II. Higher-Order Linear Ordinary Differential Equations
7. Nonhomogeneous Equations with Variable Coefficients

C. David Levermore
Department of Mathematics
University of Maryland

August 2, 2022

Contents

Return to Main Webpage

7. NONHOMOGENEOUS EQUATIONS WITH VARIABLE COEFFICIENTS

7.1. **Introduction.** We now return to the study of nonhomogeneous linear equations with variable coefficients, which we began in Chapter 5. An n^{th}-order nonhomogeneous linear ODE has the normal form

$$\mathrm{L}(t)y = f(t)\,, \tag{7.1}$$

where the differential operator $\mathrm{L}(t)$ has the normal form

$$\mathrm{L}(t) = \mathrm{D}^n + a_1(t)\mathrm{D}^{n-1} + \cdots + a_{n-1}(t)\mathrm{D} + a_n(t)\,. \tag{7.2}$$

We will assume throughout this chapter that the coefficients $a_1(t), a_2(t), \cdots, a_n(t)$ and the forcing $f(t)$ are continuous over an interval (t_L, t_R), so that Therorem 1.1 can be applied.

Recall the strategy for constructing general solutions of the nonhomogeneous equation (7.1) that we developed in Section 5.1:

(1) find a general solution $Y_H(t)$ of the associated homogeneous equation $\mathrm{L}(t)y = 0$;
(2) find a particular solution $Y_P(t)$ of equation (7.1);
(3) then $Y_H(t) + Y_P(t)$ is a general solution of (7.1).

If we can find a fundamental set $Y_1(t)$, $Y_2(t)$, $\cdots$, $Y_n(t)$ of solutions to the associated homogeneous equation $\mathrm{L}(t)y = 0$ then a general solution of that equation is given by

$$Y_H(t) = c_1Y_1(t) + c_2Y_2(t) + \cdots + c_nY_n(t)\,.$$

In the ensuing sections we will explore two methods to construct a particular solution $Y_P(t)$ of equation (7.1) from $f(t)$ and the fundamental set $Y_1(t)$, $Y_2(t)$, $\cdots$, $Y_n(t)$. One method is called *variation of parameters*, while the other is called the *general Green function* method, which extends to equations with variable coefficients the Green function method presented in Section 6.5 for equations with constant coefficients. We will see that these two methods are related. What lies behind them is the following.

> **Important Fact:** If $\mathrm{L}(t)$ is an n^{th}-order differential operator and we know a general solution of the associated homogeneous equation $\mathrm{L}(t)y = 0$ then we can *always* reduce the construction of a general solution to a nonhomogeneous equation $\mathrm{L}(t)y = f(t)$ to the problem of finding n primitives.

Because at this point you only know how to find general solutions of homogeneous equations with constant coefficients, problems you will be given will generally fall into one of two categories. Either (1) the operator $\mathrm{L}(t)$ will have variable coefficients and you will be given a fundamental set of solutions for the associated homogeneous equation, or (2) the operator $\mathrm{L}(t)$ will have constant coefficients (i.e. $\mathrm{L}(t) = \mathrm{L}$) and you will be expected to find a fundamental set of solutions for the associated homogeneous equation. In the latter case the general Green function method reduces to the method presented in Section 6.5.

7.2. **Variation of Parameters: Second-Order Case.** We begin by deriving the method of variation of parameters for second-order equations that are in the normal form

$$y'' + a_1(t)y' + a_2(t)y = f(t)\,. \tag{7.3}$$

Suppose that $Y_1(t)$ and $Y_2(t)$ are linearly independent solutions of the associated homogeneous equation $\mathrm{L}(t)y = 0$, where $\mathrm{L}(t) = \mathrm{D}^2 + a_1(t)\mathrm{D} + a_2(t)$. Then a general solution of the associated homogeneous equation is given by

$$Y_H(t) = c_1Y_1(t) + c_2Y_2(t)\,. \tag{7.4}$$

The idea of the method of variation of parameters is to seek a general solution of (7.3) in the form

$$y = u_1(t)Y_1(t) + u_2(t)Y_2(t)\,. \tag{7.5}$$

In other words, we simply replace the parameters c_1 and c_2 in (7.5) with unknown functions $u_1(t)$ and $u_2(t)$. These functions are the varying parameters referred to in the name of the method. These two functions will be governed by a system of two equations, one of which is derived by requiring that (7.3) is satisfied, and the other of which is chosen to simplify the resulting system.

7.2.1. *Derivation.* Let us see how this is done. Differentiating (7.5) yields

$$y' = u_1(t)Y_1'(t) + u_2(t)Y_2'(t) + u_1'(t)Y_1(t) + u_2'(t)Y_2(t)\,. \tag{7.6}$$

We now choose to impose the condition

$$u_1'(t)Y_1(t) + u_2'(t)Y_2(t) = 0\,, \tag{7.7}$$

whereby (7.6) simplifies to

$$y' = u_1(t)Y_1'(t) + u_2(t)Y_2'(t)\,. \tag{7.8}$$

Differentiating (7.8) then yields

$$y'' = u_1(t)Y_1''(t) + u_2(t)Y_2''(t) + u_1'(t)Y_1'(t) + u_2'(t)Y_2'(t)\,. \tag{7.9}$$

By imposing condition (7.7) we have greatly simplified this expression for y''.

Because condition (7.7) relates $u_1'(t)$ and $u_2'(t)$, these functions are now dependent. They will be uniquely determined by any other condition that we impose. If y given by (7.5) is to be a solution of the differential equation (7.3) then that second condition must come directly from the differential equation. That is what we do next.

By substituting (7.5), (7.8), and (7.9) into the differential equation (7.3), grouping the terms that multiply $u_1(t)$, $u_1'(t)$, $u_2(t)$, and $u_2'(t)$, and using the fact that $\mathrm{L}(t)Y_1(t) = 0$

and $\mathrm{L}(t)Y_2(t) = 0$, we obtain

$$\begin{aligned}
f(t) = \mathrm{L}(t)y &= y'' + a_1(t)y' + a_2(t)y \\
&= \big[u_1(t)Y_1''(t) + u_2(t)Y_2''(t) + u_1'(t)Y_1'(t) + u_2'(t)Y_2'(t)\big] \\
&\quad + a_1(t)\big[u_1(t)Y_1'(t) + u_2(t)Y_2'(t)\big] + a_2(t)\big[u_1(t)Y_1(t) + u_2(t)Y_2(t)\big] \\
&= u_1(t)\big[Y_1''(t) + a_1(t)Y_1'(t) + a_2(t)Y_1(t)\big] + u_1'(t)Y_1'(t) \\
&\quad + u_2(t)\big[Y_2''(t) + a_1(t)Y_2'(t) + a_2(t)Y_2(t)\big] + u_2'(t)Y_2'(t) \\
&= u_1(t)\big[\mathrm{L}(t)Y_1(t)\big] + u_1'(t)Y_1'(t) + u_2(t)\big[\mathrm{L}(t)Y_2(t)\big] + u_2'(t)Y_2'(t) \\
&= u_1'(t)Y_1'(t) + u_2'(t)Y_2'(t)\,.
\end{aligned} \tag{7.10}$$

This is the second condition we impose on $u_1'(t)$ and $u_2'(t)$.

