II. Higher-Order Linear Ordinary Differential Equations
8. Application: Mechanical Vibrations

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8. APPLICATION: MECHANICAL VIBRATIONS

8.1. **Spring-Mass Systems.** Consider a spring hanging from a support. When an object of mass m is attached to the free end of the spring, the object will eventually come to rest at a lower position. Let y_o and y_r be the vertical rest positions of the free end of the spring without and with the mass attached. We will assume that the mass is constrained to only move vertically and want to describe the vertical postition $y(t)$ of the mass as a function of time t when the mass is initially displaced from y_r, or is given some initial velocity, or is driven by an external force $F_{\text{ext}}(t)$.

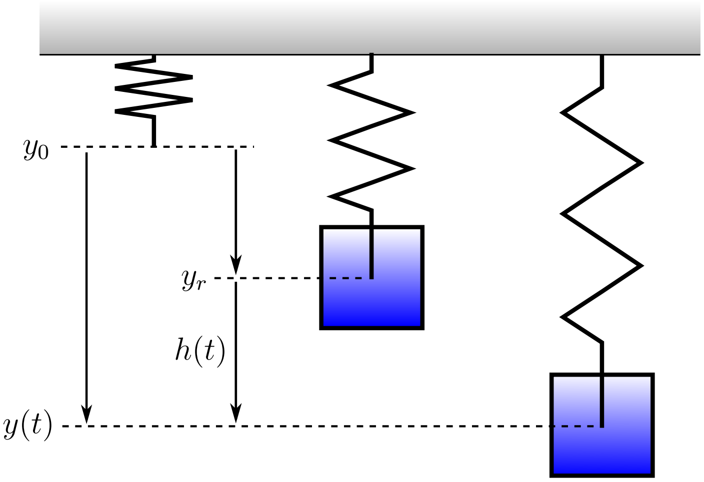

FIGURE 8.1. A spring-mass system

The forces acting on the mass that we will consider are the gravitational force F_{grav}, the spring force F_{spr}, the damping force F_{damp}, and the external force F_{ext}. Newton's law of motion states that

$$m\ddot{y} = F_{\text{grav}} + F_{\text{spr}} + F_{\text{damp}} + F_{\text{ext}}\,. \tag{8.1}$$

Here we are denoting time derivatives with dots, which is common in mechanics.

Before starting any problem be sure that you are using a consistant system of units. Three common systems are CGS, MKS, and Imperial.

- In CGS units length is given in centimeters (cm), time in seconds (sec), mass in grams (g), and force in dynes (1 dyne = 1 g cm/sec^2).
- In MKS units length is given in meters (m), time in seconds (sec), mass in kilograms (kg), and force in Newtons (1 Newton = 1 kg m/sec^2).
- In British (Imperial) units length is given in feet (ft), time in seconds (sec), mass in slugs (sl), and force in pounds (1 lb = 1 sl ft/sec^2).

Remark. Metric systems are widely used in engineering and science. The British system is seldom used by the British now.

The gravitational force F_{grav} is simply the downward weight of the mass. If we assume a uniform *gravitational acceleration* g then

$$F_{\text{grav}} = -mg\,, \tag{8.2}$$

where $g = 980$ cm/sec^2 in CGS units, $g = 9.8$ m/sec^2 in MKS units, and $g = 32$ ft/sec^2 in British units.

The spring force is modeled by Hooke's law

$$F_{\text{spr}} = -k(y - y_o)\,, \tag{8.3}$$

where k is the so-called *spring constant* or *spring coefficient*. This is a fairly good model provided $y - y_o$ does not get too large. When there is no external force, the mass has a rest position $y_r < y_o$ that satisfies

$$0 = F_{\text{grav}} + F_{\text{spr}} \quad \text{at } y = y_r\,.$$

Hence, we have

$$mg = -k(y_r - y_o) = k(y_o - y_r) = k|y_r - y_o|\,. \tag{8.4}$$

Sometimes we will be given $|y_r - y_o|$ and have to figure out k from this relation.

The damping force is modeled by

$$F_{\text{damp}} = -c\,\dot{y}\,, \tag{8.5}$$

where $c \geq 0$ is the so-called *damping coefficient*. This is not as good a model for the damping force as Hooke's Law was for the spring force, but we will use it because of its simplicity. Sometimes we will be given $|F_{\text{damp}}|$ at a particular speed and have to determine c from this relation.

If we place (8.2), (8.3), and (8.5) into Newton's law of motion (8.1) and neglect the external force, we obtain

$$m\ddot{y} + c\dot{y} + ky = ky_o - mg\,.$$

We see from (8.4) that $ky_o - mg = ky_r$, where y_r is the rest position. We thereby have

$$m\ddot{y} + c\dot{y} + ky = ky_r\,.$$

This clearly has the particular solution $y = y_r$. If we let $y(t) = y_r + h(t)$ then $h(t)$ satisfies the homogeneous equation

$$m\ddot{h} + c\dot{h} + kh = 0\,.$$

Here $h(t)$ is simply the *displacement* of the mass from its rest position y_r. If an external force is present, this becomes

$$m\ddot{h} + c\dot{h} + kh = F_{\text{ext}}(t)\,. \tag{8.6}$$

We will study the motion of this spring-mass system building up its complexity from the simplest case.

