t)$ happens to satisfy the so-called *homogeneous initial conditions*

$$Y_P(t_I) = Y_P'(t_I) = \cdots = Y_P^{(m-1)}(t_I) = 0\,.$$

EXERCISES ON NONHOMOGENEOUS EQUATIONS

For this first group of problems:
(a) Show the given Y_P is a solution to the non-homogeneous equation
(b) Find a general solution to the equation.

(1) $\ddot{w} - \dot{w} - 2w = 2e^u$ with $W_P(u) = -e^u$.
Solution

(2) $y'' + 4y = \sin(t)$ with $Y_P(t) = \frac{1}{3}\sin(t)$.
Solution

(3) $w'' + w = 2\cos(x)$ with $W_P(x) = x\sin(x)$.
Solution

(4) $y'' + 2y' + y = 3e^{2t}$ with $Y_P(t) = \frac{1}{3}e^{2t}$.
Solution

(5) $x'' + 2x' + 2x = e^{-t}$ with $X_P(t) = e^{-t}$.
Solution

(6) $y'' - 6y' + 9y = 2e^{3t}$ with $Y_P(t) = t^2e^{3t}$.
Solution

(7) $\ddot{w} - 2\dot{w} + w = -e^z/z^2$ with $W_P(z) = e^z\log(z)$.
Solution

(8) $t^2y'' - (t^2 + 2t)y' + (t + 2)y = 2t^3$ with $Y_P(t) = -2t^2$ (For part (b) note that $y_1(t) = t$ and $y_2(t) = te^t$ are both solutions to the homogeneous equation).
Solution

In the next group of problems show the solution given is a solution to the equation and then use it to find the solution to the given initial value problem.

(9) $y'' + 4y' + 4y = 9e^x$ with $Y_P(x) = e^x$ where $y(0) = 3$ and $y'(0) = 4$.
Short Answer
Solution

(10) $y'' - 4y = 5e^{3t}$ with $Y_P(t) = e^{3t}$ where $y(0) = 3$ and $y'(0) = 0$.
Short Answer
Solution

(11) $w'' - 2w' - 3w = 3e^{2u}$ with $W_P(u) = e^{3u} - e^{2u}$ where $w(0) = 2$ and $w'(0) = 1$.
Short Answer
Solution

(12) $y'' - 2y' + 6y = -29\cos(t)$ with $Y_p(t) = 2\sin(t) - 5\cos(t)$ where $y(0) = 1$ and $y'(0) = 2$.

Short Answer
Solution

(13) $z'' - z = 4we^w$ with $Z_P(w) = w^2e^w - we^w$ where $z(0) = 2$ and $z'(0) = -1$.

Short Answer
Solution

(14) $y^{(4)} - 3y'' - 4y = 2t$ with $Y_P(t) = -t/2$ where $y(0) = 10$, $y'(0) = 3/2$, $y''(0) = 0$, and $y'''(0) = -2$.

Short Answer
Solution

(15) $x''' - x'' - x' + x = 3e^{2u}$ with $X_P(u) = e^{2u}$ where $x(0) = 1$, $x'(0) = 2$ and $x''(0) = 4$.

Short Answer
Solution

(16) $t^2y'' - ty' + y = t^2$ with $Y_P(t) = t^2$ and $y_1(t) = t$ is a solution to the homogeneous equation, and where $y(1) = 1$ and $y'(1) = 0$ (Note: you will have to first find a second independent solution to the homogeneous equation).

Short Answer
Solution

Use reduction of order to find general solutions to the following non-homgeneous equations. That is given a solution y_1 to the corresponding homogeneous equation set $y_2 = y_1v$, and reduce the second order non-homogeneous equations to a first order equation.

(17) $z^2w'' - 2zw' + 2w = 4z^2$ where $w_1(z) = z$.

Short Answer
Solution

(18) $x^2\ddot{w} + 7x\dot{w} + 5w = x$ where $w_1(x) = \frac{1}{x}$.

Short Answer
Solution

In the following please justify your responses

(19) The recipe for finding a general solution to a non-homogeneous equation is to first find a single solution to the non-homogeneous equation Y_P and then add it to the general solution for the corresponding homogeneous equation Y_H.

(a) Justify why $Y_P(t) + Y_H(t)$ will be a solution for any homogeneous solution Y_H.

(b) Suppose that we also have an initial condition. Can we solve the initial value problem by adding a solution to the non-homogeneous equation and adding it to the solution to the corresponding homogeneous initial value problem? Why or why not?

Solution

(20) This problem touches on the method of annihilators. The next section depends on having the non-homogeneous part be a solution to some homogeneous equation. It gives us a way to find a non-homogeneous solution when the non-homogeneous part has the special property that it is annihilated by a differential operator.

(a) Suppose that $y'' - y = g(t)$ is such that $g(t)$ is a solution to the homogeneous equation $y'' - y = 0$. Justify why a solution to $y'' - y = g(t)$ is also a solution to the homogeneous equation $y^{(4)} - 2y'' + y = 0$.

(b) More generally suppose that L_1 and L_2 are differential operators such that $\mathrm{L}_1 Y_P = g$ and $\mathrm{L}_2 g = 0$. Justify why Y_P is a solution to $\mathrm{L}_2\mathrm{L}_1 y = 0$.

(c) If Y is the general solution to $\mathrm{L}_2\mathrm{L}_1 y = 0$, and z is the general solution to $\mathrm{L}_1 y = 0$, and g is a solution to $\mathrm{L}_2 y = 0$, justify why we can find coefficients such that $Y - z$ is a solution to $\mathrm{L}_1 y = g$.

Solution

(21) This is a two-part problem.

a) What should $h(x)$ be so that $y(x)$ is a solution to

$$y'' - 2y' + y = h(x),$$

when $i)y(x) = \cos(2x)$? $ii)y(x) = \sin(2x)$?

b) Using the results obtained from a), find a particular solution $y_p(x)$ and a general soluton to each of the following equations: $i)y'' - 2y' + y = \cos(2x)$ $ii)y'' - 2y' + y = \sin(2x)$.

(**Hint** : Here you will have to make an educated guess for a particular solution. Think about a linear combination of solutions from i) and ii) and solve for the unknown coefficients of that linear combination.)

Remark : This foreshadows the appearance of the method of ”Undetermined Coefficients” from Chapter 6.

Short Answer
Solution

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