Upon combining (7.7) and (7.10), we see that $u_1'(t)$ and $u_2'(t)$ satisfy

$$\begin{aligned}
u_1'(t)Y_1(t) + u_2'(t)Y_2(t) &= 0\,, \\
u_1'(t)Y_1'(t) + u_2'(t)Y_2'(t) &= f(t)\,.
\end{aligned} \tag{7.11}$$

This is a linear algebraic system of two equations for $u_1'(t)$ and $u_2'(t)$. Because

$$\big(Y_1(t)Y_2'(t) - Y_2(t)Y_1'(t)\big) = \mathrm{Wr}[Y_1, Y_2](t) \neq 0\,,$$

we can always solve this system to find

$$u_1'(t) = -\frac{Y_2(t)f(t)}{\mathrm{Wr}[Y_1, Y_2](t)}\,, \qquad u_2'(t) = \frac{Y_1(t)f(t)}{\mathrm{Wr}[Y_1, Y_2](t)}\,. \tag{7.12}$$

The solutions of (7.12) expressed as indefinite integrals are

$$u_1(t) = -\int \frac{Y_2(t)f(t)}{\mathrm{Wr}[Y_1, Y_2](t)}\,\mathrm{d}t\,, \qquad u_2(t) = \int \frac{Y_1(t)f(t)}{\mathrm{Wr}[Y_1, Y_2](t)}\,\mathrm{d}t\,.$$

If $u_{1P}(t)$ and $u_{2P}(t)$ are any primitives of the respective right-hand sides of (7.12), we see that

$$u_1(t) = c_1 + u_{1P}(t)\,, \qquad u_2(t) = c_2 + u_{2P}(t)\,,$$

whereby (7.5) yields the general solution

$$y = c_1Y_1(t) + u_{1P}(t)Y_1(t) + c_2Y_2(t) + u_{2P}(t)Y_2(t)\,.$$

Notice that this decomposes as $y = Y_H(t) + Y_P(t)$ where

$$Y_H(t) = c_1Y_1(t) + c_2Y_2(t)\,, \qquad Y_P(t) = u_{1P}(t)Y_1(t) + u_{2P}(t)Y_2(t)\,. \tag{7.13}$$

There are two approaches to applying variation of parameters. One is to memorize the formulas (7.12). This is not the best approach for two reasons. First, it is easy to confuse which of the two formulas gets the minus sign. Second, and more importantly, these formulas do not cleanly generalize to the higher-order case. The other approach is to construct the linear system (7.11), which then can be solved for $u_1'(t)$ and $u_2'(t)$. The work it takes to solve this system is about the same as that to generate the right-hand sides of (7.12). The linear system (7.11) is symmetric in $u_1'(t)$ and $u_2'(t)$, so is less subject to sign errors. Moreover, it also has a clean generalization to the higher-order case. Whichever approach you take, you will be led to the same two integrals.

7.2.2. *Recipe.* When applying the variation of parameters method, there is no need to repeat the above derivation. Rather there is a simple way to generate the linear system (7.11) for $u_1'(t)$ and $u_2'(t)$. Given $Y_1(t)$ and $Y_2(t)$, any fundamental set of solutions to the associated homogeneous equation, proceed as follows.

1. Write the form of the solution you seek:
$$y = u_1(t)Y_1(t) + u_2(t)Y_2(t) .$$
2. Write the linear algebraic system for $u_1'(t)$ and $u_2'(t)$:
$$\begin{aligned} u_1'(t)Y_1(t) + u_2'(t)Y_2(t) &= 0 , \\ u_1'(t)Y_1'(t) + u_2'(t)Y_2'(t) &= f(t) . \end{aligned}$$
The form of the left-hand sides of this system mimics the form of the solution you seek. The first equation simply replaces $u_1(t)$ and $u_2(t)$ with $u_1'(t)$ and $u_2'(t)$, while the second also replaces $Y_1(t)$ and $Y_2(t)$ with $Y_1'(t)$ and $Y_2'(t)$. The $f(t)$ on the right-hand side will be correct only if the equation $\mathrm{L}(t)y = f(t)$ is in normal form!
3. Solve the linear algebraic system to find explicit expressions for $u_1'(t)$ and $u_2'(t)$. This is always very easy to do, especially if you start with the first equation.
4. Find primitives $u_{1P}(t)$ and $u_{2P}(t)$ of these expressions. If you cannot find a primitive analytically then express that primitive in terms of a definite integral. We then have
$$u_1(t) = c_1 + u_{1P}(t) , \qquad u_2(t) = c_2 + u_{2P}(t) ,$$
where c_1 and c_2 are the arbitrary constants of integration.
5. Upon placing this result into the form of the solution written down in step 1, you will obtain the general solution $y = Y_H(t) + Y_P(t)$, where
$$Y_H(t) = c_1Y_1(t) + c_2Y_2(t) , \qquad Y_P(t) = u_{1P}(t)Y_1(t) + u_{2P}(t)Y_2(t) .$$
For initial-value problems c_1 and c_2 are determined from the initial conditions.

Example. Find a general solution of
$$y'' + y = \sec(t) .$$

Remark. Notice that while this equation does have constant coefficients, the forcing does not have the characteristic form needed to use the methods of either Key Identity Evaluations or Undetermined Coefficients. You should be able to recognize this right away. While you can use the Green function method to solve this problem, here we will solve it using variation of parameters.

Solution. This equation is already in normal form. Its forcing is $\sec(t) = 1/\cos(t)$, which is not defined where $\cos(t) = 0$. This happens where $t = (k + \frac{1}{2})\pi$ for some integer k. Therefore there is no solution at these points.