8.2. **Unforced, Undamped Motion.** In this case $F_{\text{ext}} = 0$ and $c = 0$, so that (8.6) reduces to

$$m\ddot{h} + kh = 0\,,$$

or in normal form

$$\ddot{h} + \frac{k}{m}\,h = 0\,. \tag{8.7}$$

Its characteristic polynomial is

$$p(z) = z^2 + \frac{k}{m}\,,$$

which has the conjugate pair of roots $\pm i\omega_o$ where we define

$$\omega_o = \sqrt{\frac{k}{m}}\,. \tag{8.8}$$

A general solution of equation (8.7) is

$$h(t) = c_1 \cos(\omega_o t) + c_2 \sin(\omega_o t)\,. \tag{8.9}$$

For the initial condtions $h(0) = h_0$ and $\dot{h}(0) = h_1$ this becomes

$$h(t) = h_0 \cos(\omega_o t) + h_1 \frac{\sin(\omega_o t)}{\omega_o}\,.$$

This motion is a periodic oscillation characterized by the single frequency ω_o. Because ω_o is given in terms of the spring constant k and the mass m through (8.8), it is called the *natural frequency* of the spring-mass system. It has units of radians/sec. The *cycle frequency* f_o, which has units of cycles/sec (= Hertz), is related by $\omega_o = 2\pi f_o$. We see from (8.9) that $h(t + T_o) = h(t)$ for every t, where the period T_o is related to ω_o and f_o by

$$\frac{\omega_o}{2\pi} = f_o = \frac{1}{T_o}\,,$$

We call T_o the *natural period* of the spring-mass system. It has units of sec. We will use ω_o rather than f_o in our presentation.

Remark. In the engineering literature f_o is often called the frequency and ω_o is called the *angular frequency.* There is some inconsistancy regarding the term angular frequency. It is given units of radians per second by some sources and units of degrees per second by others. The convention of calling ω_o the frequency and giving it units of radians per second is standard in the science and mathemaics liturature. This is what we will do.

The periodic motion described by (8.9) is called *simple harmonic motion.* It is nontrivial whenever either c_1 or c_2 is nonzero. In that case we can express it in the so-called *amplitude-phase form*

$$h(t) = A\cos(\omega_o t - \phi)\,,$$

where $A > 0$ is its *amplitude* and ϕ in $[0, 2\pi)$ is its *phase.* By the cosine addition formula the above form can be expanded as

$$h(t) = A\cos(\phi)\cos(\omega_o t) + A\sin(\phi)\sin(\omega_o t)\,.$$

By comparing this with (8.9), the linear independence of $\cos(\omega_o t)$ and $\sin(\omega_o t)$ implies

$$A\cos(\phi) = c_1\,, \qquad A\sin(\phi) = c_2\,.$$

This shows that (A, ϕ) are simply the *polar coordinates* of the point in the plane whose *Cartesian coordinates* are (c_1, c_2). Clearly $A = \sqrt{c_1^2 + c_2^2} > 0$ while ϕ satisfies

$$\cos(\phi) = \frac{c_1}{A}\,, \qquad \sin(\phi) = \frac{c_2}{A}\,.$$

There is a unique ϕ in $[0, 2\pi)$ that satisfies these equations.

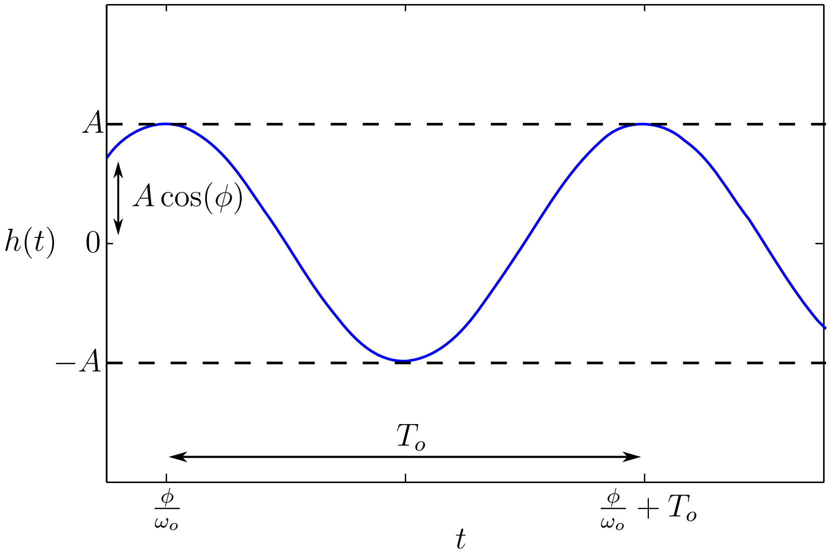

FIGURE 8.2. Simple harmonic motion: $h(t) = A\cos(\omega_0 t - \phi)$.

