II. Higher-Order Linear Ordinary Differential Equations
6. Nonhomogeneous Equations with Constant Coefficients

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6. Nonhomogeneous Equations with Constant Coefficients

This chapter gives methods by which we can construct particular solutions to an n^{th}-order nonhomogeneous linear ordinary differential equation

$$\mathrm{L}y = f(t)\,, \tag{6.1}$$

when the differential operator L has *constant coefficients* and is in normal form,

$$\mathrm{L} = \mathrm{D}^n + a_1\mathrm{D}^{n-1} + \cdots + a_{n-1}\mathrm{D} + a_n\,. \tag{6.2}$$

The previous chapter showed that constructing a particular solution of (6.1) is the key step either in finding a general solution of (6.1) or in solving an initial-value problem associated with (6.1).

The first methods we cover are *Key Identity Evaluations* and *Undetermined Coefficients*. These methods are related. They require the forcing $f(t)$ to have a special form. These methods generally provide the fastest route to find a particular solution whenever $f(t)$ has this special form. Because this special form often arises in applications, you should learn to identify it and to master at least one of these methods.

We then show how linearity can be used to construct particular solutions when the forcing $f(t)$ is a sum of terms, each of which has the special form that allows the first two methods to be applied.

The last method we cover is *Green Functions*. It can be applied to any forcing $f(t)$, but does not yield an explicit particular solution. Rather, it reduces the problem of computing a particular solution to that of evaluating n integrals. Because evaluating integrals takes time, this method should only be applied when the first two methods cannot be applied.

6.1. Degree, Characteristic, and Multiplicity.

Either *Key Identity Evaluations* or *Undetermined Coefficients* can be used to find a particular solution of equation (6.1) provided the following conditions are met.

1. The differential operator L has constant coefficients.
2. The forcing $f(t)$ can be expressed in the form

$$\begin{aligned} f(t) = &\big(\alpha_0 t^d + \alpha_1 t^{d-1} + \cdots + \alpha_d\big)e^{\mu t}\cos(\nu t) \\ &+ \big(\beta_0 t^d + \beta_1 t^{d-1} + \cdots + \beta_d\big)e^{\mu t}\sin(\nu t)\,, \end{aligned} \tag{6.3}$$

   where d is a nonnegative integer, while α_0, α_1, $\cdots$, α_d and β_0, β_1, $\cdots$, β_d, and μ and ν are real numbers. Here we assume that either $\alpha_0 \neq 0$ or $\nu\beta_0 \neq 0$, so that at least one of the t^d terms is nonzero.

When $f(t)$ can be expressed in the form (6.3) we say that it has *characteristic form*.

Remark. If $\nu = 0$ then each β_k can be anything because $\nu = 0$ implies that $\sin(\nu t) = 0$ for every t. In this case we will set every β_k in (6.3) to zero. Then the α_k are uniquely determined by the linear independence of the functions

$$t^d e^{\mu t}\,, \qquad t^{d-1}e^{\mu t}\,, \qquad \cdots \qquad t\,e^{\mu t}\,, \qquad e^{\mu t}\,.$$

Remark. If $\nu \neq 0$ then we will restrict to $\nu > 0$ because the form (6.3) remains unchanged if we replace ν with $-\nu$ and each β_k with $-\beta_k$. The restriction $\nu > 0$ implies that the α_k and β_k are uniquely determined by the linear independence of the functions

$$\begin{gathered} t^d e^{\mu t}\cos(\nu t)\,, \quad t^{d-1}e^{\mu t}\cos(\nu t)\,, \quad \cdots \quad t\,e^{\mu t}\cos(\nu t)\,, \quad e^{\mu t}\cos(\nu t)\,, \\ t^d e^{\mu t}\sin(\nu t)\,, \quad t^{d-1}e^{\mu t}\sin(\nu t)\,, \quad \cdots \quad t\,e^{\mu t}\sin(\nu t)\,, \quad e^{\mu t}\sin(\nu t)\,. \end{gathered}$$

How the methods of *Key Identity Evaluations* and *Undetermined Coefficients* are applied depends upon three numbers. The first two of these numbers can be read off from the characteristic form (6.3).

- The nonnegative integer d is called the *degree* of $f(t)$. When $d = 0$ we say that $f(t)$ has *zero degree.* When $d > 0$ we say that $f(t)$ has *positive degree.*
- The complex number $\mu + i\nu$ is called the *characteristic* of $f(t)$. When $\nu = 0$ we say that $f(t)$ has *real characteristic form.* When $\nu > 0$ we say that $f(t)$ has *complex characteristic form.*

The third number associated with $f(t)$ is a nonnegative integer m called the *multiplicity* of $\mu + i\nu$. It depends upon $p(z)$, the characteristic polynomial of L.

- We say that the characteristic $\mu + i\nu$ has multiplicity 0 if $\mu + i\nu$ is not a root of $p(z)$ — i.e. if $p(\mu + i\nu) \neq 0$.
- We say that the characteristic $\mu + i\nu$ has multiplicity m for some $m > 0$ if $\mu + i\nu$ is a root of $p(z)$ with multiplicity m — i.e. if

$$p^{(k)}(\mu + i\nu) = 0 \quad \text{for every } k < m \qquad \text{and} \qquad p^{(m)}(\mu + i\nu) \neq 0\,.$$

Remark. The multiplicity of $\mu + i\nu$ is easy to determine if the roots of $p(z)$ are already listed with their multiplicities. Then the multiplicity of $\mu + i\nu$ is simply the number of times $\mu + i\nu$ appears on the list of roots of $p(z)$.

When faced with a nonhomogeneous linear ordinary differential equation you should always begin by doing the following.

1. Check that the linear differential operator has constant coefficients. If it does not then abandon the methods in this chapter! If it does then try to find the roots of its characteristic polynomial.
2. Check that the forcing can be put into the characteristic form (6.3). If it cannot then consider using the methods in Sections 6.4 and 6.5. If it can then identify its degree d, characteristic $\mu + i\nu$, and multiplicity m.

When both of these conditions are met then quickest route to a particular solution will be provided by either *Key Identity Evaluations* or *Undetermined Coefficients.* These methods are covered in the next two sections. How they are used depends upon the values of d, $\mu + i\nu$, and m, so being able to identify these values is crucial. Other methods will be covered later that can be applied when one of these conditions is not met.

Condition (1) is always easy to verify by inspection. Condition (2) is often easy to verify by inspection, but sometimes this requires the use of trigonometric or algebraic identities. Upon doing so, the degree, characteristic, and multiplicity of the forcing should be evident.

Example. The differential equation $y'' + 2y' + 10y = 6e^{2t}$ has constant coefficients. Its characteristic polynomial is

$$p(z) = z^2 + 2z + 10 = (z+1)^2 + 3^2 \,,$$

which has the simple conjugate roots $-1 \pm i3$. Its forcing has characteristic form with degree $d = 0$ and characteristic $\mu + i\nu = 2$. Because $\mu + i\nu = 2$ is not a root of $p(z)$, it has multiplicity $m = 0$.

Example. The differential equation $v'' + 6v' + 9v = t^2 e^{-3t}$ has constant coefficients. Its characteristic polynomial is

$$p(z) = z^2 + 6z + 9 = (z+3)^2 \,,$$

which has the double real root -3. Its forcing has characteristic form with degree $d = 2$ and characteristic $\mu + i\nu = -3$. Because $\mu + i\nu = -3$ is a double root of $p(z)$, it has multiplicity $m = 2$.

Example. The differential equation $w''' + w'' - 6w' = t^3 + 7t$ has constant coefficients. Its characteristic polynomial is

$$p(z) = z^3 + z^2 - 6z = z(z+3)(z-2) \,,$$

which has the simple real roots 0, -3, and 2. Its forcing has characteristic form with degree $d = 3$ and characteristic $\mu + i\nu = 0$. Because $\mu + i\nu = 0$ is a simple root of $p(z)$, it has multiplicity $m = 1$.

Example. The differential equation $x'' + 4x = t\,e^{5t}\sin(3t)$ has constant coefficients. Its characteristic polynomial is

$$p(z) = z^2 + 4 = z^2 + 2^2 \,,$$

which has the simple conjugate roots $\pm i2$. Its forcing has characteristic form with degree $d = 1$ and characteristic $\mu + i\nu = 5 + i3$. Because $\mu + i\nu = 5 + i3$ is not a root of $p(z)$, it has multiplicity $m = 0$.

Example. The differential equation $y'' + 16y = \sin(2t)\cos(2t)$ has constant coefficients. Its characteristic polynomial is

$$p(z) = z^2 + 16 = z^2 + 4^2 \,,$$

which has the simple conjugate roots $\pm i4$. The forcing can be put into the characteristic form (6.3) by using the double-angle identity $\sin(4t) = 2\sin(2t)\cos(2t)$. The equation thereby can be expressed as $y'' + 16y = \frac{1}{2}\sin(4t)$. Therefore the forcing has characteristic form with degree $d = 0$ and characteristic $\mu + i\nu = i4$. Because $\mu + i\nu = i4$ is a simple root of $p(z)$, it has multiplicity $m = 1$.

6.2. **Key Identity Evaluations.** This method should only be applied to compute a particular solution of the nonhomogeneous linear equation (6.1) after you have:

(1) checked that the differential operator L has constant coefficients;
(2) checked that the forcing $f(t)$ has the characteristic form (6.3);
(3) identified the degree d, characteristic $\mu + i\nu$, and multiplicity m.

The method of Key Identity Evaluations rests upon the observation that for any forcing that has characteristic form with characteristic $\mu + i\nu$ we can construct an explicit particular solution of equation (6.1) by evaluating the Key Identity and some of its partial derivatives with respect to z at $z = \mu + i\nu$. Recall that if $p(z)$ is the characteristic polynomial of L then the Key Identity is

$$\mathrm{L}\big(e^{zt}\big) = p(z)e^{zt} .$$

Because the operator L does not depend on z, the k^{th} partial derivative with respect to z of the left-hand side of the Key identity is

$$\partial_z^k \mathrm{L}\big(e^{zt}\big) = \mathrm{L}\big(\partial_z^k e^{zt}\big) = \mathrm{L}\big(t^k e^{zt}\big) .$$

The k^{th} partial derivative with respect to z (taken while treating t as if it is a constant) of the right-hand side of the Key identity requires repeated applications of the product rule. We find that the Key Identity and its first four derivatives with respect to z are

$$\begin{aligned}
\mathrm{L}\big(e^{zt}\big) &= p(z)e^{zt} , \\
\mathrm{L}\big(t\, e^{zt}\big) &= p(z)\, t\, e^{zt} + p'(z)\, e^{zt} , \\
\mathrm{L}\big(t^2 e^{zt}\big) &= p(z)\, t^2 e^{zt} + 2p'(z)\, t\, e^{zt} + p''(z)\, e^{zt} , \\
\mathrm{L}\big(t^3 e^{zt}\big) &= p(z)\, t^3 e^{zt} + 3p'(z)\, t^2 e^{zt} + 3p''(z)\, t\, e^{zt} + p'''(z)\, e^{zt} , \\
\mathrm{L}\big(t^4 e^{zt}\big) &= p(z)\, t^4 e^{zt} + 4p'(z)\, t^3 e^{zt} + 6p''(z)\, t^2 e^{zt} + 4p'''(z)\, t\, e^{zt} + p^{(4)}(z)\, e^{zt} .
\end{aligned} \tag{6.4}$$

We see that when these are evaluated at $z = \mu + i\nu$ then the terms on the right-hand sides above have the same form as those appearing in the forcing (6.3).

Remark. The coefficients appearing on the right-hand side of (6.4) are the coefficients that appear in the binomial expansion. They are generated by the Pascal triangle:

$$\begin{array}{ccccccccccccc}
 & & & & & & 1 & & & & & & \\
 & & & & & 1 & & 1 & & & & & \\
 & & & & 1 & & 2 & & 1 & & & & \\
 & & & 1 & & 3 & & 3 & & 1 & & & \\
 & & 1 & & 4 & & 6 & & 4 & & 1 & & \\
 & 1 & & 5 & & 10 & & 10 & & 5 & & 1 & \\
1 & & 6 & & 15 & & 20 & & 15 & & 6 & & 1 \\
 & & & & & & \vdots & & & & & &
\end{array}$$

Recall that each entry in the Pascal triangle is the sum of the two entries appearing immediately above it. By using this rule it is easy to extend (6.4) to as many derivatives of the Key Identity with respect to z as needed.

6.2.1. *Basic Steps.* Given a nonhomogeneous equation $\mathrm{L}y = f(t)$ that has a forcing $f(t)$ in characteristic form with degree d, characteristic $\mu + i\nu$, and multiplicity m, the method of Key Identity Evaluations finds an explicit particular solution Y_P as follows.

1. Write down the Key Identity through its $(m + d)^{\text{th}}$ derivative with respect to z.
2. Evaluate the m^{th} through the $(m + d)^{\text{th}}$ derivatives of the Key Identity at the characteristic by setting $z = \mu + i\nu$.
3. Find a linear combination of the resulting $d+1$ equations whose right-hand side has real part equal to $f(t)$ and read off Y_P.

In particular, if the characteristic $\mu + i\nu$ is not a root of $p(z)$ then we need through the d^{th} derivative of the Key Identity with respect to z. These should be evaluated at $z = \mu + i\nu$. A linear combination of the resulting $d + 1$ equations can be found so that the real part of its right-hand side equals any $f(t)$ given by (6.3). We can then read off Y_P from this linear combination.

More generally, if the characteristic $\mu + i\nu$ is a root of $p(z)$ of multiplicity m then we need through the $(m + d)^{\text{th}}$ derivative of the Key Identity with respect to z. These should be evaluated at $z = \mu + i\nu$. The right-hand sides of the first m of these will vanish when evaluated at $z = \mu + i\nu$ because $\mu + i\nu$ is a root of $p(z)$ of multiplicity m, whereby

$$p^{(k)}(\mu + i\nu) = 0 \quad \text{for every } k < m \qquad \text{and} \qquad p^{(m)}(\mu + i\nu) \neq 0\,.$$

Therefore only the m^{th} through the $(m + d)^{\text{th}}$ derivatives of the Key Identity will have nonzero right-hand sides when evaluated at $z = \mu + i\nu$. A linear combination of the resulting $d + 1$ equations can be found so that its right-hand side has real part equal any to $f(t)$ given by (6.3). We can then read off Y_P from this linear combination.

The following subsections illustrate the method of Key Identity Evaluations through a sequence of examples that have forcings with increasing complexity.

Remark. The method Key Identity Evaluations is fairly painless when d is small. For the equations faced in this course both m and d will be small, so $m + d$ seldom will be larger than 3, and more commonly will be 0, 1, or 2.

6.2.2. *Forcings with Zero Degree and Real Characteristic.* The simplest case to treat is when the forcing $f(t)$ has zero degree ($d = 0$) and real characteristic ($\nu = 0$). In that case the characteristic form for $f(t)$ given by (6.3) reduces to

$$f(t) = \alpha e^{\mu t}\,, \tag{6.5}$$

where α and μ are real numbers. We assume that $\alpha \neq 0$, so that $f(t) \neq 0$.

When $d = 0$ and the characteristic μ has multiplicity m then the first two steps of the method of Key Identity Evaluations become the following.

1. Write down the Key Identity through its m^{th} derivative with respect to z.
2. Evaluate the m^{th} derivative of the Key Identity at $z = \mu$.

The result is single real equation from which a particular solution can be easily found.

When $d = 0$ and the characteristic μ has multiplicity $m = 0$ then the Key Identity evaluated at $z = \mu$ gives

$$\mathrm{L}\big(e^{\mu t}\big) = p(\mu)\, e^{\mu t}\,.$$

Because $m = 0$ we know that $p(\mu) \neq 0$, so we see that

$$\mathrm{L}\left(\alpha\, \frac{e^{\mu t}}{p(\mu)}\right) = \alpha e^{\mu t}\,,$$

whereby a particular solution is given by

$$Y_P(t) = \alpha\, \frac{e^{\mu t}}{p(\mu)}\,. \tag{6.6a}$$

Rather than memorizing formula (6.6a), it may be easier and almost as fast to retrace the steps by which it was derived. We illustrate this with an example.

Example. Find a general solution of

$$y'' + 2y' + 10y = 6e^{2t}\,.$$

Solution. This equation has constant coefficients. Its operator is $\mathrm{L} = \mathrm{D}^2 + 2\mathrm{D} + 10$. Its characteristic polynomial is

$$p(z) = z^2 + 2z + 10 = (z+1)^2 + 9 = (z+1)^2 + 3^2\,,$$

which has the simple conjugate roots $-1 \pm i3$. Hence, a general solution of the associated homogeneous equation is

$$Y_H(t) = c_1 e^{-t}\cos(3t) + c_2 e^{-t}\sin(3t)\,.$$

To find a particular solution, first notice that the forcing has characteristic form with degree $d = 0$ and characteristic $\mu + i\nu = 2$. Because the characteristic 2 is not a root of $p(z)$, it has multiplicity $m = 0$.

Because $d = m = 0$ we need only the Key Identity, which is

$$\mathrm{L}\big(e^{zt}\big) = \big(z^2 + 2z + 10\big)\, e^{zt}\,.$$

Evaluating this at $z = 2$ yields

$$\mathrm{L}\big(e^{2t}\big) = \big(2^2 + 2\cdot 2 + 10\big)\, e^{2t} = 18e^{2t}\,.$$

After dividing by 3 we read off that a particular solution of $\mathrm{L}(y) = 6e^{2t}$ is $Y_P(t) = \frac{1}{3}e^{2t}$. Therefore a general solution is

$$y(t) = c_1 e^{-t}\cos(3t) + c_2 e^{-t}\sin(3t) + \tfrac{1}{3}e^{2t}\,.$$

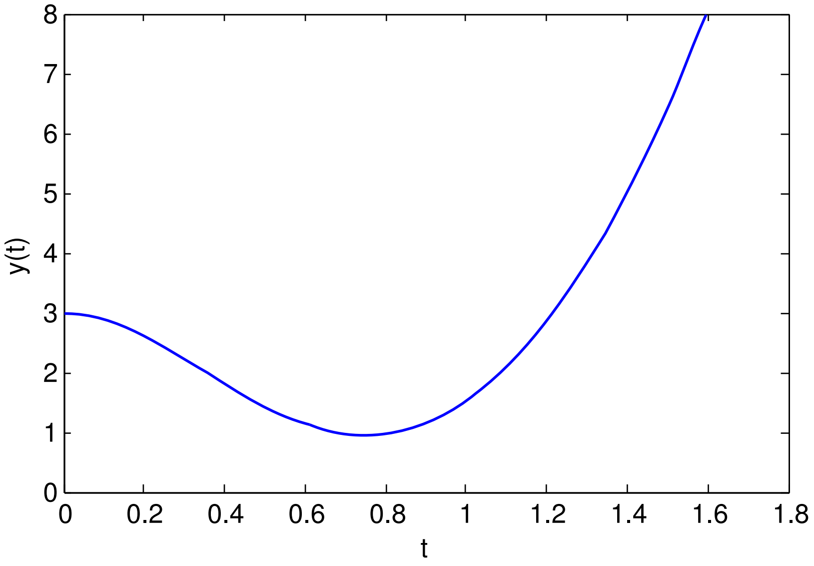

FIGURE 6.1. Solution to $y'' + 2y' + 10y = 6e^{2t}$ shown for the initial conditions $y(0) = 3$ and $y'(0) = 0$.