Because this equation has constant coefficients, it is easily found that
$$Y_H(t) = c_1 \cos(t) + c_2 \sin(t) .$$

Hence, we will seek a general solution of the form

$$y = u_1(t)\cos(t) + u_2(t)\sin(t)\,,$$

where

$$\begin{gathered} u_1'(t)\cos(t) + u_2'(t)\sin(t) = 0\,, \\ -u_1'(t)\sin(t) + u_2'(t)\cos(t) = \sec(t)\,. \end{gathered}$$

Because $\sec(t) = 1/\cos(t)$, this system can solved to obtain

$$u_1'(t) = -\frac{\sin(t)}{\cos(t)}\,, \qquad u_2'(t) = 1\,.$$

These can be integrated analytically to obtain

$$u_1(t) = c_1 + \log(|\cos(t)|)\,, \qquad u_2(t) = c_2 + t\,.$$

Therefore a general solution is

$$y = c_1\cos(t) + c_2\sin(t) + \log(|\cos(t)|)\cos(t) + t\sin(t)\,.$$

We must exclude the points where $t = (k + \frac{1}{2})\pi$ for some integer k.

Remark. The primitives $u_1(t)$ and $u_2(t)$ that we had to find above are the same ones needed to evaluate the integrals that arise when we solve this problem with the Green function method. This will always be the case.

Remark. If we use the above general solution to solve the initial-value problem

$$y'' + y = \sec(t)\,, \qquad y(0) = 1\,, \quad y'(0) = 0\,,$$

then we obtain the solution

$$y(t) = \cos(t) + \log(|\cos(t)|)\cos(t) + t\sin(t)\,.$$

Its interval of definition is $(-\frac{1}{2}\pi, \frac{1}{2}\pi)$. However, the above expression is a continuous function over $(-\infty, \infty)$ provided that we define $\log(|0|)0 = 0$. When this expression is plotted then we obtain Figure 7.1 below, which looks fine everywhere. That the interval of definition is $(-\frac{1}{2}\pi, \frac{1}{2}\pi)$ becomes evident if we compute its derivative, which is

$$y'(t) = -\sin(t) - \log(|\cos(t)|)\sin(t) + t\cos(t)\,.$$

This function is clearly undefined where $t = (k + \frac{1}{2})\pi$ for some integer k.

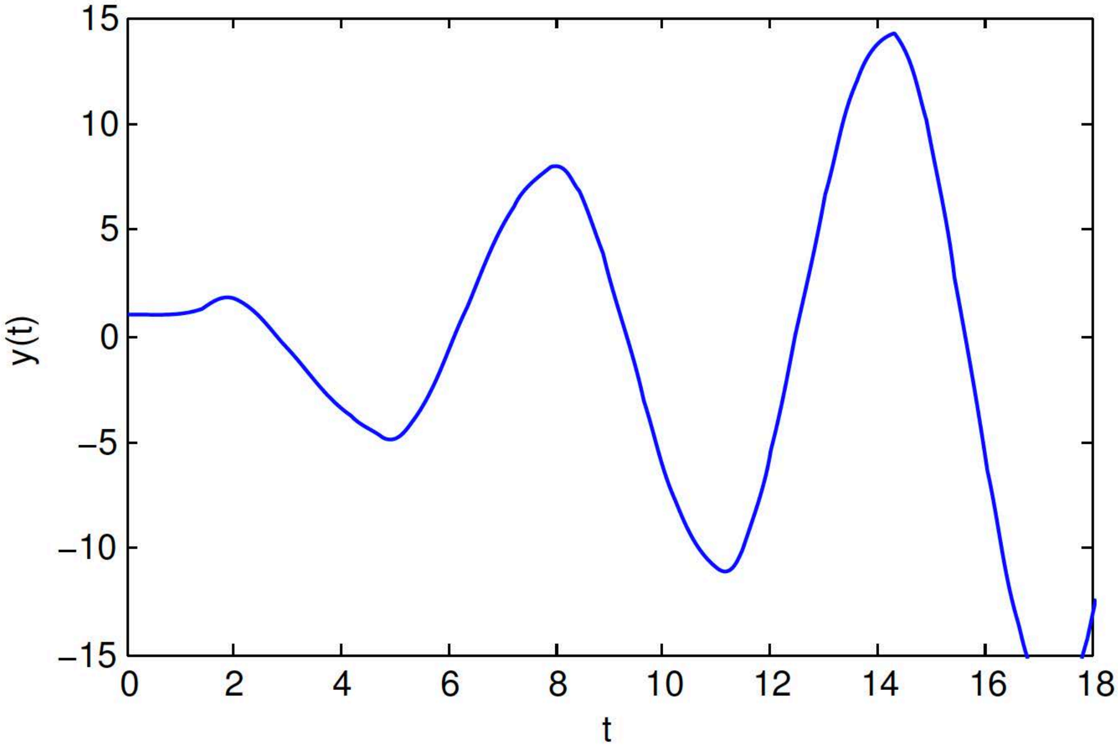

FIGURE 7.1. The graph of $y(t)$ obtained by solving the initial-value problem $y'' + y = \sec(t)$ with $y(0) = 1$ and $y'(0) = 0$. Can you see the points where $y(t)$ is not a solution?

Example. Given that t and $t^2 - 1$ are a fundamental set of solutions of the associated homogeneous equation, find a general solution of

$$(1 + t^2)x'' - 2tx' + 2x = (1 + t^2)^2 e^t \,.$$

Remark. You should be able to recognize that this equation has variable coefficients, and thereby see that one of the methods in this chapter must be used to solve this problem. Here we will use variation of parameters.

Solution. First we put this equation into its normal form,

$$x'' - \frac{2t}{1+t^2}\, x' + \frac{2}{1+t^2}\, x = (1 + t^2)e^t \,.$$

This forcing is what is used by variation of parameters. Because the coefficients and forcing are continuous over $(-\infty, \infty)$, the interval of definition is $(-\infty, \infty)$ for each of its solutions.