Example. A mass of 10 grams stretches a spring 5.0 cm when at rest. At $t = 0$ the mass is set in motion from 2.5 cm below its rest position with a downward velocity of 35.0 cm/sec. Neglect damping and external forces.

a) What is the displacement of the mass as a function of time?
b) What are the amplitude, phase, frequency, and period of the motion?
c) At what positive time does the mass first pass through its rest position?

Solution. Because $g = 980$ cm/sec^2, we can find k by setting

$$k \cdot 5 = mg = 10 \cdot 980 \quad \text{dynes}\,,$$

whereby

$$k = \frac{10 \cdot 980}{5} \quad \text{dynes/cm}\,.$$

Because we are neglecting damping and external forces, the equation of motion takes the form

$$m\ddot{h} + k\,h = 0\,,$$

which becomes

$$10\ddot{h} + \frac{10 \cdot 980}{5} h = 0 .$$

Bringing this into normal form gives

$$\ddot{h} + \frac{980}{5} h = 0 ,$$

which becomes

$$\ddot{h} + 196\, h = 0 .$$

Because $\omega_o^2 = 196$, we see that $\omega_o = 14$ 1/sec. Therefore a general solution of the equation of motion is

$$h(t) = c_1 \cos(14t) + c_2 \sin(14t) .$$

Because

$$\dot{h}(t) = -14c_1 \sin(14t) + 14c_2 \cos(14t) ,$$

the initial conditions $h(0) = -2.5$ and $\dot{h}(0) = -35.0$ imply that

$$h(0) = c_1 = -\tfrac{5}{2} , \qquad \dot{h}(0) = 14c_2 = -35 ,$$

which implies $c_1 = -\frac{5}{2}$ and $c_2 = -\frac{5}{2}$. Therefore the displacement of the mass as a function of time is

$$h(t) = -\tfrac{5}{2} \cos(14t) - \tfrac{5}{2} \sin(14t) .$$

This can be put into amplitude-phase form by observing that the point in the plane whose Cartesian coordinates are $(-\frac{5}{2}, -\frac{5}{2})$ has polar coordinates $(\frac{5\sqrt{2}}{2}, \frac{5\pi}{4})$. From this we see the following.

a) The displacement of the mass as a function of time is

$$h(t) = -\tfrac{5}{2} \cos(14t) - \tfrac{5}{2} \sin(14t) = \tfrac{5\sqrt{2}}{2} \cos(14t - \tfrac{5\pi}{4}) \quad \text{cm} .$$

b) The amplitude of the motion is $\frac{5\sqrt{2}}{2}$ cm, the phase is $\frac{5\pi}{4}$, the frequency is 14 1/sec, and the period is $\frac{\pi}{7}$ sec.

c) The mass passes through is rest position when $h(t) = 0$, which from the amplitude-phase form is when t satisfies

$$14t - \tfrac{5\pi}{4} = n\pi + \tfrac{\pi}{2} \quad \text{for some integer } n ,$$

which means that

$$14t = n\pi + \tfrac{7\pi}{4} \quad \text{for some integer } n ,$$

The first such time that is positive corresponds to $n = -1$, which is $t = \frac{3\pi}{56}$.

8.3. **Unforced, Damped Motion.** In this case $F_{\text{ext}} = 0$ and $c > 0$, so that (8.6) reduces to

$$m\ddot{h} + c\dot{h} + kh = 0\,,$$

which has the normal form

$$\ddot{h} + \frac{c}{m}\,\dot{h} + \frac{k}{m}\,h = 0\,. \tag{8.10}$$

Its characteristic polynomial is

$$p(z) = z^2 + \frac{c}{m}\,z + \frac{k}{m}\,.$$

If we complete the square then this has the form

$$p(z) = (z+\eta)^2 + \omega_o^2 - \eta^2\,. \tag{8.11}$$

where the damping rate η and the natural frequency ω_o are defined by

$$\eta = \frac{c}{2m}\,, \qquad \omega_o = \sqrt{\frac{k}{m}}\,.$$

It is clear there are three cases to consider.

- When $0 < \eta < \omega_o$ there is a conjugate pair of roots $-\eta \pm i\omega_\eta$ where

$$\omega_\eta = \sqrt{\omega_o^2 - \eta^2}\,. \tag{8.12}$$

- When $\eta = \omega_o$ there is a real double root $-\eta$, $-\eta$.
- When $\eta > \omega_o$ there are two simple real roots $-\eta \pm \sqrt{\eta^2 - \omega_o^2}$.