When $d = 0$ and the characteristic μ has multiplicity $m > 0$ then formula (6.6a) breaks down because $p(\mu) = 0$. When μ is a root of $p(z)$ of multiplicity m, we know

$$p^{(k)}(\mu) = 0 \quad \text{for every } k < m \qquad \text{and} \qquad p^{(m)}(\mu) \neq 0\,.$$

We see that when the derivatives of the Key Identity (6.4) are evaluated at $z = \mu$ then the right-hand sides will vanish until the m^{th} derivative of the Key Identity, which becomes

$$\mathrm{L}\big(t^m e^{\mu t}\big) = p^{(m)}(\mu)\, e^{\mu t}\,.$$

Because we know that $p^{(m)}(\mu) \neq 0$, we see that

$$\mathrm{L}\left(\alpha\, \frac{t^m e^{\mu t}}{p^{(m)}(\mu)}\right) = \alpha e^{\mu t}\,.$$

whereby a particular solution is given by

$$Y_P(t) = \alpha\, \frac{t^m e^{(\mu t}}{p^{(m)}(\mu)}\,. \tag{6.6b}$$

Rather than memorizing formula (6.6b), it may be easier and almost as fast to retrace the steps by which it was derived. We illustrate this with an example.

Example. Find a general solution of

$$v'' - 6v' + 9v = 4e^{3t}\,.$$

Solution. This equation has constant coefficients. Its operator is $\mathrm{L} = \mathrm{D}^2 - 6\mathrm{D} + 9$. Its characteristic polynomial is

$$p(z) = z^2 - 6z + 9 = (z-3)^2\,,$$

which has the double root 3. Hence, a general solution of the associated homogeneous equation is

$$V_H(t) = c_1 e^{3t} + c_2 t\, e^{3t}\,.$$

To find a particular solution, first notice that the forcing has characteristic form with degree $d = 0$ and characteristic $\mu + i\nu = 3$. Because the characteristic 3 is a double root of $p(z)$, it has multiplicity $m = 2$.

Because $d = 0$ and $m = 2$ we only need the second derivative of the Key Identity. The Key Identity and its first two derivatives with respect to z are

$$\begin{aligned} \mathrm{L}\big(e^{zt}\big) &= \big(z^2 - 6z + 9\big)\, e^{zt}\,, \\ \mathrm{L}\big(t\, e^{zt}\big) &= \big(z^2 - 6z + 9\big)\, t\, e^{zt} + (2z-6)\, e^{zt}\,, \\ \mathrm{L}\big(t^2 e^{zt}\big) &= \big(z^2 - 6z + 9\big)\, t^2 e^{zt} + 2(2z-6)\, t\, e^{zt} + 2e^{zt}\,. \end{aligned}$$

Evaluating the second derivative at $z = 3$ yields

$$\mathrm{L}\big(t^2 e^{3t}\big) = 2e^{3t}\,.$$

After multiplying by 2 we see that a particular solution of $\mathrm{L}(v) = 4e^{3t}$ is $V_P(t) = 2t^2 e^{3t}$. Therefore a general solution is

$$v(t) = c_1 e^{3t} + c_2 t\, e^{3t} + 2t^2 e^{3t}\,.$$

Remark. Had you failed to notice that the characteristic $\mu + i\nu = 3$ has multiplicity $m = 2$ and evaluated the Key Identity at $z = 3$ you would have found that $\mathrm{L}\big(e^{3t}\big) = 0$, which tells you that $m > 0$!

6.2.3. *Forcings with Zero Degree and Complex Characteristic.* Forcings with zero degree ($d = 0$) and complex characteristic ($\nu > 0$) often arise in applications. In that case the characteristic form for $f(t)$ given by (6.3) reduces to

$$f(t) = \alpha e^{\mu t}\cos(\nu t) + \beta e^{\mu t}\sin(\nu t)\,, \tag{6.7}$$

where α, β, μ, and ν are real numbers. We assume that either $\alpha \neq 0$ or $\beta\nu \neq 0$, so that $f(t) \neq 0$. We will need two facts.

Fact 1. We can express $f(t)$ given by (6.7) in terms of $e^{(\mu+i\nu)t}$ as

$$f(t) = \mathrm{Re}\big(\gamma\, e^{(\mu+i\nu)t}\big) \qquad \text{where} \quad \gamma = \alpha - i\beta\,. \tag{6.8}$$

This is called a *phasor form* for $f(t)$ where the complex number γ is called its *phasor*.

Reason. We can see that the phasor form (6.8) is equivalent to (6.7) because

$$\begin{aligned} \mathrm{Re}\big(\gamma\, e^{(\mu+i\nu)t}\big) &= \mathrm{Re}\big(\gamma\, e^{\mu t} e^{i\nu t}\big) = e^{\mu t}\,\mathrm{Re}\big(\gamma\, e^{i\nu t}\big) \\ &= e^{\mu t}\,\mathrm{Re}\Big((\alpha - i\beta)\,\big(\cos(\nu t) + i\sin(\nu t)\big)\Big) \\ &= e^{\mu t}\big(\alpha\cos(\nu t) + \beta\sin(\nu t)\big) \\ &= \alpha e^{\mu t}\cos(\nu t) + \beta e^{\mu t}\sin(\nu t)\,. \end{aligned}$$

Here we have used the fact that $e^{\mu t}$ is real in the first line and the Euler identity $e^{i\nu t} = \cos(\nu t) + i\sin(\nu t)$ in the second line. In the third line we computed just the real part of the product $(\alpha - i\beta)\,\big(\cos(\nu t) + i\sin(\nu t)\big)$ because that is all that we needed. □

Fact 2. If L has real coefficients then if $Z(t)$ and $h(t)$ are complex-valued functions that satisfy the single complex equation $\mathrm{L}(Z(t)) = h(t)$ then

$$\mathrm{L}(\mathrm{Re}(Z(t))) = \mathrm{Re}(h(t))\,, \qquad \mathrm{L}(\mathrm{Im}(Z(t))) = \mathrm{Im}(h(t))\,.$$

These two real equations are respectively the real and imaginary parts of the single complex equation $\mathrm{L}(Z(t)) = h(t)$, and are thereby equivalent to that equation.

Reason. This fact was introduced in Chapter [??].

We see from **Fact 2** that if $f(t)$ has the phasor form (6.8) with phasor γ and $Z(t)$ satisfies

$$\mathrm{L}\big(Z(t)\big) = \gamma\, e^{(\mu+i\nu)t}\,, \tag{6.9a}$$

then a particular solution $Y_P(t)$ of $\mathrm{L}y = f(t)$ is given by

$$Y_P(t) = \mathrm{Re}(Z(t))\,. \tag{6.9b}$$

When $d = 0$ and the characteristic $\mu + i\nu$ has $\nu > 0$ and multiplicity m then the method of Key Identity Evaluations becomes the following.

1. Express $f(t)$ in its phasor form (6.8).
2. Write down the Key Identity through its m^{th} derivative with respect to z.
3. Evaluate the m^{th} derivative of the Key Identity at $z = \mu + i\nu$.
4. Multiply the resulting equation by a factor that brings it into form (6.9a).
5. Evaluate $Y_P(t)$ given by equation (6.9b).

These steps are easy to carry out.

If the forcing has degree $d = 0$ and characteristic $\mu + i\nu$ with $\nu > 0$ and multiplicity $m = 0$ then we just need to evaluate the Key Identity at $z = \mu + i\nu$. This gives

$$\mathrm{L}\big(e^{(\mu+i\nu)t}\big) = p(\mu + i\nu)\, e^{(\mu+i\nu)t}\,.$$

Because $m = 0$ we know that $p(\mu + i\nu) \neq 0$ and see that

$$\mathrm{L}\left(\gamma\, \frac{e^{(\mu+i\nu)t}}{p(\mu + i\nu)}\right) = \gamma\, e^{(\mu+i\nu)t}\,,$$

whereby (6.9) implies that a particular solution is given by

$$Y_P(t) = \mathrm{Re}\left(\frac{\gamma\, e^{(\mu+i\nu)t}}{p(\mu + i\nu)}\right)\,. \tag{6.10a}$$

Rather than memorizing formula (6.10a), it is easier and almost as fast to retrace the steps by which it was derived. We illustrate this with an example.

Example. Find a general solution of

$$y'' + 2y' + 10y = \cos(2t)\,.$$

Solution. This equation has constant coefficients. Its operator is $\mathrm{L} = \mathrm{D}^2 + 2\mathrm{D} + 10$. Its characteristic polynomial is

$$p(z) = z^2 + 2z + 10 = (z + 1)^2 + 3^2\,,$$

which has the simple conjugate roots $-1 \pm i3$. Hence, a general solution of the associated homogeneous equation is

$$Y_H(t) = c_1 e^{-t}\cos(3t) + c_2 e^{-t}\sin(3t)\,.$$

To find a particular solution, first notice that the forcing has characteristic form with degree $d = 0$ and characteristic $\mu + i\nu = i2$. Because the characteristic $i2$ is not a root of $p(z)$, it has multiplicity $m = 0$.

Because the characteristic is complex, we express the forcing in its phasor form

$$\cos(2t) = \mathrm{Re}(e^{i2t})\,.$$

Because $d = m = 0$ we need only the Key Identity, which is

$$\mathrm{L}\big(e^{zt}\big) = \big(z^2 + 2z + 10\big)\,e^{zt}\,.$$

Evaluating this at $z = i2$ yields

$$\mathrm{L}\big(e^{i2t}\big) = \big(-2^2 + 2\cdot i2 + 10\big)\,e^{i2t} = (6 + i4)e^{i2t}\,.$$

After dividing by $6 + i4$ we obtain

$$\mathrm{L}\bigg(\frac{e^{i2t}}{6 + i4}\bigg) = e^{i2t}\,.$$

Because the forcing has the phasor form $\cos(2t) = \mathrm{Re}(e^{i2t})$, by taking real parts we see that a particular solution is given by

$$\begin{aligned} Y_P(t) &= \mathrm{Re}\bigg(\frac{e^{i2t}}{6 + i4}\bigg) = \mathrm{Re}\bigg(\frac{6 - i4}{6^2 + 4^2}\,e^{i2t}\bigg) = \tfrac{1}{52}\mathrm{Re}\big((6 - i4)e^{i2t}\big) \\ &= \tfrac{1}{52}\mathrm{Re}\big((6 - i4)(\cos(2t) + i\sin(2t))\big) = \tfrac{6}{52}\cos(2t) + \tfrac{4}{52}\sin(2t)\,. \end{aligned}$$

Therefore a general solution is

$$y = c_1 e^{-t}\cos(3t) + c_2 e^{-t}\sin(3t) + \tfrac{3}{26}\cos(2t) + \tfrac{1}{13}\sin(2t)\,.$$

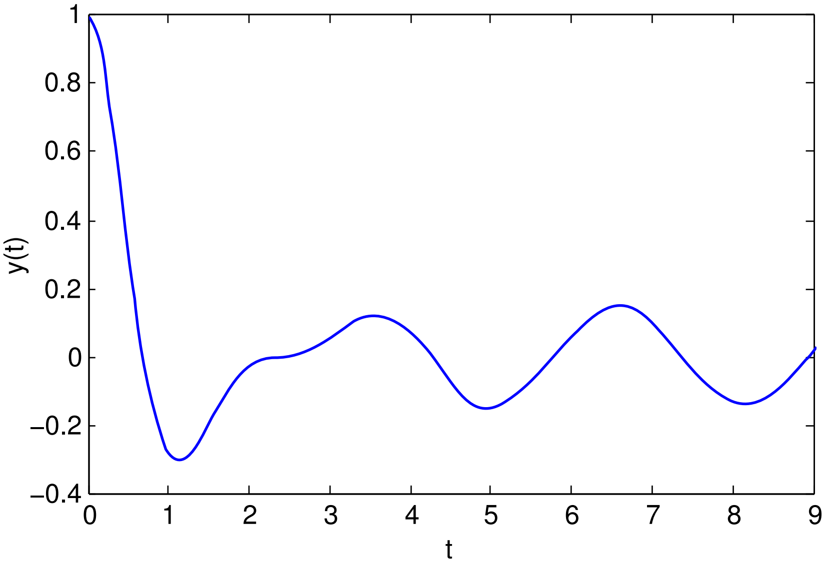

FIGURE 6.2. Solution to $y'' + 2y' + 10y = \cos(2t)$ shown for the initial conditions $y(0) = 1$ and $y'(0) = 0$.

If the forcing has degree $d = 0$ and characteristic $\mu + i\nu$ with multiplicity $m > 0$ then formula (6.10a) breaks down because $p(\mu + i\nu) = 0$. Because $\mu + i\nu$ is a root of $p(z)$ of multiplicity m, we know that

$$p^{(k)}(\mu + i\nu) = 0 \quad \text{for every } k < m \qquad \text{and} \qquad p^{(m)}(\mu + i\nu) \neq 0\,.$$

We thereby see that when the derivatives of the Key Identity (6.4) are evaluated at $z = \mu + i\nu$ then the right-hand sides will vanish until the m^{th} derivative of the Key Identity, which becomes

$$\mathrm{L}\big(t^m e^{(\mu+i\nu)t}\big) = p^{(m)}(\mu + i\nu)\, e^{(\mu+i\nu)t}\,.$$

Because we know that $p^{(m)}(\mu + i\nu) \neq 0$, we see that

$$\mathrm{L}\left(\gamma\, \frac{t^m e^{(\mu+i\nu)t}}{p^{(m)}(\mu + i\nu)}\right) = \gamma\, e^{(\mu+i\nu)t}\,,$$

whereby a particular solution is given by

$$Y_P(t) = \mathrm{Re}\left(\frac{\gamma\, t^m e^{(\mu+i\nu)t}}{p^{(m)}(\mu + i\nu)}\right)\,. \tag{6.10b}$$

Rather than memorizing formula (6.10b), it is almost as fast to retrace the steps by which it was derived. We illustrate this with examples.

Example. Find a general solution of

$$x'' + 9x = 4\cos(3t)\,.$$

Solution. This equation has constant coefficients. Its operator is $\mathrm{L} = \mathrm{D}^2 + 9$. Its characteristic polynomial is

$$p(z) = z^2 + 9 = z^2 + 3^2 \,,$$

which has the simple conjugate roots $\pm i3$. Hence, a general solution of the associated homogeneous equation is

$$X_H(t) = c_1 \cos(3t) + c_2 \sin(3t) \,.$$

To find a particular solution, first notice that the forcing has characteristic form with degree $d = 0$ and characteristic $\mu + i\nu = i3$. Because the characteristic $i3$ is a simple root of $p(z)$, it has multiplicity $m = 1$.

Because the characteristic is complex, we express the forcing in its phasor form

$$4\cos(3t) = \mathrm{Re}(4e^{i3t}) \,.$$

Because $d = 0$ and $m = 1$ we only need the first derivative of the Key Identity. The Key Identity and its first derivative with respect to z are

$$\begin{aligned} \mathrm{L}\big(e^{zt}\big) &= \big(z^2 + 9\big)\, e^{zt} \,, \\ \mathrm{L}\big(t\, e^{zt}\big) &= \big(z^2 + 9\big)\, t\, e^{zt} + 2z\, e^{zt} \,. \end{aligned}$$

Evaluating the first derivative at $z = i3$ yields

$$\mathrm{L}\big(t\, e^{i3t}\big) = 2 \cdot i3 e^{i3t} = i6 e^{i3t} \,.$$

After dividing by $i6$ and multiplying by 4 we obtain

$$\mathrm{L}\left(\frac{4t\, e^{i3t}}{i6}\right) = 4e^{i3t} \,.$$

Because the forcing has the phasor form $4\cos(3t) = \mathrm{Re}(4e^{i3t})$, by taking real parts we see that a particular solution is given by

$$\begin{aligned} X_P(t) &= \mathrm{Re}\left(\frac{4t\, e^{i3t}}{i6}\right) = -\tfrac{2}{3} t\, \mathrm{Re}\big(i\, e^{i3t}\big) \\ &= -\tfrac{2}{3} t\, \mathrm{Re}\big(i\,(\cos(3t) + i\sin(3t))\big) = \tfrac{2}{3} t\, \sin(3t) \,. \end{aligned}$$

Therefore a general solution is

$$x = c_1 \cos(3t) + c_2 \sin(3t) + \tfrac{2}{3} t\, \sin(3t) \,.$$

Example. Find a general solution of

$$w'' + 2w' + 10w = 5e^{-t} \sin(3t) \,.$$

Solution. This equation has constant coefficients. Its operator is $\mathrm{L} = \mathrm{D}^2 + 2\mathrm{D} + 10$. Its characteristic polynomial is

$$p(z) = z^2 + 2z + 10 = (z + 1)^2 + 3^2 \,,$$

which has the simple conjugate roots $-1 \pm i3$. Hence, a general solution of the associated homogeneous equation is

$$W_H(t) = c_1 e^{-t} \cos(3t) + c_2 e^{-t} \sin(3t) \,.$$

To find a particular solution, first notice that the forcing has characteristic form with degree $d = 0$ and characteristic $\mu + i\nu = -1 + i3$. Because the characteristic $-1 + i3$ is a simple root of $p(z)$, it has multiplicity $m = 1$.