Because t and $t^2 - 1$ are a fundamental set of solutions of the associated homogeneous equation, we have

$$X_H(t) = c_1 t + c_2(t^2 - 1) \,.$$

Hence, we will seek a general solution of the form

$$x = u_1(t)t + u_2(t)(t^2 - 1)\,,$$

where (using the forcing from the normal form)

$$\begin{aligned} u_1'(t)t + u_2'(t)(t^2 - 1) &= 0\,,\\ u_1'(t)1 + u_2'(t)2t &= (1 + t^2)e^t\,. \end{aligned}$$

This system can be solved to obtain

$$u_1'(t) = -(t^2 - 1)e^t\,, \qquad u_2'(t) = te^t\,.$$

These can be integrated analytically "by parts" to obtain

$$u_1(t) = c_1 - (t - 1)^2 e^t\,, \qquad u_2(t) = c_2 + (t - 1)e^t\,.$$

Therefore a general solution is

$$\begin{aligned} x &= c_1 t + c_2(t^2 - 1) - (t - 1)^2 e^t t + (t - 1)e^t(t^2 - 1)\\ &= c_1 t + c_2(t^2 - 1) + (t - 1)^2 e^t\,. \end{aligned}$$

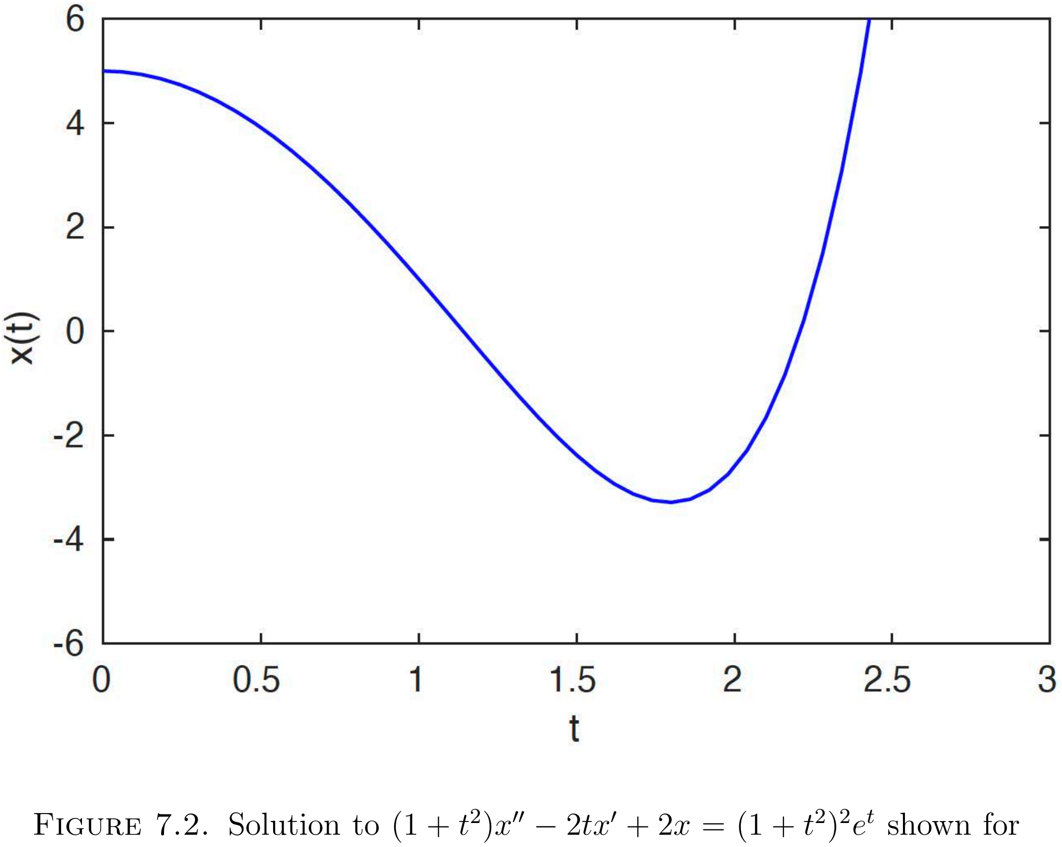

Figure 7.2. Solution to $(1 + t^2)x'' - 2tx' + 2x = (1 + t^2)^2 e^t$ shown for the initial conditions $x(0) = 5$ and $x'(0) = 0$.

7.3. **General Green Functions: Second-Order Case.** We now give a Green function method for constructing particular solutions of second-order equations that are in the normal form

$$y'' + a_1(t)y' + a_2(t)y = f(t)\,. \tag{7.14}$$

Unlike the Green function method given in Section 6.5, this approach is not restricted to equations with constant coefficients. However, it will reduce to that earlier approach for equations with constant coefficients.

7.3.1. *Formula.* This Green function method derives from the variation of parameters method given in the previous section. Suppose that $Y_1(t)$ and $Y_2(t)$ are a fundamental set of solutions to the associated homogeneous equation $\mathrm{L}(t)y = 0$. The starting point of our derivation will be the particular solution $Y_P(t)$ given in (7.13) — namely,

$$Y_P(t) = u_{1P}(t)Y_1(t) + u_{2P}(t)Y_2(t)\,,$$

where $u_{1P}(t)$ and $u_{2P}(t)$ are primitives that satisfy

$$u_{1P}'(t) = -\frac{Y_2(t)f(t)}{\mathrm{Wr}[Y_1, Y_2](t)}\,, \qquad u_{2P}'(t) = \frac{Y_1(t)f(t)}{\mathrm{Wr}[Y_1, Y_2](t)}\,.$$

If we express $u_{1P}(t)$ and $u_{2P}(t)$ as the definite integrals

$$u_{1P}(t) = -\int_{t_I}^{t} \frac{Y_2(s)f(s)}{\mathrm{Wr}[Y_1, Y_2](s)}\,\mathrm{d}s\,, \qquad u_{2P}(t) = \int_{t_I}^{t} \frac{Y_1(s)f(s)}{\mathrm{Wr}[Y_1, Y_2](s)}\,\mathrm{d}s\,, \tag{7.15}$$

where t_I is any initial time inside the interval (t_L, t_R), then the particular solution $Y_P(t)$ can be expressed as

$$Y_P(t) = \int_{t_I}^{t} G(t, s)f(s)\,\mathrm{d}s\,, \tag{7.16}$$

where $G(t, s)$ is given by

$$G(t, s) = \frac{Y_1(s)Y_2(t) - Y_1(t)Y_2(s)}{\mathrm{Wr}[Y_1, Y_2](s)} = \frac{\det\begin{pmatrix} Y_1(s) & Y_2(s) \\ Y_1(t) & Y_2(t) \end{pmatrix}}{\det\begin{pmatrix} Y_1(s) & Y_2(s) \\ Y_1'(s) & Y_2'(s) \end{pmatrix}}\,. \tag{7.17}$$

The method thereby reduces the problem of finding a particular solution $Y_P(t)$ for any forcing $f(t)$ to that of evaluating the integral in (7.16), which by formula (7.17) is equivalent to evaluating the two definite integrals in (7.15). Of course, evaluating these integrals explicitly can be quite difficult or impossible. You may have to leave your answer in terms of one or both of these definite integrals. Formulas (7.16) and (7.17) have natural generalizations to higher-order equations with variable coefficients.