These are the *under damped*, *critically damped*, and *over damped* cases respectively.

Notice that there is no need to memorize formulas here. The state of a system can be read off from the roots of its characteristic polynomial:

- a conjugate pair of roots means that the system is under damped;
- a double real root means that the system is critically damped;
- two simple real roots means that the system is over damped.

Below we examine how the system behaves in each of these cases.

We are in the *under damped* case when the characteristic polynomial has the conjugate pair of roots $-\eta \pm i\omega_\eta$ with $\eta > 0$ and $\omega_\eta > 0$. Then a general solution is

$$h(t) = c_1 e^{-\eta t}\cos(\omega_\eta t) + c_2 e^{-\eta t}\sin(\omega_\eta t)\,, \tag{8.13}$$

Whenever either c_1 or c_2 is nonzero this can be put into the amplitude-phase form

$$h(t) = Ae^{-\eta t}\cos(\omega_\eta t - \phi)\,,$$

where (A, ϕ) are the polar coordinates of the point in the plane that has the Cartesian coordinates (c_1, c_2). We thereby see that $A = \sqrt{c_1^2 + c_2^2} > 0$ and $0 \le \phi < 2\pi$ such that

$$\cos(\phi) = \frac{c_1}{A}\,, \qquad \sin(\phi) = \frac{c_2}{A}\,.$$

Therefore $h(t)$ is an exponentially decaying simple harmonic motion with the time-dependent amplitude $Ae^{-\eta t}$, frequency ω_η, and phase ϕ like that shown in Figure 8.3

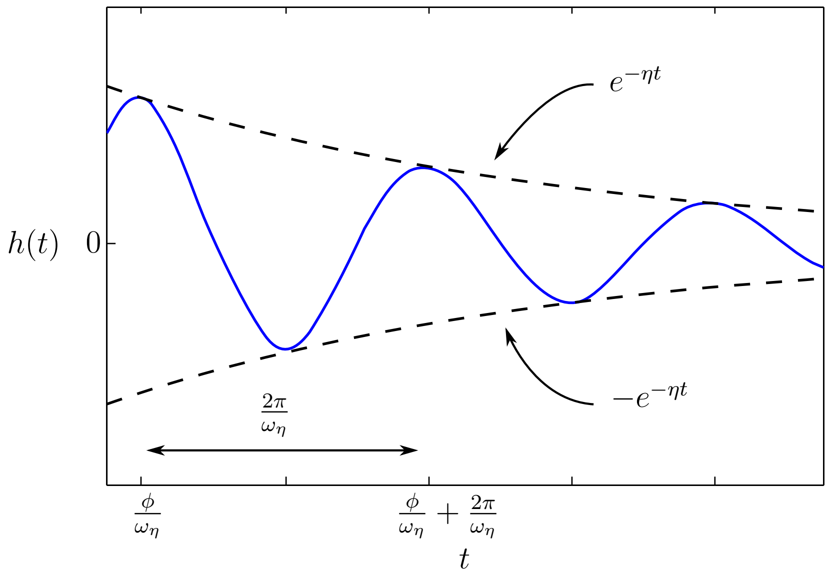

FIGURE 8.3. Under damped motion: $h(t) = e^{-\eta t}\cos(\omega_\eta t - \phi)$.

Here ω_η given by (8.12) is called the *damped frequency* or *quasi frequency* and the associated period $T_\eta = 2\pi/\omega_\eta$ is called the *damped period* or *quasi period*. Notice that

$$0 < \omega_\eta < \omega_o\,, \qquad T_o < T_\eta\,.$$

In other words, the damped frequency is always less than the natural frequency, while the damped period is always greater than the natural period.

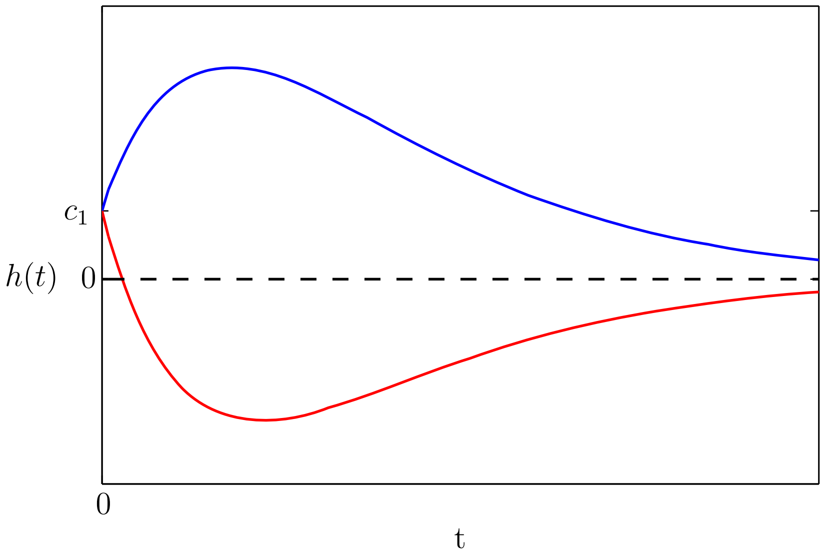

FIGURE 8.4. Critically damped motion: $h(t) = c_1 e^{-\eta t} + c_2 t\, e^{-\eta t}$. The red and blue curves have the same initial position c_1, but opposite initial velocity.