Because the characteristic is complex, we express the forcing in its phasor form

$$5e^{-t}\sin(3t) = 5e^{-t}\mathrm{Re}(-ie^{i3t})\,.$$

Because $d = 0$ and $m = 1$ we only need the first derivative of the Key Identity. The Key Identity and its first derivative with respect to z are

$$\begin{aligned} \mathrm{L}\big(e^{zt}\big) &= \big(z^2 + 2z + 10\big)\,e^{zt}\,, \\ \mathrm{L}\big(t\,e^{zt}\big) &= \big(z^2 + 2z + 10\big)\,t\,e^{zt} + (2z + 2)\,e^{zt}\,. \end{aligned}$$

Evaluating the first derivative at $z = -1 + i3$ yields

$$\mathrm{L}\big(t\,e^{(-1+i3)t}\big) = (2(-1 + i3) + 2)\,e^{(-1+i3)t} = i6e^{(-1+i3)t}\,.$$

Because the forcing has the phasor form $5\mathrm{Re}(-i5e^{(-1+i3)t})$, after dividing by $i6$ and multiplying by $-i5$ we obtain

$$\mathrm{L}\big(-\tfrac{5}{6}t\,e^{(-1+i3)t}\big) = -i5e^{(-1+i3)t}\,.$$

By taking real parts we see that a particular solution is given by

$$\begin{aligned} W_P(t) &= -\tfrac{5}{6}t\,e^{-t}\mathrm{Re}\big(e^{i3t}\big) \\ &= -\tfrac{5}{6}t\,e^{-t}\mathrm{Re}\big(\cos(3t) + i\sin(3t)\big) = -\tfrac{5}{6}t\,e^{-t}\cos(3t)\,. \end{aligned}$$

Therefore a general solution is

$$w = c_1e^{-t}\cos(3t) + c_2e^{-t}\sin(3t) - \tfrac{5}{6}t\,e^{-t}\cos(3t)\,.$$

6.2.4. *Zero Degree Formula.* For those who like to memorize formulas, there is a single formula that recovers all of the zero degree cases. For a forcing of degree $d = 0$ and characteristic $\mu + i\nu$ with multiplicity m in the form

$$f(t) = \alpha e^{\mu t}\cos(\nu t) + \beta e^{\mu t}\sin(\nu t) = e^{\mu t}\mathrm{Re}\big((\alpha - i\beta)e^{i\nu t}\big)\,,$$

a particular solution is given by the so-called *zero degree formula*

$$Y_P(t) = t^m e^{\mu t}\mathrm{Re}\left(\frac{(\alpha - i\beta)e^{i\nu t}}{p^{(m)}(\mu + i\nu)}\right)\,. \tag{6.11}$$

Remark. When $\nu = \beta = 0$ and $m = 0$ this recovers formula (6.6a),

$$Y_P(t) = e^{\mu t}\,\frac{\alpha}{p(\mu)}\,.$$

When $\nu = \beta = 0$ and $m > 0$ it recovers formula (6.6b),

$$Y_P(t) = t^m e^{\mu t}\,\frac{\alpha}{p^{(m)}(\mu)}\,.$$

When $\nu > 0$ and $m = 0$ it recovers formula (6.10a),

$$Y_P(t) = e^{\mu t}\mathrm{Re}\left(\frac{(\alpha - i\beta)e^{i\nu t}}{p(\mu + i\nu)}\right)\,. \tag{6.12}$$

When $\nu > 0$ and $m > 0$ it is equivalent to formula (6.10b).

Remark. The zero degree formula is related to the so-called *exponential response formula* that gives a particular solution of equations that have a purely exponential forcing — i.e. equations in the form $\mathrm{L}y = e^{(\mu+i\nu)t}$. Specifically, it gives the particular (generally complex-valued) solution

$$Y_P(t) = \frac{t^m e^{(\mu+i\nu)t}}{p^{(m)}(\mu+i\nu)}\,,$$

where m is the multiplicity of the characteristic $\mu + i\nu$.

Remark. When $\nu = \beta = 0$ the zero degree formula just requires real arithmetic to evaluate $p^{(m)}(\mu)$. When $\nu > 0$ the zero degree formula requires complex arithmetic both to evaluate $p^{(m)}(\mu+i\nu)$ and to take the real part. In either case it is easy to apply.

We now illustrate use of the zero degree formulas on the five examples already treated by Key Identity Evalutations. Here we will build upon the work done in those examples by focusing on finding only a particular solution rather than a general solution.

Example. Find a particular solution of

$$y'' + 2y' + 10y = 6e^{2t}\,.$$

Solution. For this problem $f(t) = 6e^{2t}$ and $p(z) = z^2 + 2z + 10$, so that $\mu = 2$, $\nu = 0$, $\alpha = 6$, $\beta = 0$, and $m = 0$, whereby $p(2) = 2^2 + 2\cdot 2 + 10 = 18$ and

$$Y_P(t) = e^{2t}\,\frac{6}{p(2)} = e^{2t}\,\frac{6}{18} = \tfrac{1}{3}e^{2t}\,.$$

Example. Find a particular solution of

$$v'' - 6v' + 9v = 4e^{3t}\,.$$

Solution. For this problem $f(t) = 4e^{3t}$ and $p(z) = z^2 - 6z + 9$, so that $\mu = 2$, $\nu = 0$, $\alpha = 4$, $\beta = 0$, $m = 2$, and $p''(z) = 2$, whereby

$$V_P(t) = t^2e^{3t}\,\frac{4}{p''(3)} = t^2e^{3t}\,\frac{4}{2} = 2t^2e^{3t}\,.$$

Example. Find a general solution of

$$y'' + 2y' + 10y = \cos(2t)\,.$$

Solution. For this problem $f(t) = \cos(2t)$ and $p(z) = z^2 + 2z + 10$, so that $\mu = 0$, $\nu = 2$, $\alpha = 1$, $\beta = 0$, and $m = 0$, whereby $p(i2) = 6 + i4$ and

$$\begin{aligned} Y_P(t) &= \mathrm{Re}\left(\frac{e^{i2t}}{6+i4}\right) = \frac{1}{36+16}\,\mathrm{Re}\big((6-i4)e^{i2t}\big) \\ &= \frac{1}{52}\,\mathrm{Re}\big((6-i4)\big(\cos(2t) + i\sin(2t)\big)\big) \\ &= \tfrac{3}{26}\cos(2t) + \tfrac{1}{13}\sin(2t)\,. \end{aligned}$$

Example. Find a general solution of

$$x'' + 9x = 4\cos(3t)\,.$$

Solution. For this problem $f(t) = 4\cos(3t)$ and $p(z) = z^2 + 9$, so that $\mu = 0$, $\nu = 3$, $\alpha = 4$, $\beta = 0$, $m = 1$, and $p'(z) = 2z$, whereby $p'(i3) = i6$ and

$$X_P(t) = t\,\mathrm{Re}\left(\frac{4e^{i3t}}{i6}\right) = \tfrac{2}{3}t\,\sin(3t)\,.$$

Example. Find a particular solution of

$$w'' + 2w' + 10w = 5e^{-t}\sin(3t)\,.$$

Solution. For this problem $f(t) = 5e^{-t}\sin(3t)$ and $p(z) = z^2+2z+10$, so that $\mu = -1$, $\nu = 3$, $\alpha = 0$, $\beta = 5$, $m = 1$, and $p'(z) = 2z + 2$, whereby $p'(-1 + i3) = i6$ and

$$W_P(t) = t\,e^{-t}\mathrm{Re}\left(\frac{-i5e^{i3t}}{i6}\right) = -\tfrac{5}{6}t\,e^{-t}\cos(3t)\,.$$

6.2.5. *Forcings with Positive Degree.* When $d > 0$ the method of Key Identity Evaluations reduces to $d+1$ equations that have to be combined into a single equation whose right-hand side is $f(t)$. For example, if the degree $d = 1$ and the characteristic $\mu + i\nu$ has multiplicity $m = 0$ then the two equations are just the Key Identity and its first derivative evaluated at $\mu + i\nu$, which are

$$\begin{aligned}
\mathrm{L}\big(e^{(\mu+i\nu)t}\big) &= p(\mu + i\nu)e^{(\mu+i\nu)t}\,,\\
\mathrm{L}\big(t\,e^{(\mu+i\nu)t}\big) &= p(\mu + i\nu)t\,e^{(\mu+i\nu)t} + p'(\mu + i\nu)e^{(\mu+i\nu)t}\,,
\end{aligned}$$

Because $m = 0$ we know that $p(\mu + i\nu) \neq 0$, the first of these equations recovers the exponential response formula

$$\mathrm{L}\left(\frac{e^{(\mu+i\nu)t}}{p(\mu + i\nu)}\right) = e^{(\mu+i\nu)t}\,,$$

while the second yields

$$\mathrm{L}\left(\frac{t\,e^{(\mu+i\nu)t}}{p(\mu + i\nu)}\right) = t\,e^{(\mu+i\nu)t} + \frac{p'(\mu + i\nu)}{p(\mu + i\nu)}\,e^{(\mu+i\nu)t}\,.$$

Upon multiplying the exponential response formula by $p'(\mu + i\nu)/p(\mu + i\nu)$ and subtracting it from the one above we obtain

$$\mathrm{L}\left(\frac{t\,e^{(\mu+i\nu)t}}{p(\mu + i\nu)} - \frac{p'(\mu + i\nu)}{p(\mu + i\nu)}\,\frac{e^{(\mu+i\nu)t}}{p(\mu + i\nu)}\right) = t\,e^{(\mu+i\nu)t}\,.$$

By linearly combining the real and imaginary parts of this formula with the real and imaginary parts of the exponential response formula we can obtain an explicit particular solution for any forcing $f(t)$ of characteristic form that has degree $d = 1$ and characteristic $\mu + i\nu$ with multiplicity $m = 0$.

The best approach to using the method of Key Identity Evaluations is to mimic the steps that we used to derive the above formulas rather than remembering the formulas themselves. That approach works when $d > 1$ as well as when $m > 0$. We now illustrate this approach.

Example. Find a general solution of

$$x'' + 2x' + 10x = 4te^{2t}\,.$$

Solution. This equation has constant coefficients. Its operator is $\mathrm{L} = \mathrm{D}^2 + 2\mathrm{D} + 10$. Its characteristic polynomial is

$$p(z) = z^2 + 2z + 10 = (z+1)^2 + 3^2\,,$$

which has the simple conjugate roots $-1 \pm i3$. Hence, a general solution of the associated homogeneous equation is

$$X_H(t) = c_1 e^{-t}\cos(3t) + c_2 e^{-t}\sin(3t)\,.$$

To find a particular solution, first notice that the forcing has characteristic form with and $d = 1$ and characteristic $\mu + i\nu = 2$. Because the characteristic 2 is not a root of $p(z)$, it has multiplicity $m = 0$.

Because $m = 0$ and $m + d = 1$, we will need the Key Identity and its first derivative with respect to z. These are

$$\begin{aligned}\mathrm{L}\big(e^{zt}\big) &= (z^2 + 2z + 10)\,e^{zt}\,,\\ \mathrm{L}\big(t\,e^{zt}\big) &= (z^2 + 2z + 10)\,t\,e^{zt} + (2z+2)\,e^{zt}\,.\end{aligned}$$

Evaluate these at $z = 2$ to obtain

$$\mathrm{L}\big(e^{2t}\big) = 18e^{2t}\,, \qquad \mathrm{L}\big(t\,e^{2t}\big) = 18\,t\,e^{2t} + 6\,e^{2t}\,.$$

Because we want to isolate the $t\,e^{2t}$ term on the right-hand side, subtract one-third the first equation from the second to get

$$\mathrm{L}\big(t\,e^{2t} - \tfrac{1}{3}e^{2t}\big) = \mathrm{L}\big(t\,e^{2t}\big) - \tfrac{1}{3}\mathrm{L}\big(e^{2t}\big) = 18\,t\,e^{2t}\,.$$

After multiplying this by $\frac{2}{9}$ we can read off that

$$X_P(t) = \tfrac{2}{9}te^{2t} - \tfrac{2}{27}e^{2t}\,.$$

Therefore a general solution is

$$x = c_1 e^{-t}\cos(3t) + c_2 e^{-t}\sin(3t) + \tfrac{2}{9}te^{2t} - \tfrac{2}{27}e^{2t}\,.$$

The general characteristic form given by (6.3) for a forcing $f(t)$ with positive degree ($d > 0$) and complex characteristic ($\nu > 0$) is

$$\begin{aligned}f(t) = {}&\big(\alpha_0 t^d + \alpha_1 t^{d-1} + \cdots + \alpha_d\big)e^{\mu t}\cos(\nu t)\\ &+ \big(\beta_0 t^d + \beta_1 t^{d-1} + \cdots + \beta_d\big)e^{\mu t}\sin(\nu t)\,,\end{aligned} \tag{6.13}$$

where α_0, α_1, $\cdots$, α_d, β_0, β_1, $\cdots$, β_d, μ, and ν are real numbers. We assume that either $\alpha_0 \neq 0$ or $\beta_0\nu \neq 0$, so that at least one of the t^d terms is nonzero. We will need the following generalization of **Fact 1**.

Fact 3. We can express $f(t)$ given by (6.13) in terms of $e^{(\mu+i\nu)t}$ as

$$f(t) = \mathrm{Re}\Big(\big(\gamma_0 t^d + \gamma_1 t^{d-1} + \cdots + \gamma_d\big)\,e^{(\mu+i\nu)t}\Big) \qquad \text{where} \quad \gamma_k = \alpha_k - i\beta_k\,.$$

This is a *generalized phasor form* for $f(t)$ where the complex numbers γ_k are its *phasors*.

Reason. We can see that this phasor form is equivalent to (6.13) because

$$\begin{aligned}\mathrm{Re}\big(\gamma_k\, t^{d-k}e^{(\mu+i\nu)t}\big) &= \mathrm{Re}\big(\gamma_k\, t^{d-k}e^{\mu t}e^{i\nu t}\big) = t^{d-k}e^{\mu t}\,\mathrm{Re}\big(\gamma_k\, e^{i\nu t}\big)\\ &= t^{d-k}e^{\mu t}\,\mathrm{Re}\big((\alpha_k - i\beta_k)\,\big(\cos(\nu t) + i\sin(\nu t)\big)\big)\\ &= t^{d-k}e^{\mu t}\big(\alpha_k\cos(\nu t) + \beta_k\sin(\nu t)\big)\\ &= \alpha_k t^{d-k}e^{\mu t}\cos(\nu t) + \beta_k t^{d-k}e^{\mu t}\sin(\nu t)\,.\end{aligned}$$

Here we computed just the real part of the product $(\alpha_k - i\beta_k)\,\big(\cos(\nu t) + i\sin(\nu t)\big)$ because that is all that we needed. ☐

Example. Find a general solution of

$$y'' + 4y = t\,\cos(2t)\,.$$

Solution. This equation has constant coefficients. Its operator is $\mathrm{L} = \mathrm{D}^2 + 4$. Its characteristic polynomial is

$$p(z) = z^2 + 4 = z^2 + 2^2\,,$$

which has the simple conjugate roots $\pm i2$. Hence, a general solution of the associated homogeneous equation is

$$Y_H(t) = c_1\cos(2t) + c_2\sin(2t)\,.$$

To find a particular solution, first notice that the forcing has characteristic form with degree $d = 1$ and characteristic $\mu + i\nu = i2$. Because the characteristic $i2$ is a simple root of $p(z)$, it has multiplicity $m = 1$.

Because the characteristic is complex, we express the forcing in its phasor form

$$t\,\cos(2t) = \mathrm{Re}(t\,e^{i2t})\,.$$

Because $m = 1$ and $m + d = 2$, we will need the first two derivatives of the Key Identity. The Key Identity and its first two derivatives with respect to z are

$$\begin{aligned}\mathrm{L}\big(e^{zt}\big) &= (z^2+4)e^{zt}\,,\\ \mathrm{L}\big(t\,e^{zt}\big) &= (z^2+4)t\,e^{zt} + 2z\,e^{zt}\,,\\ \mathrm{L}\big(t^2e^{zt}\big) &= (z^2+4)t^2e^{zt} + 4z\,t\,e^{zt} + 2e^{zt}\,.\end{aligned}$$

We evaluate the first and second derivative of the Key Identity at $z = i2$ to obtain

$$\mathrm{L}\big(t\,e^{i2t}\big) = i4\,e^{i2t}\,,\qquad \mathrm{L}\big(t^2e^{i2t}\big) = i8\,t\,e^{i2t} + 2e^{i2t}\,.$$

Because the forcing has the phasor form $\mathrm{Re}(t\,e^{i2t})$, we want to isolate the $t\,e^{i2t}$ term on the right-hand side. This is done by multiplying the first equation above by $i\frac{1}{2}$ and adding it to the second to find

$$\mathrm{L}\big((t^2 + i\tfrac{1}{2}t)e^{i2t}\big) = \mathrm{L}\big(t^2e^{i2t}\big) + i\tfrac{1}{2}\mathrm{L}\big(t\,e^{i2t}\big) = i8\,t\,e^{i2t}\,.$$

Now divide this by $i8$ to obtain

$$\mathrm{L}\bigg(\frac{t^2 + i\frac{1}{2}t}{i8}\,e^{i2t}\bigg) = t\,e^{i2t}\,,$$

from which we read off that

$$Y_P(t) = \mathrm{Re}\left(\frac{t^2 + i\frac{1}{2}t}{i8}\, e^{i2t}\right) = \frac{t}{16}\mathrm{Re}\big((1 - i2t)e^{i2t}\big) = \frac{t}{16}\big(\cos(2t) + 2t\sin(2t)\big)\,.$$

Therefore a general solution is

$$y = c_1\cos(2t) + c_2\sin(2t) + \tfrac{1}{16}t\,\cos(2t) + \tfrac{1}{8}t^2\sin(2t)\,.$$

Remark. The characterisitc $\mu + i\nu = i2$ has multiplicity $m = 1$ because $p(z) = z^2 + 4 = 0$ at $z = i2$. This means the right-hand side of the Key Identity will vanish at $z = i2$, which tells us something we already know, namely, that $\mathrm{L}(e^{i2t}) = 0$. Moreover, it means the term involving $t\,e^{zt}$ on the right-hand side of the derivative of the Key Identity and the term involving t^2e^{zt} on the right-hand side of the second derivative of the Key Identity will also vanish at $z = i2$.