We will see that (7.16) is an extension of the Green function formula (6.28) from Section 6.5 to second-order equations with variable coefficients. Specifically, when the differential equation (7.13) has constant coefficients and $g(t)$ is the associated Green function from Section 6.5 we have the relation

$$G(t, s) = g(t - s)\,.$$

As with formula (6.28) from Section 6.5, formula (7.16) generates the unique particular solution $Y_P(t)$ of (7.14) that satifies the initial-value problem

$$\mathrm{L}(t)Y_P(t) = f(t)\,, \qquad Y_P(t_I) = 0\,, \qquad Y_P'(t_I) = 0\,. \tag{7.18}$$

Therefore we call $G(t,s)$ the *Green function* for the operator $\mathrm{L}(t)$.

Before justifying the foregoing claims, let us illustrate how to construct and use this Green function.

Example. Given that t and t^2-1 are a fundamental set of solutions of the associated homogeneous equation, find a particular solution of

$$(1+t^2)x'' - 2tx' + 2x = (1+t^2)^2 e^t\,.$$

Solution. First, we put this equation into its normal form

$$x'' - \frac{2t}{1+t^2}\,x' + \frac{2}{1+t^2}\,x = (1+t^2)e^t\,.$$

Because t and t^2-1 are a fundamental set of solutions of the associated homogeneous equation, the Green function $G(t,s)$ is given by (7.17) as

$$G(t,s) = \frac{\det\begin{pmatrix} s & s^2-1 \\ t & t^2-1 \end{pmatrix}}{\det\begin{pmatrix} s & s^2-1 \\ 1 & 2s \end{pmatrix}} = \frac{(t^2-1)s - t(s^2-1)}{2s^2-(s^2-1)} = \frac{(t^2-1)s - t(s^2-1)}{s^2+1}\,.$$

Then formula (7.16) with $t_I = 1$ and $f(s) = (1+s^2)e^s$ yields

$$\begin{aligned} X_P(t) = \int_1^t G(t,s)\,f(s)\,\mathrm{d}s &= \int_1^t \frac{(t^2-1)s - t(s^2-1)}{s^2+1}\,(1+s^2)e^s\,\mathrm{d}s \\ &= (t^2-1)\int_1^t se^s\,\mathrm{d}s - t\int_1^t (s^2-1)e^s\,\mathrm{d}s\,. \end{aligned}$$

Notice that the same two integrals that arose when we treated this equation by variation of parameters on pages 7-8. As was done there, some integration-by-parts shows that

$$\int_1^t se^s\,\mathrm{d}s = (t-1)e^t\,, \qquad \int_1^t (s^2-1)e^s\,\mathrm{d}s = (t-1)^2e^t\,.$$

Therefore the particular solution is

$$X_P(t) = (t^2-1)(t-1)e^t - t(t-1)^2e^t = (t-1)^2e^t\,.$$

It is clear that this solution satisfies $X_P(1) = X_P'(1) = 0$. Had we chosen a different value for the initial time t_I we would have obtained a different particular solution $X_P(t)$.

7.3.2. *Justification.* Here we show that formula (7.16) generates the unique particular solution $Y_P(t)$ of (7.14) that satifies the initial-value problem (7.18). It is clear from (7.16) that $Y_P(t_I) = 0$. To show that $Y_P'(t_I) = 0$ we will use the fact from multivariable calculus that for any continuously differentiable $K(t,s)$ we have

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_{t_I}^{t} K(t,s)\,\mathrm{d}s = K(t,t) + \int_{t_I}^{t} \partial_t K(t,s)\,\mathrm{d}s\,.$$

We see from (7.17) that $G(t,t) = 0$. Upon differentiating (7.16) with respect to t and using the above calculus fact, we see that

$$Y_P'(t) = G(t,t)f(t) + \int_{t_I}^{t} \partial_t G(t,s) f(s)\,\mathrm{d}s = \int_{t_I}^{t} \partial_t G(t,s) f(s)\,\mathrm{d}s\,.$$

It follows that $Y_P'(t_I) = 0$, thereby showing that $Y_P(t)$ satifies the initial conditions (7.18).

The Green function $G(t,s)$ is defined by (7.17) whenever t and s are both in the interval (t_L, t_R) overwhich Y_1 and Y_2 exist. At first it might seem that $G(t,s)$ must depend upon the fundamental set of solutions that is used to construct it. We now show that this is not the case. Let us fix s and consider $G(t,s)$ as a function of t. It is clear from (7.17) that $G(t,s)$ is a linear combination of $Y_1(t)$ and $Y_2(t)$. Because $Y_1(t)$ and $Y_2(t)$ are solutions of the associated homogeneous equation $\mathrm{L}(t)y = 0$, it follows that $G(t,s)$ is too — namely, that $\mathrm{L}(t)G(t,s) = 0$. It is also clear from (7.17) that $G(t,s)\big|_{t=s} = 0$. By differentiating (7.17) with respect to t we obtain

$$\partial_t G(t,s) = \frac{Y_1(s)Y_2'(t) - Y_1'(t)Y_2(s)}{\mathrm{Wr}[Y_1,Y_2](s)} = \frac{\det\begin{pmatrix} Y_1(s) & Y_2(s) \\ Y_1'(t) & Y_2'(t) \end{pmatrix}}{\det\begin{pmatrix} Y_1(s) & Y_2(s) \\ Y_1'(s) & Y_2'(s) \end{pmatrix}}\,. \tag{7.19}$$

It is clear from this that $\partial_t G(t,s)\big|_{t=s} = 1$. Collecting these facts we see for every s that $G(t,s)$ as a function of t satisfies the initial-value problem

$$\mathrm{L}(t)G(t,s) = 0\,, \qquad G(t,s)\big|_{t=s} = 0\,, \quad \partial_t G(t,s)\big|_{t=s} = 1\,. \tag{7.20}$$

This is really a family of initial-value problems — one for each s in which s plays the role of the initial time. The uniqueness theorem implies that $G(t,s)$ is uniquely determined by this family of initial-value problems. Thus, $G(t,s)$ depends only upon the operator $\mathrm{L}(t)$. In particular, it does not depend upon which fundamental set of solutions, Y_1 and Y_2, was used to construct it.