We are in the *critically damped* case when the characteristic polynomial has the double real root $-\eta$ with $\eta > 0$. Then a general solution is

$$h(t) = c_1 e^{-\eta t} + c_2 t\, e^{-\eta t}\,, \tag{8.14}$$

Therefore the displacement has at most one zero and decays like $t\, e^{-\eta t}$ whenever $c_2 \neq 0$.

We are in the *over damped* case when the characteristic polynomial has the two simple real roots $-\eta_+$ and $-\eta_-$, with $\eta_\pm > 0$ given by

$$\eta_\pm = \eta \pm \sqrt{\eta^2 - \omega_o^2}\,. \tag{8.15}$$

The subscript $+$ or $-$ indicates the sign of the square root taken in the above formula. Notice that $0 < \eta_- < \eta < \eta_+ < 2\eta$. Then a general solution is

$$h(t) = c_1 e^{-\eta_+ t} + c_2 e^{-\eta_- t}\,, \tag{8.16}$$

Therefore the displacement has at most one zero and decays like $e^{-\eta_- t}$ whenever $c_2 \neq 0$. Because $\eta_- < \eta$ we see that in this case the decay of the displacement is slower than in either the under or critically damped cases.

Remark. The spring system is said to be extremely over damped when η is much greater than ω_o. In that case we can use the approximation

$$\sqrt{\eta^2 - \omega_o^2} = \eta\sqrt{1 - \frac{\omega_o^2}{\eta^2}} \approx \eta\left(1 - \frac{\omega_o^2}{2\eta^2}\right) = \eta - \frac{\omega_o^2}{2\eta}\,,$$

to approximate η_- and η_+ in (8.15) by

$$\eta_- \approx \frac{\omega_o^2}{2\eta}\,, \qquad \eta_+ \approx 2\eta - \frac{\omega_o^2}{2\eta}\,.$$

In this regime η_- is much smaller than η and η_+ is nearly 2η. These decay rates are very different from each other with

$$\frac{\eta_-}{\eta_+} \approx \frac{\omega_o^2}{4\eta^2}\,, \qquad \text{which is much smaller than } 1\,.$$

Remark. This damped spring system is a good model for shock absorbers in a car. When the shock absorbers are over damped we get a jarring ride, while when they are under damped we get a bouncy ride. Ideally shock absorbers are tuned to be critically damped, which gives the least jarring and least bouncy ride. In practice they are tuned to be slightly overdamped.

8.4. **Forced, Undamped Motion.** In this case $F_{\text{ext}} \neq 0$ and $c = 0$, so that (8.6) reduces to

$$m\ddot{h} + kh = F_{\text{ext}}(t)\,.$$

We will study *simple harmonic external forcings* in the form

$$F_{\text{ext}}(t) = F\cos(\omega t)\,.$$

The equation then has the normal form

$$\ddot{h} + \omega_o^2 h = a\cos(\omega t)\,, \tag{8.17}$$

where the natural frequency $\omega_o > 0$ and the driving acceleration a are given by

$$\omega_o = \sqrt{\frac{k}{m}}\,, \qquad a = \frac{F}{m}\,.$$

Equation (8.17) may be solved either by Key Identity Evaluations or by Undetermined Coefficients. The characteristic polynomial is $p(z) = z^2 + \omega_o^2$, which has roots $\pm i\omega_o$. The forcing has degree $d = 0$, characteristic $\pm i\omega$, multiplicity $m = 0$ when $\omega \neq \omega_o$, and multiplicity $m = 1$ when $\omega = \omega_o$.

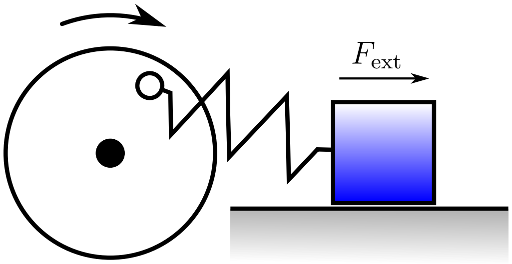

Figure 8.5. A driven oscillator

For $\omega \neq \omega_o$, if we impose the initial conditions $h(0) = h_0$ and $\dot{h}(0) = h_1$ then the solution is found to be

$$h(t;\omega) = h_0\cos(\omega_o t) + h_1\frac{\sin(\omega_o t)}{\omega_o} + a\,\frac{\cos(\omega t) - \cos(\omega_o t)}{\omega_o^2 - \omega^2}\,. \tag{8.18}$$

This is not simple harmonic motion because two frequencies are involved. Such motion is called biharmonic. In general, whenever more than one frequency is involved the motion is called polyharmonic.