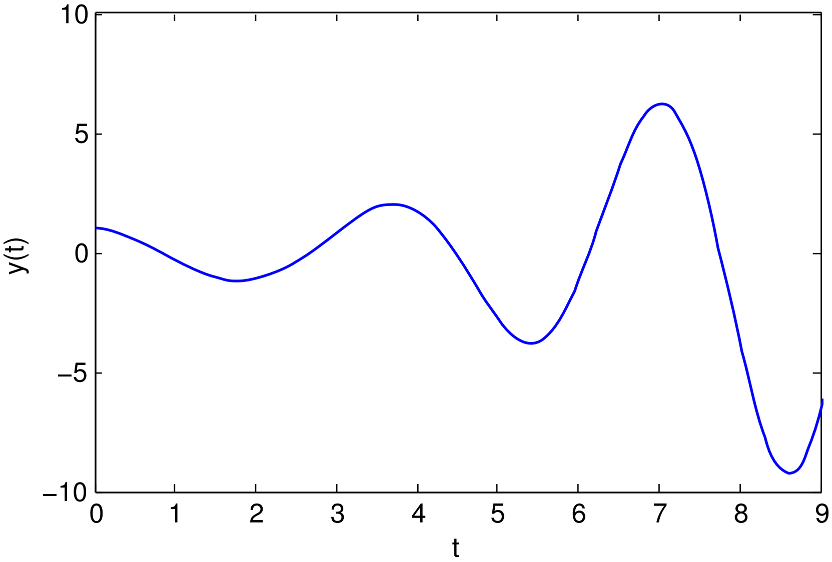

FIGURE 6.3. The solution to $y'' + 4y = t\cos(2t)$ shown for the initial conditions $y(0) = 1$ and $y'(0) = 0$

6.3. **Undetermined Coefficients.** This method should only be applied to compute a particular solution of the nonhomogeneous linear equation (6.1) after you have:

(1) checked that the differential operator L has constant coefficients;

(2) checked that the forcing $f(t)$ has the characteristic form (6.3);

(3) identified the degree d, characteristic $\mu + i\nu$, and multiplicity m.

These are the same first steps required by the method of Key Identity Evalutaions.

The method of Undetermined Coefficients is based on the observation that if the characteristic $\mu + i\nu$ of the forcing $f(t)$ is not a root of the characteristic polynomial $p(z)$ of the operator L then equation (6.1) has a particular solution of the form

$$\begin{aligned} Y_P(t) = &\big(A_0 t^d + A_1 t^{d-1} + \cdots + A_d\big) e^{\mu t} \cos(\nu t) \\ &+ \big(B_0 t^d + B_1 t^{d-1} + \cdots + B_d\big) e^{\mu t} \sin(\nu t)\,, \end{aligned} \tag{6.14a}$$

where A_0, A_1, $\cdots$, A_d, and B_0, B_1, $\cdots$, B_d are real constants. Notice that when $\nu = 0$ the terms involving B_0, B_1, $\cdots$, B_d all vanish.

More generally, if the characteristic $\mu + i\nu$ of the forcing $f(t)$ is a root of $p(z)$ of multiplicity m then equation (6.1) has a particular solution of the form

$$\begin{aligned} Y_P(t) = &\big(A_0 t^{m+d} + A_1 t^{m+d-1} + \cdots + A_d t^m\big) e^{\mu t} \cos(\nu t) \\ &+ \big(B_0 t^{m+d} + B_1 t^{m+d-1} + \cdots + B_d t^m\big) e^{\mu t} \sin(\nu t)\,, \end{aligned} \tag{6.14b}$$

where A_0, A_1, $\cdots$, A_d, and B_0, B_1, $\cdots$, B_d are real constants. Notice that when $\nu = 0$ the terms involving B_0, B_1, $\cdots$, B_d all vanish. If we set $m = 0$ in (6.14b) then it reduces to (6.14a).

Remark. Notice how the powers of t in (6.14b) run from t^{m+d} down to t^m. This is how the degree and multiplicity enter into the form.

6.3.1. *Basic Steps.* Given a nonhomogeneous equation $\mathrm{L}y = f(t)$ with forcing $f(t)$ that has characteristic form with degree d, characteristic $\mu + i\nu$, and multiplicity m, the method of undetermined coefficients seeks a particular solution $Y_P(t)$ in the form (6.14b) with A_0, A_1, $\cdots$, A_d, and B_0, B_1, $\cdots$, B_d as unknowns to be determined. These are the "undetermined coefficients" of the method. There are $2d + 2$ unknowns when $\nu > 0$, and only $d + 1$ unknowns when $\nu = 0$ because in that case the terms involving B_0, B_1, $\cdots$, B_d vanish. These unknowns are determined as follows.

1. Substitute $Y_P(t)$ given by form (6.14b), directly into $\mathrm{L}Y_P$, and collect like terms.
2. Set $\mathrm{L}Y_P = f(t)$ and match the coefficients of the linearly independent functions that appears on either side. (Examples will make this clearer.)
3. Solve the resulting linear algebraic system for the unknowns in the form (6.14b).

This linear algebraic system will consist of either $2d{+}2$ equations for the $2d{+}2$ unknowns A_0, A_1, $\cdots$, A_d, and B_0, B_1, $\cdots$, B_d (when $\nu > 0$) or $d + 1$ equations for the $d + 1$ unknowns A_0, A_1, $\cdots$, A_d (when $\nu = 0$). Because these unknowns are the parameters of the family (6.14b), this method is also sometimes called "Undetermined Parameters." We do not call it by that name here in order to avoid confusion with the method of "Variation of Parameters," which we will study later.

The following subsections illustrate the method of Undertermined Coefficients through a sequence of examples that have forcings with increasing complexity. In order to contrast the two methods, we will use the same examples that we had previously treated by Key Identity Evaluations.

Remark. The methods of Undetermined Coefficients and Key Identity Evaluations are each fairly painless when m and d are both small and $\nu = 0$. When m and d are both small and $\nu > 0$ then Key Identity Evaluations is usually faster. For the equations faced in this course both m and d will be small, so $m + d$ will seldom be larger than 3, and more commonly be 0, 1, or 2.

6.3.2. *Forcings with Zero Degree and Real Characteristic.* The simplest case to treat is when the forcing $f(t)$ has zero degree ($d = 0$) and real characteristic ($\nu = 0$). In that case the characteristic form for $f(t)$ given by (6.3) reduces to

$$f(t) = \alpha e^{\mu t}\,, \tag{6.15}$$

where α and μ are real numbers. We assume that $\alpha \neq 0$, so that $f(t) \neq 0$.

When $d = 0$, $\nu = 0$ and the characteristic μ has multiplicity m then the form for the particular solution (6.14b) reduces to

$$Y_P(t) = At^m e^{\mu t}\,. \tag{6.16}$$

The the first step of the method of Undetermined Coefficients is to substitute this directly into $\mathrm{L}Y_P$, and collect like terms. We will illustrate this with examples.

Example. Find a general solution of

$$y'' + 2y' + 10y = 6e^{2t}\,.$$

Solution. This equation has constant coefficients. Its characteristic polynomial is

$$p(z) = z^2 + 2z + 10 = (z + 1)^2 + 9 = (z + 1)^2 + 3^2\,,$$

which has the simple conjugate roots $-1 \pm i3$. Hence, a general solution of the associated homogeneous equation is

$$Y_H(t) = c_1 e^{-t}\cos(3t) + c_2 e^{-t}\sin(3t)\,.$$

To find a particular solution, first notice that the forcing has characteristic form with degree $d = 0$ and characteristic $\mu + i\nu = 2$. Because the characteristic 2 is not a root of $p(z)$, it has multiplicity $m = 0$.

Because $d = 0$, $\mu + i\nu = 2$, and $m = 0$, we see from (6.16) that Y_P has the form

$$Y_P(t) = Ae^{2t}\,.$$

Because

$$Y_P'(t) = 2Ae^{2t}\,, \qquad Y_P''(t) = 4Ae^{2t}\,,$$

we see that

$$\begin{aligned} \mathrm{L}Y_P(t) &= Y_P''(t) + 2Y_P'(t) + 10Y_P(t) \\ &= 4Ae^{2t} + 4Ae^{2t} + 10Ae^{2t} = 18Ae^{2t}\,. \end{aligned}$$

If we set $\mathrm{L}Y_P(t) = 6e^{2t}$ then we see that $18A = 6$, whereby $A = \frac{1}{3}$. Hence, our particular solution is

$$Y_P(t) = \tfrac{1}{3}e^{2t}\,.$$

Therefore a general solution is

$$y = c_1 e^{-t}\cos(3t) + c_2 e^{-t}\sin(3t) + \tfrac{1}{3}e^{2t}\,.$$

Example. Find a general solution of

$$v'' - 6v' + 9v = 4e^{3t}\,.$$

Solution. This equation has constant coefficients. The characteristic polynomial is

$$p(z) = z^2 - 6z + 9 = (z-3)^2 .$$

It has the double root 3. Hence, a general solution of the associated homogeneous equation is

$$V_H(t) = c_1 e^{3t} + c_2 t\, e^{3t} .$$

To find a particular solution, first notice that the forcing has characteristic form with degree $d = 0$ and characteristic $\mu + i\nu = 3$. Because the characteristic 3 is a double root of $p(z)$, it has multiplicity $m = 2$.

Because $d = 0$, $\mu + i\nu = 3$, and $m = 2$, we see from (6.16) that V_P has the form

$$V_P(t) = At^2 e^{3t} .$$

Because

$$\begin{aligned} V_P'(t) &= 3At^2 e^{3t} + 2At\, e^{3t} , \\ V_P''(t) &= 9At^2 e^{3t} + 12At\, e^{3t} + 2Ae^{3t} , \end{aligned}$$

we see that

$$\begin{aligned} \mathrm{L}V_P(t) &= V_P''(t) - 6V_P'(t) + 9V_P(t) \\ &= \big(9At^2 e^{3t} + 12At\, e^{3t} + 2Ae^{3t}\big) \\ &\quad - 6\big(3At^2 e^{3t} + 2At\, e^{3t}\big) + 9At^2 e^{3t} \\ &= (9 - 18 + 9)At^2 e^{3t} + (12 - 12)t\, e^{3t} + 2Ae^{3t} \\ &= 2Ae^{3t} . \end{aligned}$$

If we set $\mathrm{L}V_P(t) = 4e^{3t}$ then we see that $2A = 4$, whereby $A = 2$. Hence, our particular solution is

$$V_P(t) = 2t^2 e^{3t} .$$

Therefore a general solution is

$$v = c_1 e^{3t} + c_2 t\, e^{3t} + 2t^2 e^{3t} .$$

6.3.3. *Forcings with Zero Degree and Complex Characteristic.* Forcings with zero degree ($d = 0$) and complex characteristic ($\nu > 0$) often arise in applications. In this case the characteristic form (6.3) with degree $d = 0$ and characteristic $\mu + i\nu$ reduces to

$$f(t) = \alpha e^{\mu t} \cos(\nu t) + \beta e^{\mu t} \sin(\nu t) , \tag{6.17}$$

where α, β, μ, and ν are real numbers. We assume that either $\alpha \neq 0$ or $\beta\nu \neq 0$, so that $f(t) \neq 0$.

When $d = 0$ and the characteristic $\mu + i\nu$ has multiplicity m then the form for the particular solution (6.14b) reduces to

$$Y_P(t) = At^m e^{\mu t} \cos(\nu t) + Bt^m e^{\mu t} \sin(\nu t) . \tag{6.18}$$

The the first step of the method of Undetermined Coefficients is to substitute this directly into $\mathrm{L}Y_P$, and collect like terms.

Remark. Unlike Key Identity Evaluations, the method of Undetermined Coefficients method does not use complex arithmatic. However, its calculations can be much longer.

The main reason for this that the step of plugging the form (6.18) directly into $\mathrm{L}Y_P$ and collecting like terms can be lengthy. You may have noticed a hint of this in the examples with real characteristics given in the previous subsection. This will become more evident in the examples with complex characteristics given below.

Example. Find a general solution of

$$y'' + 2y' + 10y = \cos(2t)\,.$$

Solution. This equation has constant coefficients. Its characteristic polynomial is

$$p(z) = z^2 + 2z + 10 = (z+1)^2 + 3^2\,,$$

which has the simple conjugate roots $-1 \pm i3$. Hence, a general solution of the associated homogeneous equation is

$$Y_H(t) = c_1 e^{-t}\cos(3t) + c_2 e^{-t}\sin(3t)\,.$$

To find a particular solution, first notice that the forcing has characteristic form with degree $d = 0$ and characteristic $\mu + i\nu = i2$. Because the characteristic $i2$ is not a root of $p(z)$, it has mutplicity $m = 0$.

Because $d = 0$, $\mu + i\nu = i2$, and $m = 0$ we see from (6.18) that Y_P has the form

$$Y_P(t) = A\cos(2t) + B\sin(2t)\,.$$

Because

$$Y_P'(t) = -2A\sin(2t) + 2B\cos(2t)\,, \qquad Y_P''(t) = -4A\cos(2t) - 4B\sin(2t)\,,$$

we see that

$$\begin{aligned} \mathrm{L}Y_P(t) &= Y_P''(t) + 2Y_P'(t) + 10Y_P(t) \\ &= -4A\cos(2t) - 4B\sin(2t) - 4A\sin(2t) + 4B\cos(2t) \\ &\quad + 10A\cos(2t) + 10B\sin(2t) \\ &= (6A + 4B)\cos(2t) + (6B - 4A)\sin(2t)\,. \end{aligned}$$

If we set $\mathrm{L}Y_P(t) = \cos(2t)$ then by equating the coefficients of the linearly independent functions $\cos(2t)$ and $\sin(2t)$ we see that

$$6A + 4B = 1\,, \qquad -4A + 6B = 0\,.$$

Upon solving this system we find that $A = \frac{3}{26}$ and $B = \frac{1}{13}$, whereby our particular solution is

$$Y_P(t) = \tfrac{3}{26}\cos(2t) + \tfrac{1}{13}\sin(2t)\,.$$

Therefore a general solution is

$$y = c_1 e^{-t}\cos(3t) + c_2 e^{-t}\sin(3t) + \tfrac{3}{26}\cos(2t) + \tfrac{1}{13}\sin(2t)\,.$$

Example. Find a general solution of

$$x'' + 9x = 4\cos(3t)\,.$$

Solution. This equation has constant coefficients. Its characteristic polynomial is

$$p(z) = z^2 + 9 = z^2 + 3^2\,,$$

which has the simple conjugate roots $\pm i3$. Hence, a general solution of the associated homogeneous equation is

$$X_H(t) = c_1 \cos(3t) + c_2 \sin(3t)\,.$$

To find a particular solution, first notice that the forcing has characteristic form with degree $d = 0$ and characteristic $\mu + i\nu = i3$. Because the characteristic $i3$ is a simple root of $p(z)$, it has multiplicity $m = 1$.

Because $d = 0$, $\mu + i\nu = i3$, and $m = 1$ we see from (6.18) that X_P has the form

$$X_P(t) = At\,\cos(3t) + Bt\,\sin(3t)\,.$$

Because

$$\begin{aligned} X_P'(t) &= -3At\,\sin(3t) + A\cos(3t) + 3Bt\,\cos(3t) + B\sin(3t)\,,\\ X_P''(t) &= -9At\,\cos(3t) - 6A\sin(3t) - 9Bt\,\sin(3t) + 6B\cos(3t)\,, \end{aligned}$$

we see that

$$\begin{aligned} \mathrm{L}X_P(t) &= X_P''(t) + 9X_P(t)\\ &= \big(-9At\,\cos(3t) - 6A\sin(3t) - 9Bt\,\sin(3t) + 6B\cos(3t)\big)\\ &\quad + 9\big(At\,\cos(3t) + Bt\,\sin(3t)\big)\\ &= (-9+9)At\,\cos(3t) - 6A\sin(3t) + (-9+9)Bt\,\sin(3t) + 6B\cos(3t)\\ &= -6A\sin(3t) + 6B\cos(3t)\,. \end{aligned}$$

If we set $\mathrm{L}X_P(t) = 4\cos(3t)$ then by equating the coefficients of the linearly independent functions $\cos(3t)$ and $\sin(3t)$ we see that

$$-6A = 0\,, \qquad 6B = 4\,.$$

We find that $A = 0$ and $B = \frac{2}{3}$, whereby our particular solution is

$$X_P(t) = \tfrac{2}{3}t\,\sin(3t)\,.$$

Therefore a general solution is

$$x = c_1\cos(3t) + c_2\sin(3t) + \tfrac{2}{3}t\,\sin(3t)\,.$$

Example. Find a general solution of

$$w'' + 2w' + 10w = 5e^{-t}\sin(3t)\,.$$

Solution. This equation has constant coefficients. Its characteristic polynomial is

$$p(z) = z^2 + 2z + 10 = (z+1)^2 + 3^2\,,$$

which has the simple conjugate roots $-1 \pm i3$. Hence, a general solution of the associated homogeneous equation is

$$W_H(t) = c_1 e^{-t}\cos(3t) + c_2 e^{-t}\sin(3t)\,.$$

To find a particular solution, first notice that the forcing has characteristic form with degree $d = 0$ and characteristic $\mu + i\nu = -1 + i3$. Because the characteristic $-1 + i3$ is a simple root of $p(z)$, it has multiplicity $m = 1$.