When $\mathrm{L}(t)$ has constant coefficients then it is easy to check that the family of initial-value problems (7.20) is satisfied by $G(t,s) = g(t-s)$, where $g(t)$ is the Green function that was defined by the initial-value problem (6.29) in Section 6.5. Formula (7.16) thereby extends the Green function formula (6.28) from Section 6.5 to second-order equations with variable coefficients.

When $\mathrm{L}(t)$ has constant coefficients the fastest way to compute the Green function is to solve the single initial-value problem (6.29) from Section 6.5. When $\mathrm{L}(t)$ has variable coefficients we first have to find a fundamental set of solutions, $Y_1(t)$ and $Y_2(t)$, to the associated homogeneous equation. We can then construct the Green function either by formula (7.17) or by solving the family of initial-value problems (7.20). The later approach goes as follows. Because $\mathrm{L}(t)G(t,s) = 0$ for every s we know that there exist $C_1(s)$ and $C_2(s)$ such that

$$G(t,s) = Y_1(t)C_1(s) + Y_2(t)C_2(s)\,. \tag{7.21}$$

The initial conditions of (7.20) then imply that

$$\begin{aligned} 0 &= \quad G(t,s)\big|_{t=s} = Y_1(s)C_1(s) + Y_2(s)C_2(s)\,,\\ 1 &= \partial_t G(t,s)\big|_{t=s} = Y_1'(s)C_1(s) + Y_2'(s)C_2(s)\,. \end{aligned}$$

The solution of this linear algebraic system is

$$C_1(s) = -\frac{Y_2(s)}{Y_1(s)Y_2'(s) - Y_1'(s)Y_2(s)}\,, \quad C_2(s) = \frac{Y_1(s)}{Y_1(s)Y_2'(s) - Y_1'(s)Y_2(s)}\,,$$

which when plugged into (7.21) yields (7.17).

Example. Given that t and t^2-1 are a fundamental set of solutions of the associated homogeneous equation, find a particular solution of

$$(1+t^2)x'' - 2tx' + 2x = (1+t^2)^2 e^t\,.$$

Solution. First, we put this equation into its normal form

$$x'' - \frac{2t}{1+t^2}\,x' + \frac{2}{1+t^2}\,x = (1+t^2)e^t\,.$$

Because t and t^2-1 are a fundamental set of solutions of the associated homogeneous equation, by (7.21) the Green function has the form

$$G(t,s) = t\,C_1(s) + (t^2-1)\,C_2(s)\,,$$

where the initial conditions of (7.20) imply

$$\begin{aligned} 0 &= \quad G(t,s)\big|_{t=s} = s\,C_1(s) + (s^2-1)\,C_2(s)\,,\\ 1 &= \partial_t G(t,s)\big|_{t=s} = 1\,C_1(s) + 2s\,C_2(s)\,. \end{aligned}$$

These may be solved to obtain

$$C_1(s) = -\frac{s^2-1}{s^2+1}\,, \qquad C_2(s) = \frac{s}{s^2+1}\,,$$

whereby

$$G(t,s) = -t\,\frac{s^2-1}{s^2+1} + (t^2-1)\,\frac{s}{s^2+1} = \frac{(t^2-1)s - t(s^2-1)}{s^2+1}\,.$$

We then compute $X_P(t)$ by formula (7.16) as before.

7.4. **Order Reduction: Second-Order Case.** We now give a Green function method for constructing particular solutions of second-order equations that are in the normal form

$$y'' + a_1(t)y' + a_2(t)y = f(t)\,. \tag{7.22}$$

While there is no general recipe for constructing solutions to this equation, if we are able to find one solution of the associated homogeneous equation then we can reduce the problem of finding solutions of (7.22) to that of solving a first-order linear differential equation. This technique is called *order reduction*.

Suppose that $Y(t)$ solves the associated homogeneous equation,

$$Y''(t) + a_1(t)Y'(t) + a_2(t)Y(t) = 0\,. \tag{7.23}$$

If we set $y = Y(t)u(t)$ and compute its first two derivatives we obtain

$$\begin{aligned} y' &= Y'(t)u + Y(t)u'\,, \\ y'' &= Y''(t)u + 2Y'(t)u' + Y(t)u''\,. \end{aligned}$$

When these expressions are substituted into (7.22) all the terms involving u will drop out because $Y(t)$ satsifies the associated homogeneous equation (7.23). The result is the first-order, nonhomogeneous linear differential equation for u' given by

$$Y(t)u'' + \big(2Y'(t) + a_1(t)Y(t)\big)u' = f(t)\,.$$

In other words, $w = u'$ solves the the first-order, nonhomogeneous linear differential equation

$$Y(t)w' + \big(2Y'(t) + a_1(t)Y(t)\big)w = f(t)\,.$$

This can be solved by the methods of Chapter 2 in Part I of the course.

Remark. Euler used the order reduction method to find general solutions of nonhomogeneous linear ordinary differential equations. Lagrange introduced variation of parameters as a more efficient method to do this. The Green function method recasts variation of parameters into a form that is more efficient for solving initial-value problems.

7.5. **Variation of Parameters: Higher-Order Case.** The method of variation of parameters extends to higher-order linear equations in the normal form

$$\mathrm{L}(t)y = f(t)\,, \tag{7.24}$$

where the linear differential operator $\mathrm{L}(t)$ is given by

$$\mathrm{L}(t) = \mathrm{D}^n + a_1(t)\mathrm{D}^{n-1} + \cdots + a_{n-1}(t)\mathrm{D} + a_n(t)\,.$$

While this material was not covered in class and you will not be tested on it, a summary is given here for the sake of completeness.

Suppose that $Y_1(t), Y_2(t), \cdots, Y_n(t)$ are a fundamental set of solutions to the associated homogeneous equation $\mathrm{L}(t)y = 0$ Then a general solution of the associated homogeneous equation is given by

$$Y_H(t) = c_1Y_1(t) + c_2Y_2(t) + \cdots + c_nY_n(t)\,.$$

The idea of the method of variation of parameters is to seek solutions of (7.24) in the form

$$y = u_1(t)Y_1(t) + u_2(t)Y_2(t) + \cdots + u_n(t)Y_n(t)\,, \tag{7.25}$$

where $u_1'(t), u_2'(t), \cdots, u_n'(t)$ satisfy the linear algebraic system

$$\begin{aligned}
u_1'(t)Y_1(t) \quad + \quad u_2'(t)Y_2(t) \quad + \cdots + \quad u_n'(t)Y_n(t) \quad &= 0\,,\\
u_1'(t)Y_1'(t) \quad + \quad u_2'(t)Y_2'(t) \quad + \cdots + \quad u_n'(t)Y_n'(t) \quad &= 0\,,\\
&\vdots\\
u_1'(t)Y_1^{(n-2)}(t) + u_2'(t)Y_2^{(n-2)}(t) + \cdots + u_n'(t)Y_n^{(n-2)}(t) &= 0\,,\\
u_1'(t)Y_1^{(n-1)}(t) + u_2'(t)Y_2^{(n-1)}(t) + \cdots + u_n'(t)Y_n^{(n-1)}(t) &= f(t)\,.
\end{aligned} \tag{7.26}$$