For $\omega = \omega_o$, if we impose the inital conditions $h(0) = h_0$ and $\dot{h}(0) = h_1$ then the solution is found to be

$$h(t;\omega_o) = h_0\cos(\omega_o t) + h_1\frac{\sin(\omega_o t)}{\omega_o} + a\,\frac{t\,\sin(\omega_o t)}{2\omega_o}\,. \tag{8.19}$$

This is also not simple harmonic motion. In fact, its amplitude grows linearly in t! This is the phenomenon of *resonance*, which occurs when the driving frequency ω equals the natural frequency ω_o of the system. Because the l'Hopital rule implies

$$\lim_{\omega\to\omega_o}\frac{\cos(\omega t) - \cos(\omega_o t)}{\omega_o^2 - \omega^2} = \lim_{\omega\to\omega_o}\frac{-t\,\sin(\omega t)}{-2\omega} = \frac{t\,\sin(\omega_o t)}{2\omega_o}\,,$$

we see that formula (8.19) is obtained by taking the limit $\omega \to \omega_o$ in formula (8.18).

We can understand the onset of resonance as $\omega \to \omega_o$ by using the identity

$$\cos(\omega t) - \cos(\omega_o t) = -2\sin\left(\frac{\omega - \omega_o}{2}\, t\right) \sin\left(\frac{\omega + \omega_o}{2}\, t\right),$$

to re-express formula (8.18) as

$$h(t;\omega) = h_0 \cos(\omega_o t) + h_1 \frac{\sin(\omega_o t)}{\omega_o} + A_\omega(t)\, \sin\left(\frac{\omega + \omega_o}{2}\, t\right),$$

where

$$A_\omega(t) = \frac{2a}{\omega^2 - \omega_o^2}\, \sin\left(\frac{\omega - \omega_o}{2}\, t\right).$$

When $\omega - \omega_o$ is very small compared to ω and ω_o then $A_\omega(t)$ will be a very slowly varying function of t compared to $\sin\big((\omega + \omega_o)t/2\big)$. In that case $\sin\big((\omega + \omega_o)t/2\big)$ will oscillate very many times during a period over which $A_\omega(t)$ oscillates just once. These rapid oscillations will have an amplitude of $|A_\omega(t)|$ that slowly oscillates between 0 and $2a/(\omega^2 - \omega_o^2)$. This slow oscillation is the phenomenon of *beats*. The so-called *beating frequency* is $\omega - \omega_o$, which is small, while the so-called *beating period* is $2\pi/(\omega - \omega_o)$, which is large. As $\omega \to \omega_o$ the beating frequency vanishes, the beating period diverges to infinity, while by the l'Hospital rule we see that

$$\lim_{\omega\to\omega_o} A_\omega(t) = \lim_{\omega\to\omega_o} \frac{2a \sin\left(\dfrac{\omega - \omega_o}{2}\, t\right)}{\omega^2 - \omega_o^2} = \lim_{\omega\to\omega_o} \frac{a \cos\left(\dfrac{\omega - \omega_o}{2}\, t\right) t}{2\omega} = \frac{a\, t}{2\omega_o}.$$

This is in accord with the amplitude we found in formula (8.19).

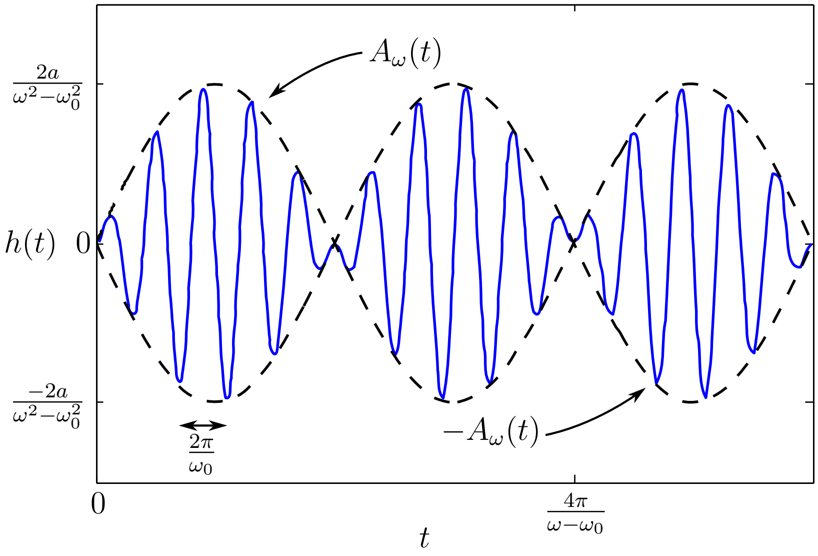

FIGURE 8.6. Motion that shows the phenomenon of beats for a forced undamped oscillator with $\omega < \omega_o$.