Because $d = 0$, $\mu + i\nu = -1 + i3$, and $m = 1$ we see from (6.18) that W_P has the form

$$W_P(t) = At\, e^{-t}\cos(3t) + Bt\, e^{-t}\sin(3t)\,.$$

Because

$$\begin{aligned}
W_P'(t) &= Ae^{-t}\cos(3t) - At\, e^{-t}\cos(3t) - 3At\, e^{-t}\sin(3t)\\
&\quad + Be^{-t}\sin(3t) - Bt\, e^{-t}\sin(3t) + 3Bt\, e^{-t}\cos(3t)\\
&= Ae^{-t}\cos(3t) + Be^{-t}\sin(3t) - (A-3B)t\, e^{-t}\cos(3t) - (3A+B)t\, e^{-t}\sin(3t)\,,\\
W_P''(t) &= -Ae^{-t}\cos(3t) - 3Ae^{-t}\sin(3t) - Be^{-t}\sin(3t) + 3Be^{-t}\cos(3t)\\
&\quad - (A-3B)e^{-t}\cos(3t) + (A-3B)t\, e^{-t}\cos(3t) + 3(A-3B)t\, e^{-t}\sin(3t)\\
&\quad - (3A+B)e^{-t}\sin(3t) + (3A+B)t\, e^{-t}\sin(3t) - 3(3A+B)t\, e^{-t}\cos(3t)\\
&= -2(A-3B)e^{-t}\cos(3t) - 2(3A+B)e^{-t}\sin(3t)\\
&\quad - (8A+6B)t\, e^{-t}\cos(3t) + (6A-8B)t\, e^{-t}\sin(3t)\,,
\end{aligned}$$

we see that

$$\begin{aligned}
\mathrm{L}W_P(t) &= W_P''(t) + 2W_P'(t) + 10W_P(t)\\
&= \big(-2(A-3B)e^{-t}\cos(3t) - 2(3A+B)e^{-t}\sin(3t)\\
&\qquad - (8A+6B)t\, e^{-t}\cos(3t) + (6A-8B)t\, e^{-t}\sin(3t)\big)\\
&\quad + 2\big(Ae^{-t}\cos(3t) + Be^{-t}\sin(3t)\\
&\qquad\quad - (A-3B)t\, e^{-t}\cos(3t) - (3A+B)t\, e^{-t}\sin(3t)\big)\\
&\quad + 10\big(At\, e^{-t}\cos(3t) + Bt\, e^{-t}\sin(3t)\big)\\
&= (-2A+6B+2A)e^{-t}\cos(3t) + (-6A-2B+2B)e^{-t}\sin(3t)\\
&\quad + (-8A-6B-2A+6B+10A)t\, e^{-t}\cos(3t)\\
&\quad + (6A-8B-6A-2B+10B)t\, e^{-t}\sin(3t)\\
&= 6Be^{-t}\cos(3t) - 6Ae^{-t}\sin(3t)\,.
\end{aligned}$$

If we set $\mathrm{L}W_P(t) = 5e^{-t}\sin(3t)$ then by equating the coefficients of the linearly independent functions $e^{-t}\cos(3t)$ and $e^{-t}\sin(3t)$ we see that

$$6B = 0\,, \qquad -6A = 5\,.$$

We find that $B = 0$ and $A = -\frac{5}{6}$, whereby our particular solution is

$$W_P(t) = -\tfrac{5}{6}t\, e^{-t}\cos(3t)\,.$$

Therefore a general solution is

$$w = c_1e^{-t}\cos(3t) + c_2e^{-t}\sin(3t) - \tfrac{5}{6}t\, e^{-t}\cos(3t)\,.$$

Remark. Key Identity Evaluations was much faster than Undetermined Coefficients for the last example. This is always the case when $\mu \neq 0$, $\nu > 0$ and $m > 0$.

6.3.4. *Forcings with Positive Degree.* When $d > 0$ then there is a greater number of coefficients that need to be determined. If the forcing has real characteristic ($\nu = 0$) with multiplicity m then the form for the particular solution (6.14b) reduces to

$$Y_P(t) = \big(A_0 t^{m+d} + A_1 t^{m+d-1} + \cdots + A_d t^m\big)e^{\mu t}\,. \tag{6.19}$$

This form has $d+1$ undetermined coefficients: $A_0, \cdots, A_d$. If the forcing has complex characteristic ($\nu > 0$) with multiplicity m then the form for the particular solution (6.14b) is

$$\begin{aligned} Y_P(t) = \big(A_0 t^{m+d} + A_1 t^{m+d-1} + \cdots + A_d t^m\big)e^{\mu t}\cos(\nu t)& \\ + \big(B_0 t^{m+d} + B_1 t^{m+d-1} + \cdots + B_d t^m\big)e^{\mu t}\sin(\nu t)&\,. \end{aligned} \tag{6.20}$$

This form has $2d+2$ undetermined coefficients: $A_0, \cdots, A_d$ and $B_0, \cdots, B_d$.

Example. Find a general solution of

$$x' + 2x' + 10x = 4te^{2t}\,.$$

Solution. This equation has constant coefficients. Its characteristic polynomial is

$$p(z) = z^2 + 2z + 10 = (z+1)^2 + 3^2\,,$$

which has the simple conjugate roots $-1 \pm i3$. Hence, a general solution of the associated homogeneous equation is

$$X_H(t) = c_1 e^{-t}\cos(3t) + c_2 e^{-t}\sin(3t)\,.$$

To find a particular solution, first notice that the forcing has characteristic form with degree $d = 1$ and characteristic $\mu + i\nu = 2$. Because the characteristic 2 is not a root of $p(z)$, it has multiplicity $m = 0$.

Because $d = 1$, $\mu + i\nu = 2$, and $m = 0$, we see from (6.19) that X_P has the form

$$X_P(t) = (A_0 t + A_1)e^{2t}\,.$$

Because

$$X_P'(t) = 2(A_0 t + A_1)e^{2t} + A_0 e^{2t}\,, \qquad X_P''(t) = 4(A_0 t + A_1)e^{2t} + 4A_0 e^{2t}\,,$$

we see that

$$\begin{aligned} \mathrm{L}X_P(t) &= X_P''(t) + 2X_P'(t) + 10X_P(t) \\ &= 4(A_0 t + A_1)e^{2t} + 4A_0 e^{2t} + 4(A_0 t + A_1)e^{2t} + 2A_0 e^{2t} + 10(A_0 t + A_1)e^{2t} \\ &= 18(A_0 t + A_1)e^{2t} + 6A_0 e^{2t} \\ &= 18A_0 te^{2t} + (18A_1 + 6A_0)e^{2t}\,. \end{aligned}$$

If we set $\mathrm{L}X_P(t) = 4te^{2t}$ then by equating the coefficients of the linearly independent functions te^{2t} and e^{2t} we see that

$$18A_0 = 4\,, \qquad 18A_1 + 6A_0 = 0\,.$$

Upon solving this linear algebraic system for A_0 and A_1 we first find that $A_0 = \frac{2}{9}$ and then that $A_1 = -\frac{1}{3}A_0 = -\frac{2}{27}$. Hence, our particular solution is

$$X_P(t) = \tfrac{2}{9}te^{2t} - \tfrac{2}{27}e^{2t}\,.$$

Therefore a general solution is

$$x = c_1 e^{-t}\cos(3t) + c_2 e^{-t}\sin(3t) + \tfrac{2}{9} t e^{2t} - \tfrac{2}{27} e^{2t}\,.$$

Example. Find a general solution of

$$y'' + 4y = t\,\cos(2t)\,.$$

Solution. This equation has constant coefficients. Its characteristic polynomial is

$$p(z) = z^2 + 4 = z^2 + 2^2\,,$$

which has the simple conjugate roots $\pm i2$. Hence, a general solution of the associated homogeneous equation is

$$Y_H(t) = c_1\cos(2t) + c_2\sin(2t)\,.$$

To find a particular solution, first notice that the forcing has characteristic form with degree $d = 1$ and characteristic $\mu + i\nu = i2$. Because the characteristic $i2$ is a simple root of $p(z)$, it has multiplicity $m = 1$.

Because $d = 1$, $\mu + i\nu = i2$, and $m = 1$, we see from (6.20) that Y_P has the form

$$Y_P(t) = (A_0 t^2 + A_1 t)\cos(2t) + (B_0 t^2 + B_1 t)\sin(2t)\,.$$

Because

$$\begin{aligned}
Y_P'(t) &= -2(A_0t^2 + A_1t)\sin(2t) + (2A_0t + A_1)\cos(2t) \\
&\quad + 2(B_0t^2 + B_1t)\cos(2t) + (2B_0t + B_1)\sin(2t) \\
&= \big(2B_0t^2 + 2(B_1 + A_0)t + A_1\big)\cos(2t) - \big(2A_0t^2 + 2(A_1 - B_0)t - B_1\big)\sin(2t)\,, \\
Y_P''(t) &= -2\big(2B_0t^2 + 2(B_1 + A_0)t + A_1\big)\sin(2t) + \big(4B_0t + 2(B_1 + A_0)\big)\cos(2t) \\
&\quad - 2\big(2A_0t^2 + 2(A_1 - B_0)t - B_1\big)\cos(2t) - \big(4A_0t + 2(A_1 - B_0)\big)\sin(2t) \\
&= -\big(4A_0t^2 + (4A_1 - 8B_0)t - 4B_1 - 2A_0\big)\cos(2t) \\
&\quad - \big(4B_0t^2 + (4B_1 + 8A_0)t + 4A_1 - 2B_0\big)\sin(2t)\,,
\end{aligned}$$

we see that

$$\begin{aligned}
\mathrm{L}Y_P(t) &= Y_P''(t) + 4Y_P(t) \\
&= -\Big[\big(4A_0t^2 + (4A_1 - 8B_0)t - 4B_1 - 2A_0\big)\cos(2t) \\
&\qquad + \big(4B_0t^2 + (4B_1 + 8A_0)t + 4A_1 - 2B_0\big)\sin(2t)\Big] \\
&\quad + 4\big[(A_0t^2 + A_1t)\cos(2t) + (B_0t^2 + B_1t)\sin(2t)\big] \\
&= \big(8B_0t + 4B_1 + 2A_0\big)\cos(2t) - \big(8A_0t + 4A_1 - 2B_0\big)\sin(2t)\,.
\end{aligned}$$

If we set $\mathrm{L}Y_P(t) = t\,\cos(2t)$ then by equating the coefficients of the linearly independent functions $\cos(2t)$, $t\,\cos(2t)$, $\sin(2t)$, and $t\,\sin(2t)$, we see that

$$4B_1 + 2A_0 = 0\,, \qquad 8B_0 = 1\,, \qquad 4A_1 - 2B_0 = 0\,, \qquad 8A_0 = 0\,.$$

The solution of this system is $A_0 = 0$, $B_0 = \frac{1}{8}$, $A_1 = \frac{1}{16}$, and $B_1 = 0$, whereby our particular solution is

$$Y_P(t) = \tfrac{1}{16} t\,\cos(2t) + \tfrac{1}{8} t^2\sin(2t)\,.$$

Therefore a general solution is

$$y = c_1 \cos(2t) + c_2 \sin(2t) + \tfrac{1}{16} t \cos(2t) + \tfrac{1}{8} t^2 \sin(2t) \,.$$

Remark. The above example is typical of a case when Key Identity Evaluations is far faster than Undetermined Coefficients. This is because the forcing has a conjugate pair characteristic $\mu \pm i\nu = \pm i2$, positive degree $d = 1$, and small multiplicity $m = 1$. This advantage becomes much more dramatic for larger d. Key Identity Evaluations will usually be as fast or faster than Undetermined Coefficients. If you master both methods you will develop a sense about which one is most efficient for any given problem.

6.4. **Forcings of Composite Characteristic Form.** The methods of Key Identity Evaluations and Undetermined Coefficients can be applied multiple times to construct a particular solution of $\mathrm{L}y = f(t)$ whenever

(1) the differential operator L has constant coefficients,
(2) the forcing $f(t)$ is a sum of terms each of which has characteristic form (6.3) with a distinct characteristic.

When the second of these conditions is satisfied the forcing is said to have *composite characteristic form*. The first of these conditions is always easy to verify by inspection. Verification of the second usually can also be done by inspection, but sometimes it might require the use of a trigonometric or some other identity. You should be able to identify when a forcing $f(t)$ can be expressed as a sum of terms that have the characteristic form (6.3), and when it is, to read-off the degree and characteristic of each component.

Example. The forcing of the equation $\mathrm{L}y = \cos(t)^2$ can be written as a sum of terms that have the characteristic form (6.3) by using the identity $\cos(t)^2 = (1 + \cos(2t))/2$. We see that

$$\mathrm{L}y = \cos(t)^2 = \tfrac{1}{2} + \tfrac{1}{2}\cos(2t) \,.$$

Each term on the right-hand side above has the characteristic form (6.3); the first with degree $d = 0$ and characteristic $\mu + i\nu = 0$, and the second with degree $d = 0$ and characteristic $\mu + i\nu = i2$.

Example. The forcing of the equation $\mathrm{L}y = \sin(2t)\cos(3t)$ can be written as a sum of terms that have the characteristic form (6.3) by using the identity

$$\sin(2t)\cos(3t) = \tfrac{1}{2}\big(\sin(3t + 2t) - \sin(3t - 2t)\big) = \tfrac{1}{2}\big(\sin(5t) - \sin(t)\big) \,.$$

We see that

$$\mathrm{L}y = \sin(2t)\cos(3t) = \tfrac{1}{2}\sin(5t) - \tfrac{1}{2}\sin(t) \,.$$

Each term on the right-hand side above has the characteristic form (6.3); the first with degree $d = 0$ and characteristic $\mu + i\nu = i5$, and the second with degree $d = 0$ and characteristic $\mu + i\nu = i$.

Example. The forcing of the equation $\mathrm{L}y = \tan(t)$ cannot be written as a sum of terms in characteristic form (6.3) because every such function is smooth (infinitely differentiable) while $\tan(t)$ is not defined at $t = \frac{\pi}{2} + m\pi$ for every integer m.

Given a nonhomogeneous equation $\mathrm{L}y = f(t)$ in which the forcing $f(t)$ is a sum of terms, each of which has the characteristic form (6.3), we must first identify the characteristic of each term and group all the terms with the same characteristic together. We then decompose $f(t)$ as

$$f(t) = f_1(t) + f_2(t) + \cdots + f_g(t)\,, \tag{6.21}$$

where each $f_j(t)$ contains all the terms of a given characteristic. Each $f_j(t)$ will then have the characteristic form (6.3) for some degree $d = 0$ and some characteristic $\mu + i\nu$. Then we can apply either Key Identity Evaluations or Undetermined Coefficients to find particular solutions Y_{jP} to each of

$$\mathrm{L}Y_{1P}(t) = f_1(t)\,, \quad \mathrm{L}Y_{2P}(t) = f_2(t)\,, \quad \cdots \quad \mathrm{L}Y_{gP}(t) = f_g(t)\,. \tag{6.22}$$

Then a particular solution of $\mathrm{L}y = f(t)$ is

$$Y_P(t) = Y_{1P}(t) + Y_{2P}(t) + \cdots + Y_{gP}(t)\,. \tag{6.23}$$

This is true because by (6.23), the linearity of L, (6.22), and (6.21), we have

$$\begin{aligned} \mathrm{L}Y_P(t) &= \mathrm{L}\big(Y_{1P}(t) + Y_{2P}(t) + \cdots + Y_{gP}(t)\big) \\ &= \mathrm{L}Y_{1P}(t) + \mathrm{L}Y_{2P}(t) + \cdots + \mathrm{L}Y_{gP}(t) \\ &= f_1(t) + f_2(t) + \cdots + f_g(t) = f(t)\,. \end{aligned}$$

Example. Find a particular solution of $\mathrm{D}^4y + 25\mathrm{D}^2y = f(t)$ where

$$f(t) = e^{2t} + 9\cos(5t) + 4t^2e^{2t} - 7t\sin(5t) + 8 - 6t\,.$$

Solution. The decomposition (6.21) becomes $f(t) = f_1(t) + f_2(t) + f_3(t)$, where

$$f_1(t) = 8 - 6t\,, \qquad f_2(t) = \big(1 + 4t^2\big)e^{2t}\,, \qquad f_3(t) = 9\cos(5t) - 7t\sin(5t)\,.$$

Here $f_1(t)$, $f_2(t)$, and $f_3(t)$ contain all the terms of $f(t)$ with characteristic 0, 2, and $i5$, respectively. They each have the characteristic form (6.3) with degree 1, 2, and 1 respectively. The characteristic polynomial is $p(z) = z^4 + 25z^2 = z^2(z^2 + 5^2)$, which has roots 0, 0, $-i5$, $i5$. We thereby see that the characteristics 0, 2, and $i5$ have multiplicities 2, 0, and 1 respectively.

By (6.22) with $\mathrm{L} = \mathrm{D}^4 + 25\mathrm{D}^2$, we seek particular solutions of the equations

$$\begin{aligned} \mathrm{L}Y_{1P}(t) &= 8 - 6t\,, \\ \mathrm{L}Y_{2P}(t) &= \big(1 + 4t^2\big)e^{2t}\,, \\ \mathrm{L}Y_{3P}(t) &= 9\cos(5t) - 7t\sin(5t)\,. \end{aligned} \tag{6.24}$$

Because the forcings of each of these equations has characteristic form, they each may be solved either by Key Identity Evaluations or by Undetermined Coefficients.

Remark. Notice that by grouping terms in $f(t)$ with the same characteristic together we are led to three nonhomogeneous equations to solve in (6.24). Had we rather chosen to write one nonhomogeneous equation for each term in $f(t)$ then we would have been led to six nonhomogeneous equations to solve, which requires much more work!

The method of Undetermined Coefficients would seek particular solutions of the nonhomogeneous equations in (6.24) that by (6.14b) have the forms

$$\begin{aligned} Y_{1P}(t) &= A_0t^3 + A_1t^2\,, \\ Y_{2P}(t) &= \big(A_0t^2 + A_1t + A_2\big)e^{2t}\,, \\ Y_{3P}(t) &= \big(A_0t^2 + A_1t\big)\cos(5t) + \big(B_0t^2 + B_1t\big)\sin(5t)\,. \end{aligned}$$

The method leads to three systems of linear algebraic equations to solve — systems of two equations, three equations, and four equations. We will not solve them here.

The method of Key Identity Evaluations is almost always the fastest way to solve nonhomogeneous equations whose forcings have composite characteristic form because the Key Identity and its derivatives only have to be computed once. In the problem at hand $m+d$ is 3, 2, and 2 for the forcings $f_1(t)$, $f_2(t)$, and $f_3(t)$ respectively. Therefore we need the Key Identity and its first three derivatives with respect to z:

$$\begin{aligned} \mathrm{L}\big(e^{zt}\big) &= \big(z^4 + 25z^2\big)\,e^{zt}\,, \\ \mathrm{L}\big(t\,e^{zt}\big) &= \big(z^4 + 25z^2\big)\,t\,e^{zt} + \big(4z^3 + 50z\big)\,e^{zt}\,, \\ \mathrm{L}\big(t^2e^{zt}\big) &= \big(z^4 + 25z^2\big)\,t^2e^{zt} + 2\big(4z^3 + 50z\big)\,t\,e^{zt} + \big(12z^2 + 50\big)\,e^{zt}\,, \\ \mathrm{L}\big(t^3e^{zt}\big) &= \big(z^4 + 25z^2\big)\,t^3e^{zt} + 3\big(4z^3 + 50z\big)\,t^2e^{zt} + 3\big(12z^2 + 50\big)\,t\,e^{zt} + 24z\,e^{zt}\,. \end{aligned}$$

For the characteristic 0 we have $m = 2$ and $m + d = 3$, so we evaluate the second through third derivative of the Key Identity at $z = 0$ to obtain

$$\mathrm{L}\big(t^2\big) = 50\,, \qquad \mathrm{L}\big(t^3\big) = 150t\,.$$

It follows that $\mathrm{L}\big(\frac{4}{25}t^2 - \frac{1}{25}t^3\big) = 8 - 6t$, whereby $Y_{1P}(t) = \frac{4}{25}t^2 - \frac{1}{25}t^3$.