Because

$$\det\begin{pmatrix} Y_1(t) & Y_2(t) & \cdots & Y_n(t)\\ Y_1'(t) & Y_2'(t) & \cdots & Y_n'(t)\\ \vdots & \vdots & \ddots & \vdots\\ Y_1^{(n-1)}(t) & Y_2^{(n-1)}(t) & \cdots & Y_n^{(n-1)}(t)\end{pmatrix} = \mathrm{Wr}[Y_1, Y_2, \cdots, Y_n](t) \neq 0\,,$$

the linear algebraic system (7.26) can be solved to obtain explicit expressions for $u_1'(t)$, $u_2'(t)$, $\cdots$, $u_n'(t)$. For example, when $n = 3$ we find

$$u_1'(t) = \frac{\mathrm{Wr}[Y_2, Y_3](t)f(t)}{\mathrm{Wr}[Y_1, Y_2, Y_3](t)}\,, \quad u_2'(t) = \frac{\mathrm{Wr}[Y_3, Y_1](t)f(t)}{\mathrm{Wr}[Y_1, Y_2, Y_3](t)}\,, \quad u_3'(t) = \frac{\mathrm{Wr}[Y_1, Y_2](t)f(t)}{\mathrm{Wr}[Y_1, Y_2, Y_3](t)}\,.$$

Find primitives $u_{1P}(t)$, $u_{2P}(t)$, $\cdots$, $u_{nP}(t)$ of these expressions. If you cannot find a primitive analytically then express that primitive as a definite integral. We then have

$$u_1(t) = c_1 + u_{1P}(t)\,, \quad u_2(t) = c_2 + u_{2P}(t)\,, \quad \cdots \quad u_n(t) = c_n + u_{nP}(t)\,,$$

where $c_1, c_2, \cdots, c_n$ are the arbitrary constants of integration. Therefore the general solution given by (7.25) is $y = Y_H(t) + Y_P(t)$, where

$$\begin{aligned}
Y_H(t) &= c_1Y_1(t) + c_2Y_2(t) + \cdots + c_nY_n(t)\,,\\
Y_P(t) &= u_{1P}(t)Y_1(t) + u_{2P}(t)Y_2(t) + \cdots + u_{nP}(t)Y_n(t)\,.
\end{aligned}$$

For initial-value problems we must determine $c_1, c_2, \cdots, c_n$ from the initial conditions.

7.6. **General Green Functions: Higher-Order Case.** This method can be used to construct a particular solution of an n^{th}-order nonhomogeneous linear ODE in the normal form (7.1). Specifically, for any initial time t_I the particular solution $y = Y_P(t)$ of (7.1) that satisfies the initial condition $Y_P(t_I) = Y_P'(t_I) = \cdots = Y_P^{(n-1)}(t_I) = 0$ is given by

$$Y_P(t) = \int_{t_I}^{t} G(t,s)f(s)\,\mathrm{d}s\,, \tag{7.27}$$

where $G(t,s)$ is given by

$$G(t,s)=\frac{1}{\mathrm{Wr}[Y_1,Y_2,\cdots,Y_n](s)}\det\begin{pmatrix} Y_1(s) & Y_2(s) & \cdots & Y_n(s) \\ Y_1'(s) & Y_2'(s) & \cdots & Y_n'(s) \\ \vdots & \vdots & \vdots & \vdots \\ Y_1^{(n-2)}(s) & Y_2^{(n-2)}(s) & \cdots & Y_n^{(n-2)}(s) \\ Y_1(t) & Y_2(t) & \cdots & Y_n(t) \end{pmatrix}.$$

The function G is called the *Green function* associated with the operator $\mathrm{L}(t)$. It can also be determined as the solution of the family of initial-value problems

$$\mathrm{L}(t)G(t,s)=0\,,\qquad G(t,s)\Big|_{t=s}=\cdots=\partial_t^{n-2}G(t,s)\Big|_{t=s}=0\,,\quad \partial_t^{n-1}G(t,s)\Big|_{t=s}=1\,.$$

The method thereby reduces the problem of finding a particular solution $Y_P(t)$ for any forcing $f(t)$ to that of evaluating the integral in (7.27). However, evaluating this integral explicitly can be quite difficult or impossible. At worst, you can leave your answer in terms of definite integrals.

EXERCISES ON NONHOMOGENEOUS EQUATIONS WITH VARIABLE COEFFICIENTS

For #1–#7, find a general solution to the differential equation by using variation of parameters.
(#1—#3 can be done by other methods as well, but stick to variation of parameters for them.)

(1) $v'' - v = e^u$

Short Answer
Solution

(2) $y'' + y = \tan(t)$

Short Answer
Solution

(3) $tx'' + x' = 1$
(The homogeneous solutions are 1 and $\ln(t)$.)

Short Answer
Solution

(4) $x^2y'' - 6xy' + 10y = x^6\cos(2x)$
(The homogeneous solutions are x^2 and x^5.)

Short Answer
Solution

(5) $xw'' - (1+x)w' + w = x^2e^x$
(The homogeneous solutions are e^x and $1+x$.)

Short Answer
Solution

(6) $x^3y'' - xy' + y = \dfrac{1}{x^2}e^{-\frac{1}{x}}$
(The homogeneous solutions are x and $xe^{-\frac{1}{x}}$.)

Short Answer
Solution

(7) $z^2w'' + zw' + \left(z^2 - \dfrac{1}{4}\right)w = z\sqrt{z}$
(The homogeneous solutions are $\frac{\cos(z)}{\sqrt{z}}$ and $\frac{\sin(z)}{\sqrt{z}}$.)

Short Answer
Solution

For #8–#14, find a general solution to the differential equation by integrating against a Green function. (Some can be done by other methods as well, we admit.)