8.5. **Forced, Damped Motion.** In this case $F_{\text{ext}} \neq 0$ and $c > 0$, so that (8.6) reduces to

$$m\ddot{h} + c\dot{h} + kh = F_{\text{ext}}(t)\,.$$

We will again study *simple harmonic external forcings* in the form

$$F_{\text{ext}}(t) = F\cos(\omega t)\,.$$

The equation then has the normal form

$$\ddot{h} + \frac{c}{m}\,\dot{h} + \frac{k}{m}\,h = \frac{F}{m}\,\cos(\omega t)\,. \tag{8.20}$$

The associated homogeneous equation is (8.10), which describes the associated unforced, damped system.

Once again we introduce the damping rate η, the natural frequency $\omega_o > 0$, and the driving acceleration a defined by

$$\eta = \frac{c}{2m}\,, \qquad \omega_o = \sqrt{\frac{k}{m}}\,, \qquad a = \frac{F}{m}\,. \tag{8.21}$$

Equation (8.20) then becomes

$$\ddot{h} + 2\eta\,\dot{h} + \omega_o^2\,h = a\,\cos(\omega t)\,. \tag{8.22}$$

The solution $h_H(t)$ of the associated homogeneous equation is then given by either (8.13), (8.14), or (8.16) depending on whether the associated unforced system is under damped, critically damped, or over damped. In each case $h_H(t)$ decays to zero as $t \to \infty$.

The nonhomogeneous equation (8.22) has constant coefficients and a forcing that is in characteristic form with degree $d = 0$, characteristic $i\omega$, and multiplicity $m = 0$. Therefore a particular solution $h_P(t)$ can be found by either Key Identity Evaluations, the Zero Degree Formula, or Undetermined Coefficients. For example, the Key Identity is

$$\mathrm{L}(e^{zt}) = \big(z^2 + 2\eta z + \omega_o^2\big)e^{zt}\,.$$

By evaluating it at $z = i\omega$ we obtain

$$\mathrm{L}(e^{i\omega t}) = \big(-\omega^2 + i2\eta\omega + \omega_o^2\big)e^{i\omega t}\,.$$

From this we see that

$$\mathrm{L}\left(\frac{ae^{i\omega t}}{\omega_o^2 - \omega^2 + i2\eta\omega}\right) = ae^{i\omega t}\,.$$

Therefore by taking real parts we find the particular solution

$$\begin{aligned} h_P(t) &= \mathrm{Re}\left(\frac{a\,e^{i\omega t}}{\omega_o^2 - \omega^2 + i2\eta\omega}\right) \\ &= \frac{a}{(\omega_o^2 - \omega^2)^2 + 4\eta^2\omega^2}\,\mathrm{Re}\Big(\big(\omega_o^2 - \omega^2 - i2\eta\omega\big)\big(\cos(\omega t) + i\sin(\omega t)\big)\Big) \\ &= \frac{a(\omega_o^2 - \omega^2)}{(\omega_o^2 - \omega^2)^2 + 4\eta^2\omega^2}\,\cos(\omega t) + \frac{a2\eta\omega}{(\omega_o^2 - \omega^2)^2 + 4\eta^2\omega^2}\,\sin(\omega t)\,, \end{aligned}$$

where η, ω_o, and a are given by (8.21).

Therefore a general solution of equation (8.22) has the form

$$h(t) = h_H(t) + h_P(t)\,,$$

where $h_H(t)$ decays to zero as $t \to \infty$ and $h_P(t)$ is the periodic solution given by (8.23). Because it decays to zero, $h_H(t)$ is called the *transient component* of the solution, or simply the *transient*. Because it is periodic, $h_P(t)$ is called the *steady-state component* of the solution. This decomposition of a solution into a transient and a steady-state component is illustrated in Figure 8.7.

Because $h_P(t)$ is the unique periodic solution of equation (8.22), it is also called the *steady-state solution* of the forced, damped system. This simple harmonic motion can be put into the amplitude-phase form

$$h_P(t) = A\cos(\omega t - \delta)\,, \tag{8.23a}$$

where

$$A = \frac{a}{\sqrt{(\omega_o^2 - \omega^2)^2 + 4\eta^2\omega^2}}\,, \qquad \delta = \cos^{-1}\left(\frac{\omega_o^2 - \omega^2}{\sqrt{(\omega_o^2 - \omega^2)^2 + 4\eta^2\omega^2}}\right). \tag{8.23b}$$

Here A is the *amplitude* of the steady-state solution while δ is the *phase shift* by which the steady-state solution lags behind its forcing.