For the characteristic 2 we have $m = 0$ and $m + d = 2$, so we evaluate the zeroth through second derivative of the Key Identity at $z = 2$ to obtain

$$\begin{aligned} \mathrm{L}\big(e^{2t}\big) &= 116\,e^{2t}\,, \\ \mathrm{L}\big(t\,e^{2t}\big) &= 116\,t\,e^{2t} + 132e^{2t}\,, \\ \mathrm{L}\big(t^2e^{2t}\big) &= 116\,t^2e^{2t} + 264\,t\,e^{2t} + 98\,e^{2t}\,. \end{aligned}$$

We eliminate $t\,e^{2t}$ from the right-hand sides by multiplying the second equation by $\frac{264}{116}$ and subtracting it from the third equation, thereby obtaining

$$\mathrm{L}\big(t^2e^{2t} - \tfrac{264}{116}t\,e^{2t}\big) = 116\,t^2e^{2t} + \big(98 - \tfrac{264}{116}132\big)\,e^{2t}\,.$$

Dividing this by 29 gives

$$\mathrm{L}\big(\tfrac{1}{29}t^2e^{2t} - \tfrac{66}{29^2}t\,e^{2t}\big) = 4\,t^2e^{2t} + \big(\tfrac{98}{29} - \tfrac{66\cdot 132}{29^2}\big)\,e^{2t}\,.$$

We eliminate e^{2t} from the right-hand side above by multiplying the first equation by $\frac{1}{116}\big(\frac{98}{29} - \frac{66\cdot 132}{29^2}\big)$ and subtracting it from the above equation, thereby obtaining

$$\mathrm{L}\Big(\tfrac{1}{29}t^2e^{2t} - \tfrac{66}{29^2}t\,e^{2t} - \tfrac{1}{116}\big(\tfrac{98}{29} - \tfrac{66\cdot 132}{29^2}\big)\,e^{2t}\Big) = 4\,t^2e^{2t}\,.$$

Next, by multiplying the first equation by $\frac{1}{116}$ and adding it to the above equation we obtain

$$\mathrm{L}\Big(\tfrac{1}{29}t^2e^{2t} - \tfrac{66}{29^2}t\,e^{2t} - \tfrac{1}{116}\big(\tfrac{98}{29} - \tfrac{66\cdot 132}{29^2} - 1\big)\,e^{2t}\Big) = \big(1 + 4\,t^2\big)\,e^{2t}\,,$$

whereby $Y_{2P}(t) = \frac{1}{29}t^2e^{2t} - \frac{66}{29^2}t\,e^{2t} - \frac{1}{116}\big(\frac{98}{29} - \frac{66\cdot 132}{29^2} - 1\big)\,e^{2t}$.

For the characteristic $i5$ we have $m = 1$ and $m + d = 2$, so we evaluate the first through second derivative of the Key Identity at $z = i5$ to obtain

$$\mathrm{L}\big(t\,e^{i5t}\big) = -i250\,e^{i5t}\,, \qquad \mathrm{L}\big(t^2e^{i5t}\big) = -i2\cdot 250\,t\,e^{i5t} - 250\,e^{i5t}\,.$$

Upon multiplying the first equation by i and adding it to the second we find that

$$\mathrm{L}\big(t^2e^{i5t} + it\,e^{i5t}\big) = -i2\cdot 250\,t\,e^{i5t}\,.$$

The first equation and the above equation imply

$$\mathrm{L}\big(i\tfrac{1}{250}t\,e^{i5t}\big) = e^{i5t}\,, \qquad \mathrm{L}\big(\tfrac{1}{500}t^2e^{i5t} + i\tfrac{1}{500}t\,e^{i5t}\big) = -i\,t\,e^{i5t}\,.$$

The real parts of the above equations are

$$\mathrm{L}\big(-\tfrac{1}{250}t\,\sin(5t)\big) = \cos(5t)\,, \qquad \mathrm{L}\big(\tfrac{1}{500}t^2\cos(5t) - \tfrac{1}{500}t\,\sin(5t)\big) = t\,\sin(5t)\,.$$

This implies that

$$\mathrm{L}\big(-\tfrac{9}{250}t\,\sin(5t) - \tfrac{7}{500}t^2\cos(5t) + \tfrac{7}{500}t\,\sin(5t)\big) = 9\cos(5t) - 7t\,\sin(5t)\,,$$

whereby $Y_{3P}(t) = -\frac{11}{500}t\,\sin(5t) - \frac{7}{500}t^2\cos(5t)$.

Finally, putting all the components together by (6.23), a particular solution is

$$\begin{aligned} Y_P(t) &= Y_{1P}(t) + Y_{2P}(t) + Y_{3P}(t) \\ &= \tfrac{4}{25}t^2 - \tfrac{1}{25}t^3 + \tfrac{1}{29}t^2e^{2t} - \tfrac{66}{29^2}t\,e^{2t} - \tfrac{1}{116}\big(\tfrac{98}{29} - \tfrac{66\cdot 132}{29^2} - 1\big)\,e^{2t} \\ &\quad - \tfrac{11}{500}t\,\sin(5t) - \tfrac{7}{500}t^2\cos(5t)\,. \end{aligned}$$

6.5. **Green Functions: Constant Coefficient Case.** This method can be used to construct a particular solution of an n^{th}-order nonhomogeneous linear ODE

$$\mathrm{L}y = f(t) \tag{6.25}$$

whenever the differential operator L has constant coefficients and is in normal form,

$$\mathrm{L} = \mathrm{D}^n + a_1\mathrm{D}^{n-1} + \cdots + a_{n-1}\mathrm{D} + a_n\,. \tag{6.26}$$

6.5.1. *Basic Steps.* The Green function method rests on the fact that for any initial time t_I the particular solution of (6.25) that solves the initial-value problem

$$\mathrm{L}Y_P(t) = f(t)\,, \qquad Y_P(t_I) = Y_P'(t_I) = \cdots = Y_P^{(n-1)}(t_I) = 0\,, \tag{6.27}$$

is given by the formula

$$Y_P(t) = \int_{t_I}^{t} g(t-s) f(s)\,\mathrm{d}s\,, \tag{6.28}$$

where $g(t)$ is the solution of the homogeneous initial-value problem

$$\mathrm{L}g = 0\,, \qquad g(0) = 0\,, \quad g'(0) = 0\,, \quad \cdots \quad g^{(n-2)}(0) = 0\,, \quad g^{(n-1)}(0) = 1\,. \tag{6.29}$$

The function g is called the *Green function* associated with the operator L.

Solving the initial-value problem (6.29) for the Green function is not difficult when the roots of the characteristic polynomials can be found. The method thereby reduces the problem of finding a particular solution $Y_P(t)$ for any forcing $f(t)$ to that of evaluating the integral in formula (6.28). However, evaluating this integral explicitly can be quite difficult or impossible. In such cases the answer might be expressed in terms of definite integrals.

Remark. Formula (6.28) will be correct only if the differential operator L given by (6.26) is normal form! Be sure to put any nonhomogeneous equation into normal form before applying the Green Function method!

We will not verify that $Y_P(t)$ given by formula (6.28) is a solution of the initial-value problem (6.27) here. That will be done in Section 6.6.3. However, here we will illustrate how the method works with a few examples.

Example. Find a particular solution of

$$y'' - y = \frac{2}{e^t + e^{-t}}\,.$$

Solution. The differential equation has constant coefficients and is already in normal form. Its characteristic polynomial is given by $p(z) = z^2 - 1 = (z-1)(z+1)$, which has simple real roots ± 1. Therefore a general solution of the associated homogeneous equation is

$$Y_H(t) = c_1 e^t + c_2 e^{-t}\,.$$

By (6.29) the Green function g is the solution of the initial-value problem

$$g'' - g = 0\,, \qquad g(0) = 0\,, \quad g'(0) = 1\,.$$

Set $g(t) = c_1 e^t + c_2 e^{-t}$. The first initial condition implies $g(0) = c_1 + c_2 = 0$. Because $g'(t) = c_1 e^t - c_2 e^{-t}$, the second condition implies $g'(0) = c_1 - c_2 = 1$. Upon solving these equations we find that $c_1 = \frac{1}{2}$ and $c_2 = -\frac{1}{2}$. Therefore the Green function is $g(t) = \frac{1}{2}(e^t - e^{-t}) = \sinh(t)$.

The particular solution given by (6.28) with $t_I = 0$ is then

$$\begin{aligned} Y_P(t) &= \int_0^t g(t-s)\, f(s)\, \mathrm{d}s = \int_0^t \frac{e^{t-s} - e^{-(t-s)}}{2} \, \frac{2}{e^s + e^{-s}} \, \mathrm{d}s \\ &= \int_0^t \frac{e^{t-s} - e^{-t+s}}{e^s + e^{-s}} \, \mathrm{d}s = e^t \int_0^t \frac{e^{-s}}{e^s + e^{-s}} \, \mathrm{d}s - e^{-t} \int_0^t \frac{e^s}{e^s + e^{-s}} \, \mathrm{d}s \,. \end{aligned}$$

The definite integrals in the above expression can be evaluated as

$$\begin{aligned} \int_0^t \frac{e^{-s}}{e^s + e^{-s}} \, \mathrm{d}s &= \int_0^t \frac{e^{-2s}}{1 + e^{-2s}} \, \mathrm{d}s = -\tfrac{1}{2} \log\bigl(1 + e^{-2s}\bigr)\Big|_0^t = -\tfrac{1}{2} \log\left(\frac{1 + e^{-2t}}{2}\right) , \\ \int_0^t \frac{e^{s}}{e^s + e^{-s}} \, \mathrm{d}s &= \int_0^t \frac{e^{2s}}{e^{2s} + 1} \, \mathrm{d}s = \tfrac{1}{2} \log\bigl(e^{2s} + 1\bigr)\Big|_0^t = \tfrac{1}{2} \log\left(\frac{e^{2t} + 1}{2}\right) . \end{aligned}$$

Therefore the expression for the particular $Y_P(t)$ becomes

$$Y_P(t) = -\tfrac{1}{2} e^t \log\left(\frac{1 + e^{-2t}}{2}\right) - \tfrac{1}{2} e^{-t} \log\left(\frac{e^{2t} + 1}{2}\right) .$$

Remark. Notice that in the above example the definite integral in the expression for $Y_P(t)$ given by (6.28) splits into two definite integrals over s whose integrands do not involve t. This kind of splitting always happens. In general, if L is an n^{th}-order operator then the expression for $Y_P(t)$ given by (6.28) *always* splits into n definite integrals over s whose integrands do not involve t.

6.5.2. *Use of Trigonometric Identities.* When the Green function involves terms like $e^{\mu t} \cos(\nu t)$ or $e^{\mu t} \sin(\nu t)$ then the task of splitting the expression for $Y_P(t)$ given by (6.28) into n definite integrals over s whose integrands do not involve t requires using the trigonometric identities

$$\begin{aligned} \cos(\phi - \psi) &= \cos(\phi)\cos(\psi) + \sin(\phi)\sin(\psi) \,, \\ \sin(\phi - \psi) &= \sin(\phi)\cos(\psi) - \cos(\phi)\sin(\psi) \,. \end{aligned} \tag{6.30}$$

These identities can be derived easily using the Euler identity $e^{i\theta} = \cos(\theta) + i\sin(\theta)$ and the law of exponents $e^{i(\phi-\psi)} = e^{i\phi} e^{-i\psi}$. Indeed, we have the identity

$$\begin{aligned} \cos(\phi - \psi) + i\sin(\phi - \psi) = e^{i(\phi-\psi)} &= e^{i\phi} e^{-i\psi} \\ &= \bigl(\cos(\phi) + i\sin(\phi)\bigr)\bigl(\cos(\psi) - i\sin(\psi)\bigr) \\ &= \bigl(\cos(\phi)\cos(\psi) + \sin(\phi)\sin(\psi)\bigr) \\ &\quad + i\bigl(\sin(\phi)\cos(\psi) - \cos(\phi)\sin(\psi)\bigr) \,. \end{aligned}$$

The identities in (6.30) follow by equating the real and imaginary parts of this identity. You should be familiar with them. The following examples show how they are used to split the expression for $Y_P(t)$ given by (6.28) when the Green function involves terms like $e^{\mu t} \cos(\nu t)$ or $e^{\mu t} \sin(\nu t)$.

Example. Find a particular solution of

$$x'' + 9x = \frac{27}{16 + 9\sin(3t)^2} \,.$$

Solution. The differential equation has constant coefficients and is already in normal form. Its characteristic polynomial is $p(z) = z^2 + 9 = z^2 + 3^2$, which has the simple conjugate roots $\pm i3$. Therefore a general solution of the associated homogeneous equation is

$$X_H(t) = c_1 \cos(3t) + c_2 \sin(3t)\,.$$

By (6.29) the Green function g is the solution of the initial-value problem

$$g'' + 9g = 0\,, \qquad g(0) = 0\,, \quad g'(0) = 1\,.$$

Set $g(t) = c_1 \cos(3t) + c_2 \sin(3t)$. The first initial condition implies $g(0) = c_1 = 0$, whereby $g(t) = c_2 \sin(3t)$. Because $g'(t) = 3c_2 \cos(3t)$, the second condition implies $g'(0) = 3c_2 = 1$, whereby $c_2 = \frac{1}{3}$. Therefore the Green function is $g(t) = \frac{1}{3}\sin(3t)$.

The particular solution given by (6.28) with $t_I = 0$ is then

$$X_P(t) = \int_0^t g(t-s) f(s)\,\mathrm{d}s = \int_0^t \frac{\sin\bigl(3(t-s)\bigr)}{3}\,\frac{27}{16 + 9\sin(3s)^2}\,\mathrm{d}s\,.$$

By (6.30) with $\phi = 3t$ and $\psi = 3s$, we see $\sin\bigl(3(t-s)\bigr) = \sin(3t)\cos(3s) - \cos(3t)\sin(3s)$. We can use this to express $X_P(t)$ as

$$X_P(t) = \sin(3t) \int_0^t \frac{9\cos(3s)}{16 + 9\sin(3s)^2}\,\mathrm{d}s - \cos(3t) \int_0^t \frac{9\sin(3s)}{16 + 9\sin(3s)^2}\,\mathrm{d}s\,.$$

The definite integrals in the above expression can be evaluated as

$$\begin{aligned}
\int_0^t \frac{9\cos(3s)}{16 + 9\sin(3s)^2}\,\mathrm{d}s &= \int_0^t \frac{\frac{9}{16}\cos(3s)}{1 + \frac{9}{16}\sin(3s)^2}\,\mathrm{d}s \\
&= \tfrac{1}{4}\tan^{-1}\bigl(\tfrac{3}{4}\sin(3s)\bigr)\Big|_0^t = \tfrac{1}{4}\tan^{-1}\bigl(\tfrac{3}{4}\sin(3t)\bigr)\,. \\
\int_0^t \frac{9\sin(3s)}{16 + 9\sin(3s)^2}\,\mathrm{d}s &= \int_0^t \frac{9\sin(3s)}{25 - 9\cos(3s)^2}\,\mathrm{d}s = \int_0^t \frac{\frac{9}{25}\sin(3s)}{1 - \frac{9}{25}\cos(3s)^2}\,\mathrm{d}s \\
&= -\tfrac{1}{10}\log\left(\frac{1 + \frac{3}{5}\cos(3s)}{1 - \frac{3}{5}\cos(3s)}\right)\Bigg|_0^t = -\tfrac{1}{10}\log\left(\frac{1 + \frac{3}{5}\cos(3t)}{1 - \frac{3}{5}\cos(3t)}\,\frac{\frac{2}{5}}{\frac{8}{5}}\right)\,.
\end{aligned}$$

Here the first integral has the form

$$\tfrac{1}{4}\int \frac{\mathrm{d}u}{1 + u^2} = \tfrac{1}{4}\tan^{-1}(u) + C\,, \qquad \text{where } u = \tfrac{3}{4}\sin(3s)\,,$$

while by using partial fractions we see that the second has the form

$$-\tfrac{1}{5}\int \frac{\mathrm{d}u}{1 - u^2} = -\tfrac{1}{10}\log\left(\frac{1 + u}{1 - u}\right) + C\,, \qquad \text{where } u = \tfrac{3}{5}\cos(3s)\,.$$

Therefore the expression for the particular solution $X_P(t)$ becomes

$$X_P(t) = \tfrac{1}{4}\sin(3t)\tan^{-1}\bigl(\tfrac{3}{4}\sin(3t)\bigr) + \tfrac{1}{10}\cos(3t)\log\left(\frac{5 + 3\cos(3t)}{5 - 3\cos(3t)}\,\frac{1}{4}\right)\,.$$

Remark. Any integral whose integrand is a rational function of sine and cosine can be evaluated. The integrals in the above example are of this type.

The next example illustrates what happens in most instances when the Green function method is applied — namely, the integrals that arise cannot be evaluated analytically.

Example. Find a particular solution of

$$v'' + 2v' + 5v = \frac{1}{1+t^2}\,.$$

Solution. The differential equation has constant coefficients and is already in normal form. Its characteristic polynomial is $p(z) = z^2 + 2z + 5 = (z+1)^2 + 2^2$, which has the simple conjugate roots $-1 \pm i2$. Therefore a general solution of the associated homogeneous equation is

$$V_H(t) = c_1 e^{-t}\cos(2t) + c_2 e^{-t}\sin(2t)\,.$$

By (6.29) the Green function g is the solution of the initial-value problem

$$g'' + 2g' + 5g = 0\,, \qquad g(0) = 0\,, \quad g'(0) = 1\,.$$

Set $g(t) = c_1 e^{-t}\cos(2t) + c_2 e^{-t}\sin(2t)$. The first initial condition implies $g(0) = c_1 = 0$, whereby $g(t) = c_2 e^{-t}\sin(2t)$. Because $g'(t) = 2c_2 e^{-t}\cos(2t) - c_2 e^{-t}\sin(2t)$, the second condition implies $g'(0) = 2c_2 = 1$, whereby $c_2 = \frac{1}{2}$. Therefore the Green function is $g(t) = \frac{1}{2}e^{-t}\sin(2t)$.