(8) $y'' + y = t + 1$

Short Answer
Solution

(9) $xw'' + w' = \frac{1}{x}$
(The homogeneous solutions are 1 and $\log(x)$. Look for a solution on $(0, \infty)$.)
Short Answer
Solution

(10) $(t+1)y'' + 2ty' - 4y = (t+1)^3$
(The homogeneous solutions are e^{-2t} and $2t^2 + 2t + 1$. Look for a solution on $(-1, \infty)$)
Short Answer
Solution

(11) $6u^2\ddot{v} + u\dot{v} + v = u + 1$
(The homogeneous solutions are $\sqrt{u}$ and $\sqrt[3]{u}$. Find a solution on $(0, \infty)$.)
Short Answer
Solution

(12) $y'' + \dfrac{2}{t}\, y' - y = \dfrac{\cos(2t)}{t}$
(The homogeneous solutions are $\frac{1}{t}\, e^t$ and $\frac{1}{t}\, e^{-t}$. Find a solution on $(0, \infty)$.)
Short Answer
Solution

(13) $\cos(x)\ddot{z} + \sin(x)\dot{z} = \cos^2(x)\sin(x)$
(The homogeneous solutions are 1 and $\sin(x)$. Look for a solution on $(-\frac{\pi}{2}, \frac{\pi}{2})$.)

Short Answer
Solution

(14) $y'' + \dfrac{4}{t}y' + \dfrac{2}{t^2}y = \log(t)$
(The homogeneous solutions are $\frac{1}{t}$ and $\frac{1}{t^2}$. Look for a solution on $(0, \infty)$.])
Short Answer
Solution

(15) Suppose the differential operator L has constant coefficients, say

$$\mathrm{L}y = D^n y + a_1 D^{n-1} y + a_2 D^{n-2} y + \cdots + a_{n-2} D^2 y + a_{n-1} D y + a_n y,$$

where all the a_i are real numbers. Let $g(x)$ be the Green function for this operator, as defined in the previous section (that is, a solution to the initial value problem $\mathrm{L}g = 0$, $g(0) = 0$, $g'(0) = 1$). Let $G(x, s)$ be the Green function as defined in this section. Show that $G(x, s) = g(x - s)$.
Solution

For #16–#17, find a general solution to the differential equation by using variation of parameters.

(16) $y''' - y' = \cos(2t)$
Short Answer
Solution

(17) $xz''' - z'' = \log(x)$
(The homogeneous solutions are 1, x, and x^3.)

Short Answer
Solution

For #18–#19, find a general solution to the differential equation by integrating against a Green function.

(18) $y''' - 19y' + 30y = e^{2t}$
(Note $z^3 - 19z + 30 = (z-2)(z-3)(z+5)$)

Short Answer
Solution

(19) $w''' - \dfrac{2}{t^2}w' = t$
(The homogeneous solutions are 1, t^3, $\log(t)$. Find a solution on $(0, \infty)$.)

Short Answer
Solution

(20) **A comparison between different solving techniques**
This is a good review of the solving techniques discussed so far for second-order, linear non-homogeneous differential equations.
a) Find a general solution to the following equation:

$$w'' + 9w = 0.$$

b) Can you think of multiple ways of solving the following differential equation $w'' + 9w = e^x$?
c) Can you think of multiple ways of solving the folllowing differential equation $w'' + 9w = \sec(3x)$?
Remark There should be at least 3 ways to approach part c) :).

Short Answer
Solution

(21) The functions v^2 and v are solutions to the following homogeneous equation

$$v^2w'' - 2vw' + 2w = 0,$$

for $v > 0$. (No need to check that v^2 and v are indeed solutions!)
a) Compute the Wronskian of the two functions and evaluate it at $v = 5$.
b) Solve the initial value problem

$$v^2w'' - 2vw' + 2w = v^3e^v, \ w(1) = w'(1) = 0, \ v \ > \ 0,$$

using both solving technques discussed in this section. You should be able to evaluate all the definite integrals.

Short Answer
Solution

(22) Write down a general solution to the non-homogeneous second-order linear equation $x'' + x = \sec(t)$, for $t \in (-\frac{\pi}{2}, \frac{\pi}{2})$, using the two methods discussed in this section. Which one is lengthier/easier to solve with?

Short Answer
Solution

(23) Here we will illustrate how the nonhomogeneity in an initial value problem can be separated in two initial value problems.

Show that the solution to the differential equation $\mathrm{L}[v] = v'' + p(x)v' + q(x)v = f(x), v(x_0) = v_0, v'(x_0) = v_0'$ can be written as $v(x) = y(x) + w(x)$, where y and w are solutions to the following initial value problems:

$$\mathrm{L}[y] = 0,\ y(x_0) = v_0,\ y'(x_0) = v_0'\ ,$$
$$\mathrm{L}[w] = f(x),\ w(x_0) = 0,\ w'(x_0) = 0.$$

Note : If a set of fundamental solutions is known for the differential equation $\mathrm{L}[y] = 0$, then $y(x)$ can be relatively easy to find. Moreover, finding $w(x)$ could be done by the method of variation of parameters or Green's function method.

Solution

(24) The method of reduction of order can also be used with non-homogeneous non-constant coefficient linear differential equations. Specifically, consider the following equation $w'' + p(u)w' + q(u)w = g(u)$, and suppose that one solution to the associated homogeneous equation is known, $w_1(u)$. Now let $w(u) = v(u)w_1(u)$ and show that if $w(u)$ is a solution to $w'' + p(u)w' + q(u)w = g(u)$, then $v(u)$ must be a solution to $w_1(u)v'' + [2w_1'(u) + p(u)w_1(u)]v' = g(u)$.

Remark : The equation $w_1(u)v'' + [2w_1'(u) + p(u)w_1(u)]v' = g(u)$ is a first order linear equation in v'. After solving this equation, integrating the result to obtain $v(u)$ and multiplying that by $w_1(u)$, we obtain the general solution $w(u) = v(u)w_1(u)$ to the original differential equation $w'' + p(u)w' + q(u)w = g(u)$.

Solution

(25) Given that $y_1(x) = e^x$, $y_2(x) = xe^x$, and $y_3(x) = e^{-x}$ are solutions to the homogeneous equation associated with $y''' - y'' - y' + y = f(x)$, determine a particular solution to the differential equation in terms of the definite integrals. Can you write down the general solution as well?

Short Answer
Solution

(26) Find a formula involving integrals for a particular solution of the integral equation

$$w''' - w'' + w' - w = f(u).$$

Short Answer
Solution

Navigation to Other Chapters

This page may not work with every browser-driven pdf viewer. When it does work then it will enable you to link directly to any chapter. When it does not work then you can link to any chapter through the main webpage.