The particular solution given by (6.28) with $t_I = \pi$ is then

$$V_P(t) = \int_\pi^t g(t-s)f(s)\,\mathrm{d}s = \int_\pi^t \tfrac{1}{2}e^{-t+s}\sin\bigl(2(t-s)\bigr)\frac{1}{1+s^2}\,\mathrm{d}s\,.$$

By (6.30) with $\phi = 2t$ and $\psi = 2s$, we see $\sin\bigl(2(t-s)\bigr) = \sin(2t)\cos(2s) - \cos(2t)\sin(2s)$. We can use this to express $V_P(t)$ as

$$V_P(t) = \tfrac{1}{2}e^{-t}\sin(2t)\int_\pi^t \frac{e^s\cos(2s)}{1+s^2}\,\mathrm{d}s - \tfrac{1}{2}e^{-t}\cos(2t)\int_\pi^t \frac{e^s\sin(2s)}{1+s^2}\,\mathrm{d}s\,.$$

The above definite integrals cannot be evaluated analytically. Whenever this is the case, the answer can be left in terms of the integrals.

Remark. The Green function method should never be used whenever the methods of Key Identity Evaluations and Undetermined Coefficients can be applied. For example, for the equation

$$y'' + 2y' + 5y = t\,,$$

the Green function method leads to the expression

$$Y_P(t) = \tfrac{1}{2}e^{-t}\sin(2t)\int_0^t e^s\cos(2s)s\,\mathrm{d}s - \tfrac{1}{2}e^{-t}\cos(2t)\int_0^t e^s\sin(2s)s\,\mathrm{d}s\,.$$

The evaluation of these integrals requires several integration-by-parts. The time it would take to do these integrals is much longer than the time it would take to carry out either of the other two methods, both of which quickly yield $Y_p(t) = \frac{1}{5}t - \frac{2}{25}$!

The above examples asked for a particular solution of the given nonhomogeneous equation. If they had asked for a general solution then we would have had to add a general solution of the associated homogeneous equation to the particular solution that we found above. A general solution for the first example is $y = Y_H(t) + Y_P(t)$, which is

$$y = c_1 e^t + c_2 e^{-t} - \tfrac{1}{2} e^t \log\left(\frac{1 + e^{-2t}}{2}\right) - \tfrac{1}{2} e^{-t} \log\left(\frac{e^{2t} + 1}{2}\right).$$

A general solution for the second example is $x = X_H(t) + X_P(t)$, which is

$$\begin{aligned} x &= c_1 \cos(3t) + c_2 \sin(3t) \\ &\quad + \tfrac{1}{4} \sin(3t) \tan^{-1}\bigl(\tfrac{3}{4} \sin(3t)\bigr) + \tfrac{1}{10} \cos(3t) \log\left(\frac{5 + 3\cos(3t)}{5 - 3\cos(3t)} \frac{1}{4}\right). \end{aligned}$$

A general solution for the third example is $v = V_H(t) + V_P(t)$, which is

$$\begin{aligned} v &= c_1 e^{-t} \cos(2t) + c_2 e^{-t} \sin(2t) \\ &\quad + \tfrac{1}{2} e^{-t} \sin(2t) \int_\pi^t \frac{e^s \cos(2s)}{1 + s^2}\, \mathrm{d}s - \tfrac{1}{2} e^{-t} \cos(2t) \int_\pi^t \frac{e^s \sin(2s)}{1 + s^2}\, \mathrm{d}s. \end{aligned}$$

Such gerenal solutions can be used to solve initial-value problems for nonhomogeneous equations as was done in Section 5.2. However, there is a more efficient approach to solving initial-value problems when the Green function method is used to generate a particular solution.

6.5.3. *Initial-Value Problems.* The Green function method is well-suited to solving initial-value problems because the particular solution $Y_P(t)$ given by formula (6.28) satisfies the initial-value problem (6.27). It is easily checked that the solution of the initial-value problem

$$\text{(6.31)} \qquad \mathrm{L}y = f(t), \qquad y(t_I) = y_0, \quad y'(t_I) = y_1, \quad \cdots \quad y^{(n-1)}(t_I) = y_{n-1},$$

is given by $y = Y_I(t) + Y_P(t)$ where $Y_I(t)$ is the solution of the associated homogeneous initial-value problem

$$\text{(6.32)} \qquad \mathrm{L}Y_I(t) = 0, \qquad Y_I(t_I) = y_0, \quad Y_I'(t_I) = y_1, \quad \cdots \quad Y_I^{(n-1)}(t_I) = y_{n-1},$$

and $Y_P(t)$ is given by (6.28), which is the solution of the initial-value problem (6.27). Indeed, by the linearity of L we have

$$\mathrm{L}y = \mathrm{L}\bigl(Y_I(t) + Y_P(t)\bigr) = \mathrm{L}Y_I(t) + \mathrm{L}Y_P(t) = 0 + f(t) = f(t),$$

while by the initial conditions in (6.32) and (6.27) we have

$$y^{(k)}(t_I) = Y_I^{(k)}(t_I) + Y_P^{(k)}(t_I) = y_k + 0 = y_k \quad \text{for } k = 0, \cdots, n-1.$$

Therefore $y = Y_I(t) + Y_P(t)$ solves the initial-value problem (6.31).

Example. Solve the initial-value problem

$$u'' - 6u' + 9u = \frac{8e^{3t}}{1 + 4t^2}, \qquad u(0) = 0, \quad u'(0) = 7.$$

Solution. The differential equation has constant coefficients and is already in normal form. Its characteristic polynomial is $p(z) = z^2 - 6z + 9 = (z-3)^2$, which has the double root 3. Therefore a general solution of the associated homogeneous equation is

$$U_H(t) = c_1 e^{3t} + c_2 t\, e^{3t} \,.$$

By (6.29) the Green function g is the solution of the initial-value problem

$$g'' - 6g' + 9g = 0 \,, \qquad g(0) = 0 \,, \quad g'(0) = 1 \,.$$

Set $g(t) = c_1 e^{3t} + c_2 t\, e^{3t}$. The first initial condition implies $g(0) = c_1 = 0$. Because then $g'(t) = c_2 e^{3t} + 3c_2 t\, e^{3t}$, the second condition implies $g'(0) = c_2 = 1$. Therefore the Green function is $g(t) = t\, e^{3t}$.

The particular solution given by (6.28) with $t_I = 0$ is

$$\begin{aligned} U_P(t) &= \int_0^t g(t-s)\, f(s)\, \mathrm{d}s = \int_0^t (t-s)\, e^{3(t-s)}\, \frac{8e^{3s}}{1+4s^2}\, \mathrm{d}s \\ &= t\, e^{3t} \int_0^t \frac{8}{1+4s^2}\, \mathrm{d}s - e^{3t} \int_0^t \frac{8s}{1+4s^2}\, \mathrm{d}s \,. \end{aligned}$$

The definite integrals in the above expression can be evaluated as

$$\begin{aligned} \int_0^t \frac{8}{1+4s^2}\, \mathrm{d}s &= 4\tan^{-1}(2s)\Big|_{s=0}^{t} = 4\tan^{-1}(2t) \,, \\ \int_0^t \frac{8s}{1+4s^2}\, \mathrm{d}s &= \log(1+4s^2)\Big|_{s=0}^{t} = \log(1+4t^2) \,. \end{aligned}$$

The expression for the particular solution $U_P(t)$ thereby becomes

$$U_P(t) = 4t\, e^{3t} \tan^{-1}(2t) - e^{3t} \log(1+4t^2) \,.$$

By (6.32) the associated homogeneous initial-value problem for $U_I(t)$ is

$$U_I'' - 6U_I' + 9U_I = 0 \,, \qquad U_I(0) = 0 \,, \quad U_I'(0) = 7 \,.$$

Upon comparing this with the initial-value problem that we used to compute $g(t)$, we see that $U_I(t) = 7g(t) = 7t\, e^{3t}$. Therefore the solution of the original initial-value problem is given by $u = U_I(t) + U_P(t)$, which is

$$u = 7t\, e^{3t} + 4t\, e^{3t} \tan^{-1}(2t) - e^{3t} \log(1+4t^2) \,.$$

6.6. **Why the Methods Work.** This section collects arguments that show why the methods of Key Identity Evaluations, Undetermined Coefficients, and Green Functions work. These arguments can be skipped at first reading. They will be easier to follow after you have fully mastered the methods. There are benefits to understanding them. For example, they will give a deeper understanding of the steps in each method. They also show how the methods of Key Identity Evaluations and Undetermined Coefficients are related.

6.6.1. *Key Identity Evaluations.* While the examples presented in Section 6.2 show how the method of Key Identity Evaluations works, they do not show why it works. Specifically, they do not show why it can find a particular solution for every forcing that has characteristic form. Here we will show why this is true. You do not need to know these arguments.

If the forcing $f(t)$ has the characteristic form (6.3) with degree d and characteristic $\mu + i\nu$ then it can be written as

$$f(t) = \mathrm{Re}\Big(\big(\gamma_0 t^d + \gamma_1 t^{d-1} + \cdots + \gamma_{d-1} t + \gamma_d\big)\, e^{(\mu+i\nu)t}\Big)\,,$$

where $\gamma_k = \alpha_k - i\beta_k$ for every $k = 0, 1, \cdots, d$.

Now let $p(z)$ be the characteristic polynomial of the operator L. The k^{th} derivative of the Key Identity with respect to z is

$$\mathrm{L}\big(t^k e^{zt}\big) = \sum_{j=0}^{k} \binom{k}{j}\, p^{(k-j)}(z)\, t^j e^{zt}\,,$$

where we recall that the binomial coefficient is defined by

$$\binom{k}{j} = \frac{k!}{(k-j)!\, j!}\,.$$

Suppose that $\mu + i\nu$ is a root of $p(z)$ of multiplicity m. For any $C_0, C_1, \cdots, C_d$ we have

$$\begin{aligned}
\mathrm{L}\left(\sum_{k=0}^{d} C_{d-k} t^{m+k} e^{(\mu+i\nu)t}\right) &= \sum_{k=0}^{d} C_{d-k} \mathrm{L}\big(t^{m+k} e^{(\mu+i\nu)t}\big) \\
&= \sum_{k=0}^{d} C_{d-k} \left(\sum_{j=0}^{m+k} \binom{m+k}{j} p^{(m+k-j)}(z)\, t^j e^{zt}\right) \\
&= \sum_{k=0}^{d} \sum_{j=0}^{k} C_{d-k} \binom{m+k}{j} p^{(m+k-j)}(\mu+i\nu)\, t^j e^{(\mu+i\nu)t} \\
&= \sum_{j=0}^{d} \left(\sum_{k=j}^{d} \binom{m+k}{j} p^{(m+k-j)}(\mu+i\nu) C_{d-k}\right) t^j e^{(\mu+i\nu)t} \\
&= \sum_{j=0}^{d} \gamma_{d-j}\, t^j e^{(\mu+i\nu)t}\,,
\end{aligned}$$

where $\gamma_0, \gamma_1, \cdots, \gamma_d$ are related to $C_0, C_1, \cdots, C_d$ by

$$\gamma_{d-j} = \sum_{k=j}^{d} \binom{m+k}{j} p^{(m+k-j)}(\mu+i\nu) C_{d-k} \qquad \text{for every } j = 0, \cdots, d\,.$$

In particular, these $d+1$ equations have the form

$$\begin{aligned}
\gamma_0 &= \binom{m+d}{d} p^{(m)}(\mu+i\nu)C_0\,,\\
\gamma_1 &= \binom{m+d}{d-1} p^{(m+1)}(\mu+i\nu)C_0 + \binom{m+d-1}{d-1} p^{(m)}(\mu+i\nu)C_1\,,\\
\gamma_2 &= \binom{m+d}{d-2} p^{(m+2)}(\mu+i\nu)C_0\\
&\quad + \binom{m+d-1}{d-2} p^{(m+1)}(\mu+i\nu)C_1 + \binom{m+d-2}{d-2} p^{(m)}(\mu+i\nu)C_2\,,\\
&\ \vdots\\
\gamma_d &= p^{(m+d)}(\mu+i\nu)C_0 + p^{(m+d-1)}(\mu+i\nu)C_1 + \cdots + p^{(m)}(\mu+i\nu)C_d\,,
\end{aligned} \tag{6.33}$$

Because $p^{(m)}(\mu+i\nu) \neq 0$, system (6.33) can be solved for C_0, C_1, $\cdots$, C_d for any given γ_0, γ_1, $\cdots$, γ_d. We first solve the first equation for C_0 and plug the result into the remaining equations. We then solve the second equation for C_1 and plug the result into the remaining equations. This continues until we finally solve the last equation for C_d.

Let C_0, C_1, $\cdots$, C_d be the solution of system (6.33) when we set $\gamma_k = \alpha_k - i\beta_k$ for every $k = 0, 1, \cdots, d$. Then a particular solution of $\mathrm{L}y = f(t)$ is given by

$$Y_P(t) = \mathrm{Re}\Big(\big(C_0 t^{m+d} + C_1 t^{m+d-1} + \cdots + C_{d-1}t^{m+1} + C_d t^m\big)\, e^{(\mu+i\nu)t}\Big)\,. \tag{6.34}$$

6.6.2. *Undetermined Coefficients.* While the examples presented in Section 6.3 show how the method of Undetermined Coefficients works, they do not show why it works. Specifically, they do not show why there is a particular solution of the form (6.14b) for every forcing that has the characteristic form (6.3). We will now show why this is the case. In fact, we will give two arguments. While you do not need to know these arguments, understanding them might help you remember the form (6.14b).

The first argument builds upon the argument in the previous subsection. There we showed that for every forcing in the characteristic form (6.3) there is a particular solution $Y_P(t)$ of the equation $\mathrm{L}y = f(t)$ given by (6.34). Then we can define real A_k and B_k for every $k = 0, \cdots, d$ by the relation $C_k = A_k - iB_k$. Upon placing these relations into (6.34), we see that this $Y_P(t)$ has the form (6.14b).

The second argument gives another insight into how the general form (6.14b) arises. We will break it into two cases.

First, suppose the forcing $f(t)$ has the characteristic form (6.3) with real characteristic μ and degree d — i.e. with $\nu = 0$. Let $p(z)$ be the characteristic polynomial of the operator L. Suppose that μ is a root of $p(z)$ of multiplicity m. Then $p(z)$ can be factored as $p(z) = (z-\mu)^m q(z)$ where $q(\mu) \neq 0$. Observe the characteristic form of $f(t)$ implies that it satisfies the homogeneous linear equation

$$(\mathrm{D} - \mu)^{d+1} f(t) = 0\,.$$

Then every solution of $\mathrm{L}y = f(t)$ also satisfies the homogeneous equation

$$(\mathrm{D}-\mu)^{d+1}\mathrm{L}y = (\mathrm{D}-\mu)^{d+1}f(t) = 0\,.$$

The characteristic polynomial of $(\mathrm{D}-\mu)^{d+1}\mathrm{L}$ is $r(z) = (z-\mu)^{d+1}p(z)$, which factors as

$$r(z) = (z-\mu)^{d+1}p(z) = (z-\mu)^{m+d+1}q(z)\,, \quad \text{where } q(\mu) \neq 0\,.$$

Therefore μ is a root of $r(z)$ of multiplicity $m+d+1$. All the other roots of $r(z)$ and their multiplicities are determined by the factors of $q(z)$. Therefore a fundamental set of solutions of the homogeneous equation $(\mathrm{D}-\mu)^{d+1}\mathrm{L}y = 0$ is

$$e^{\mu t}\,, \quad t\,e^{\mu t}\,, \quad \cdots\,, \quad t^{m+d}e^{\mu t}\,,$$

plus the solutions generated by the roots of $q(z)$. All of these solutions are also solutions of the homogenous equation $\mathrm{L}w = 0$ except

$$t^{m}e^{\mu t}\,, \quad t^{m+1}e^{\mu t}\,, \quad \cdots\,, \quad t^{m+d}e^{\mu t}\,.$$

Hence, every solution of $\mathrm{L}y = f(t)$ can be written as $y = Y_H(t) + Y_P(t)$ where $Y_H(t)$ solves the associated homogeneous equation and $Y_P(t)$ has the form (6.14b) with $\nu = 0$.

Next, suppose the forcing $f(t)$ has the characteristic form (6.3) with degree d and characteristic $\mu + i\nu$ where $\nu > 0$. Let $p(z)$ be the characteristic polynomial of the operator L. Suppose that $\mu + i\nu$ is a root of $p(z)$ of multiplicity m. Then $p(z)$ can be factored as $p(z) = \big((z-\mu)^2+\nu^2\big)^m q(z)$ where $q(\mu+i\nu) \neq 0$. Observe the characteristic form of $f(t)$ implies that it satisfies the homogeneous linear equation

$$\big((\mathrm{D}-\mu)^2+\nu^2\big)^{d+1}f(t) = 0\,.$$

Then every solution of $\mathrm{L}y = f(t)$ also satisfies the homogeneous equation

$$\big((\mathrm{D}-\mu)^2+\nu^2\big)^{d+1}\mathrm{L}y = \big((\mathrm{D}-\mu)^2+\nu^2\big)^{d+1}f(t) = 0\,.$$

The characteristic polynomial of $\big((\mathrm{D}-\mu)^2+\nu^2\big)^{d+1}\mathrm{L}$ is $r(z) = \big((z-\mu)^2+\nu^2\big)^{d+1}p(z)$, which factors as

$$r(z) = \big((z-\mu)^2+\nu^2\big)^{d+1}p(z) = \big((z-\mu)^2+\nu^2\big)^{m+d+1}q(z)\,, \quad \text{where } q(\mu+i\nu) \neq 0\,.$$

Therefore $\mu + i\nu$ is a root of $r(z)$ of multiplicity $m+d+1$. All the other roots of $r(z)$ and their multiplicities are determined by the factors of $q(z)$. Therefore a fundamental set of solutions of the homogeneous equation $\big((\mathrm{D}-\mu)^2+\nu^2\big)^{d+1}\mathrm{L}y = 0$ is

$$\begin{aligned} &e^{\mu t}\cos(\nu t)\,, \quad t\,e^{\mu t}\cos(\nu t)\,, \quad \cdots\,, \quad t^{m+d}e^{\mu t}\cos(\nu t)\,,\\ &e^{\mu t}\sin(\nu t)\,, \quad t\,e^{\mu t}\sin(\nu t)\,, \quad \cdots\,, \quad t^{m+d}e^{\mu t}\sin(\nu t)\,,\end{aligned}$$

plus the solutions generated by the roots of $q(z)$. All of these solutions are also solutions of the homogenous equation $\mathrm{L}w = 0$ except

$$\begin{aligned} &t^{m}e^{\mu t}\cos(\nu t)\,, \quad t^{m+1}e^{\mu t}\cos(\nu t)\,, \quad \cdots\,, \quad t^{m+d}e^{\mu t}\cos(\nu t)\,,\\ &t^{m}e^{\mu t}\sin(\nu t)\,, \quad t^{m+1}e^{\mu t}\sin(\nu t)\,, \quad \cdots\,, \quad t^{m+d}e^{\mu t}\sin(\nu t)\,.\end{aligned}$$

Hence, every solution of $\mathrm{L}y = f(t)$ can be written as $y = Y_H(t) + Y_P(t)$ where $Y_H(t)$ solves the associated homogeneous equation and $Y_P(t)$ has the form (6.14b).

6.6.3. *Green Functions.* While the examples presented in Section 6.5 show how the method of Green Functions works, they do not show why it works. Now let us verify that $Y_P(t)$ given by formula (6.28) indeed solves the initial-value problem (6.27) when $g(t)$ is the solution of the initial-value problem (6.29). We will use the fact from multivariable calculus that for any continuously differentiable $K(t,s)$ we have the identity

$$\mathrm{D}\int_{t_I}^{t} K(t,s)\,\mathrm{d}s = K(t,t) + \int_{t_I}^{t} \partial_t K(t,s)\,\mathrm{d}s\,, \qquad \text{where} \quad \mathrm{D} = \frac{\mathrm{d}}{\mathrm{d}t}\,. \tag{6.35}$$

Formula (6.28) for $Y_P(t)$, identity (6.35) with $K(t,s) = g(t-s)f(s)$, and the fact $g(0) = 0$, imply that

$$\mathrm{D}Y_P(t) = g(0)f(t) + \int_{t_I}^{t} \mathrm{D}g(t-s)f(s)\,\mathrm{d}s = \int_{t_I}^{t} \mathrm{D}g(t-s)f(s)\,\mathrm{d}s\,.$$

If $2 < n$ then the above formula for $\mathrm{D}Y_P(t)$, identity (6.35) with $K(t,s) = \mathrm{D}g(t-s)f(s)$, and the fact $\mathrm{D}g(0) = g'(0) = 0$ imply that

$$\mathrm{D}^2 Y_P(t) = g'(0)f(t) + \int_{t_I}^{t} \mathrm{D}^2 g(t-s)f(s)\,\mathrm{d}s = \int_{t_I}^{t} \mathrm{D}^2 g(t-s)f(s)\,\mathrm{d}s\,.$$

If we continue to argue this way then because $\mathrm{D}^{k-1}g(0) = g^{(k-1)}(0) = 0$ for $k < n$, we see that for every $k < n$

$$\mathrm{D}^k Y_P(t) = g^{(k-1)}(0)f(t) + \int_{t_I}^{t} \mathrm{D}^k g(t-s)f(s)\,\mathrm{d}s = \int_{t_I}^{t} \mathrm{D}^k g(t-s)f(s)\,\mathrm{d}s\,. \tag{6.36}$$

Similarly, because $\mathrm{D}^{n-1}g(0) = g^{(n-1)}(0) = 1$, we see that

$$\mathrm{D}^n Y_P(t) = g^{(n-1)}(0)f(t) + \int_{t_I}^{t} \mathrm{D}^n g(t-s)f(s)\,\mathrm{d}s = f(t) + \int_{t_I}^{t} \mathrm{D}^n g(t-s)f(s)\,\mathrm{d}s\,.$$

Because $\mathrm{L}g(t) = 0$, it follows that $\mathrm{L}g(t-s) = 0$. Then by the above formulas for $\mathrm{D}^k Y_P(t)$, we see that

$$\begin{aligned}
\mathrm{L}Y_P(t) = p(\mathrm{D})Y_P(t) &= \mathrm{D}^n Y_P(t) + a_1 \mathrm{D}^{n-1} Y_P(t) + \cdots + a_{n-1}\mathrm{D}Y_P(t) + a_n Y_P(t) \\
&= f(t) + \int_{t_I}^{t} \mathrm{D}^n g(t-s)f(s)\,\mathrm{d}s + \int_{t_I}^{t} a_1 \mathrm{D}^{n-1} g(t-s)f(s)\,\mathrm{d}s \\
&\quad + \cdots + \int_{t_I}^{t} a_{n-1}\mathrm{D}g(t-s)f(s)\,\mathrm{d}s + \int_{t_I}^{t} a_n g(t-s)f(s)\,\mathrm{d}s \\
&= f(t) + \int_{t_I}^{t} p(\mathrm{D})g(t-s)f(s)\,\mathrm{d}s \\
&= f(t) + \int_{t_I}^{t} \mathrm{L}g(t-s)f(s)\,\mathrm{d}s = f(t)\,.
\end{aligned}$$

Therefore, $Y_P(t)$ given by (6.27) satisfies $\mathrm{L}Y_P(t) = f(t)$. Moreover, we see from (6.36) that $Y_P(t)$ satisfies $Y_P(t_I) = Y_P'(t_I) = \cdots = Y_P^{(n-1)}(t_I) = 0$. Therefore $Y_P(t)$ solves the initial-value problem (6.27).

EXERCISES ON NONHOMOGENEOUS EQUATIONS WITH CONSTANT COEFFICIENTS

(1) Which of the following forcing terms have characteristic form? For the ones that do, give the characteristic and the degree.
  (a) u
  (b) $te^{-\frac{t}{2}}$
  (c) $(v^2+v+1)e^v$
  (d) $te^t\sin t+(t^2+1)e^t\cos t$
  (e) $\frac{\sin(t)}{t}\,e^{-3t}$
  (f) $z\cos(z)$
  (g) $\frac{\cos(x)}{e^x}+e^{2x}\sin(x)$
  (h) $t\tan(t)+t\sec(t)$
  (i) $\cos(2w)$
  (j) $\cos(3y)\cos(4y)$

Solution

For #2–#5, find a general solution to the differential equation by using undetermined coefficients.

(2) $v''+3v'+2v=200e^{3u}$

Short Answer
Solution

(3) $w''-4w'+5w=e^{2x}$

Short Answer
Solution

(4) $\ddot{y}+5\dot{y}+6y=e^{-3t}$

Short Answer
Solution

(5) $y''+2y'-3y=16e^t\cos(2t)$

Short Answer
Solution

For #6–#9, find a general solution to the differential equation by using key identity evaluations.

(6) $y''-6y'+8y=\cos(2t)$

Short Answer
Solution

(7) $y''-4y'+13y=5e^{3t}$

Short Answer

Solution

(8) $w'' + 9w = 2\sin(3z)$

Short Answer
Solution

(9) $x'' + x = t\cos(t)$

Short Answer
Solution

For #10–#20, solve the following initial value problems. Some of them will be easier with undetermined coefficients and some will be easier with key identity evaluations, but the choice of which method to use is left up to you.

(10) $y'' - y' = 8,\ \ y(0) = 2,\ \ y'(0) = 8$

Short Answer
Solution

(11) $v'' - 4v = 15e^{3u},\ \ v(0) = 5,\ v'(0) = 21$

Short Answer
Solution

(12) $y'' - 2y' + 2y = 100\cos(2t),\ \ y(0) = 0,\ \ y'(0) = -25$

Short Answer
Solution

(13) $w'' - 3w' + 2w = 4ze^{z}\cos(z),\ \ w(0) = 0,\ \ w'(0) = 0$

Short Answer
Solution

(14) $y''-5y'+6y = 2\sin(2t)\sin(3t),\ \ y(0) = \frac{1176}{9860},\ \ y'(0) = \frac{264}{9860}$ [*Hint.* These numbers were chosen so that some cancellation would occur.]

Short Answer
Solution

(15) $w'' - 3w' + w = e^{t} + e^{2t},\ \ w(0) = -2,\ \ w'(0) = 0$

Short Answer
Solution

(16) $y'' - 2y' + 10y = 3e^{t} + 37\cos(3t),\ \ y(0) = y'(0) = 1$

Short Answer
Solution

(17) $y'' - 8y' + 17y = 100x^2e^{x},\ \ y(0) = 5,\ \ y'(0) = 0$

Short Answer
Solution

(18) $y'' + 2y' + 5y = 4e^{-t}\sin(2t),\ \ y(0) = \frac{1}{2},\ \ y'(0) = \frac{3}{2}$

Short Answer
Solution

(19) $y'' - 2y' + y = (3u + 1)e^{u},\ \ y(0) = 4,\ \ y'(0) = 0$

Short Answer
Solution

(20) $y'' + y = 3t^2 \cos(2t) - 6t^2 \sin(2t), \;\; y(0) = 3, \;\; y'(0) = 0$

Short Answer
Solution

For #21–#23, find the Green function for the following differential equations. Don't try to produce a general solution.

(21) $v'' + 2v' + 5v = \log(u)$

Solution

(22) $y'' - 4y' + 4y = \frac{1}{t}$

Solution

(23) $\ddot{y} - 6\dot{y} - 7y = \sec(2t)$

Solution

For #24–#27, find general solutions to the differential equations using Green functions.
Some of them may involve antiderivatives that you can't completely evaluate; leaving things in terms of integrals is okay.

(24) $\ddot{y} = \frac{1}{t}$ [*Note.* Yes you can solve this by just integrating twice. This is an easy differential equation to solve, even with Green functions.]

Short Answer
Solution

(25) $\ddot{w} + 16w = \sec(4x)$

Short Answer
Solution

(26)

$$y'' - 3y' = \frac{1}{e^{3t} + 1}$$

Short Answer
Solution

(27)

$$\ddot{y} - 4y = \frac{1}{e^{2t} + e^{-2t}}$$

Short Answer
Solution

Remark: While we've focused so far on forcing terms that are continuous, it's possible to create solutions to differential equations with discontinuous forcing using the methods we've learned. Often physical models of real-world situations will include functions with jumps in them; this could represent the almost-instantaneous effect of flipping a switch that causes a voltage to pass through a current, for instance. Since the solution to the differential equation is supposed to represent something physical (like a position or a voltage), we assume the solution is going to be continuous. This suggests the following method of solution:

(a) solve the differential equation using the forcing term as it looks at $t = 0$, using the initial conditions, then
(b) find the value of the differential equation when the forcing term jumps, and finally
(c) use that value as an initial condition to solve a second differential equation, starting at the jump point, using the second case of the forcing term.

Use the above framework to solve the following two differential equations.

(28)

$$y'' + y = \begin{cases} 3\sin(2t) & 0 \le t < \pi \\ \sin(t) & t \ge \pi \end{cases} \quad \text{where} \quad y(0) = 1,\ y'(0) = -2$$

Short Answer
Solution

(29)

$$z'' - z = \begin{cases} 1 & 0 \le u < 1 \\ e^{-2u} & u \ge 1 \end{cases} \quad \text{where} \quad z(0) = z'(0) = 0$$

Suggestion. This could possibly get a little ugly; try using three decimal places to estimate e.]

Short Answer
Solution

(30) Consider the equation

$$v'' - 6v' + 25v = e^{x^2}.$$

a) Compute the Green function $g(x)$ associated with the differential equation.
b) Find a particular solution $V_p(x)$ in terms of definite integrals.

Short Answer
Solution

(31) Find the Green function associated with the differential operator $\mathrm{L} = \mathrm{D}^2 - 16$.

Short Answer
Solution

(32) Give a particular solution to the following equation:

$$w'' + 8w' + 25w = 10e^{-t},$$

using two of the methods discussed in this section. How do they compare to the third method discussed?

Short Answer
Solution

(33) **A bit of thinking outside the box**

(a) Consider the equation $z^{(7)} - 2z^{(6)} + z^{(5)} - 2z^{(4)} = 0$. Given that the characteristic polynomial associated with this equation is of the form $z^4(z-2)(z^2+1)$, write down a general solution to this homogeneous, constant coefficient, linear seventh-order differential equation.

(b) Now consider the equation $z^{(7)} - 2z^{(6)} + z^{(5)} - 2z^{(4)} = 4x + \cos(x) + e^{16x}$. Write down a guess for a particular solution $Z_p(x)$ to this non-homogeneous equation. There is no need to solve for the undetermined coefficients!

Short Answer
Solution

(34) Write down the form of the particular solution to

$$\ddot{y} + p(x)\dot{y} + q(x)y = g(x)$$

for the following $g(x)'s$:

a) $g(x) = e^{7x} + 6$;
b) $g(x) = 10e^{-5x} + e^{-5x}\cos(6x) - \sin(6x)$;
c) $g(x) = x^2\cos(x) - 5x\sin(x)$.

Hint Don't worry about the form of $p(x)$ or $g(x)$ for now, but concentrate on the form of the composite characteristic form instead.

Short Answer
Solution

(35) Consider the following differential equation

$$a\ddot{w} + b\dot{w} + cw = g(u),$$

where a, b, and c are positive. If $W_1(u)$ and $W_2(u)$ are solutions of the differential equation, prove that $W_1(u) - W_2(u) \to 0$ as $u \to \infty$. Does the result hold when $b = 0$?

Short Answer
Solution

(36) Here we will show an alternative method towards solving the differential equation

$$y'' + by' + cy = (\mathrm{D}^2 + b\mathrm{D} + c)y = f(x),$$

where b and c are constants, and D is the differentiation operator with respect to x. Let z_1 and z_2 be the zeros of the characteristic polynomial of the corresponding homogeneous equation. Note that these roots could be distinct real, real and equal or complex conjugates of each other.

(a) Show that the differential equation $y'' + by' + cy = (\mathrm{D}^2 + b\mathrm{D} + c)y = f(x)$, can be rewritten as $(\mathrm{D} - z_1)(\mathrm{D} - z_2)y = f(x)$ and find out what z_1 and z_2 are in terms of b and c.

(b) Let $v = (\mathrm{D} - z_2)y$. Show that the solution of $y'' + by' + cy = (\mathrm{D}^2 + b\mathrm{D} + c)y = f(x)$, can be found by solving the following system of first-order non-homogeneous differential equations:

$$(\mathrm{D} - z_1)v(x) = f(x),$$

$$(\mathrm{D} - z_2)y(x) = v(x).$$

Short Answer
Solution

(37) In the light of the previous problem, use the method outlined above to solve the following differential equation:

$$w'' - 3w' - 4w = 3e^{2u}.$$

Note : We have other methods for solving this differential equation as well, but here we would like to illustrate how annihilating the second-order operator yields a system of first-order equations.

Short Answer
Solution

(38) Find a second order homogeneous constant coefficient equation whose general solution is of the form $v(u) = c_1 \cos(5u) + c_2 \sin(5u) - e^u \sin(2u)$.

Short Answer
Solution

(39) Suppose that $w_1 = 2u\sin(3u)$ is a solution of the following equation

$$\ddot{w} + 2\dot{w} + 2w = f_1(u),$$

and $w_2 = \cos(6u) - e^{-u}\cos(u)$ is a solution of the following equation

$$\ddot{w} + 2\dot{w} + 2w = f_1(u).$$

What is the general solution of the following equation?

$$\ddot{w} + 2\dot{w} + 2w = 5f_1(u) - 2f_2(u)$$

Short Answer
Solution

(40) Find a second order homogeneous linear constant coefficient differential equation whose general solution is of the from $y(x) = c_1e^{-x} + c_2xe^{-x} + x^3 - 5x$.

Short Answer
Solution

(41) Suppose the equation $w'' - 4w' - 5w = f(x)$ has $w = 3x^3$ has one of its solutions.
(a) Which one of the following functions is also a solution?
Note : More than one answer might be possible.

- $w = e^{-u} + 3x^3$;
- $w = 2080e^{5x} + 3x^3$;
- $w = 2080e^{5x} + 6x^3$;
- $w = e^{-x} + e^{5x} + 3x^3$;
- $w = 20e^{-x} + 14e^{5x} + x^3$.

(b) What is the general solution of the equation?
(c) Find $f(x)$.
(d) Knowing that $w(0) = 3$ and that $w'(0) = 3$, solve the initial value problem.

Short Answer
Solution

(42) **More on Annihilation Operators**

As we've seen before, an annihilation operator for a given function f is a polynomial $P(\mathrm{D})$ for the differential operator D such that $P(\mathrm{D})f = 0$. In other words, f is the solution to the homogeneous equation characterized by $P(\mathrm{D})f = 0$. For example, $f(t) = e^t$ satisfied the first-order differential equation $f' - f = 0$ and the corresponding annihilation operator for f is $\mathrm{D} - 1$, in that $(\mathrm{D} - 1)f = 0$. For another example, $f(t) = 2t$, the corresponding annihilation operator is D^2 because $\mathrm{D}^2(f) = 0$.

Just to get ourselves thinking in the annihilation operator mode one more time, find the corresponding annihilation operator for the following functions and write down the corresponding homogeneous differential equation:

a) $\sin(x)$;
b) ve^v;
c) $2e^{3x} + \cos(x)$.

Short Answer
Solution

(43) Consider the following differential equation $w'' - 5w' + 6w = e^{2v}$. Write down a general solution to the differential equation using the method of annihilators and starting from the general solution, name exactly which is the particular solution.

Note : This problem can be tackled using either of the three methods discussed in this section, but here we are illustrating the use of annihilators in achieving the same goal!

Short Answer
Solution

(44) Consider the following differential equation $v''' - v' = x + 1$. Write down a general solution to the differential equation using the method of annihilators and starting from the general solution, name exactly which is the particular solution.

Note : Once again, this problem can also be tackled using either of the three methods discussed in this section, but here we are illustrating the use of annihilators in achieving the same goal!

Short Answer
Solution

(45) Using the method of annihilators, determine the general solution to the following constant coefficient differential equations:

a) $(\mathrm{D} - 4)(\mathrm{D} + 1)w = 16ue^{3u}$;
b) $(\mathrm{D} - 2)v = 3\cos(u) + 4\sin(u)$.

Short Answer

Solution

(46) Find the differential equations (in operator form) solved by the following functions:

a) $v(x) = 14 + x^2 - 21\cos(x)$;
b) $w(t) = 10t^2e^{4t}$;
c) $v(u) = -e^{3u} + ue^{3u} - 4ue^{-u}\cos 3u$.

Short Answer
Solution

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Ordinary Differential Equations