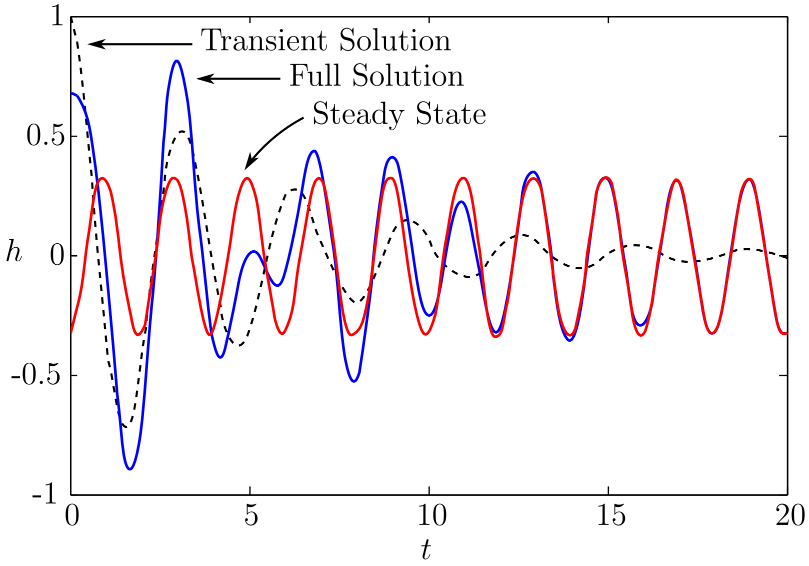

FIGURE 8.7. Typical motion for a forced, damped oscillator. The full solution (blue), the steady-state solution (red), and the transient solution (dashed) solution are shown on top of each other.

Finally, we examine how the phenomenon of resonance is modified by damping. One obvious difference is that all solutions will remain bounded for any driving frequency ω. However, there can still be a resonance-like phenomenon in which oscillations are amplified by certain driving frequencies. We see from (8.23) that the amplitude of the steady-state solution is a function of the driving frequency ω that is given by

$$A(\omega) = \frac{a}{\sqrt{(\omega_o^2 - \omega^2)^2 + 4\eta^2\omega^2}}\,.$$

It is clear that this function will be maximum when $(\omega_o^2 - \omega^2)^2 + 4\eta^2\omega^2$ is minimum. Because

$$\begin{aligned}(\omega_o^2 - \omega^2)^2 + 4\eta^2\omega^2 &= \omega^4 - 2\omega_o^2\omega^2 + \omega_o^4 + 4\eta^2\omega^2 \\ &= \omega^4 - 2\big(\omega_o^2 - 2\eta^2\big)\omega^2 + \omega_o^4\,,\end{aligned}$$

we see that when $2\eta^2 < \omega_o^2$ the minimizer of this quantity is

$$\omega = \sqrt{\omega_o^2 - 2\eta^2}\,,$$

while when $2\eta^2 \geq \omega_o^2$ the minimizer is $\omega = 0$. Therefore the function $A(\omega)$ is maximized at $\omega = \omega_{\max}(\eta)$ where

$$\omega_{\max}(\eta) = \begin{cases} \sqrt{\omega_o^2 - 2\eta^2} & \text{when } 0 < 2\eta^2 < \omega_o^2\,, \\ 0 & \text{when } \omega_o^2 \leq 2\eta^2\,. \end{cases} \tag{8.24}$$

The associated maximum of $A(\omega)$ is thereby

$$A(\omega_{\max}(\eta)) = \begin{cases} \dfrac{a}{\sqrt{4\eta^2(\omega_o^2 - \eta^2)}} & \text{when } 0 < 2\eta^2 < \omega_o^2\,, \\[2ex] \dfrac{a}{\omega_o^2} & \text{when } \omega_o^2 \leq 2\eta^2\,. \end{cases} \tag{8.25}$$

We see that a resonance-like phenomenon persists when $0 < \eta < \frac{\sqrt{2}}{2}\omega_o$ but disappears when $\frac{\sqrt{2}}{2}\omega_o \leq \eta$. This fact is illustrated in Figure 8.8.

Remark. When $0 < 2\eta^2 < \omega_o^2$ the system is under damped and

$$0 < \omega_{\max}(\eta) < \omega_\eta < \omega_o\,,$$

where ω_η is the damped frequency given by (8.12) and $\omega_{\max}(\eta)$ is given by (8.24).

Remark. When $0 < 2\eta^2 < \omega_o^2$ we can eliminate η from the equations

$$\omega = \omega_{\max}(\eta)\,, \qquad A = A(\omega_{\max}(\eta))\,,$$

and show that the points $(\omega_{\max}(\eta), A(\omega_{\max}(\eta)))$ lie on the curve $A = A_{\max}(\omega)$ where

$$A_{\max}(\omega) = \frac{a}{\sqrt{\omega_o^4 - \omega^4}}\,.$$

This curve is illustrated by a dotted line in Figure 8.8.

Exercise. Prove the assertion made in the above remark.

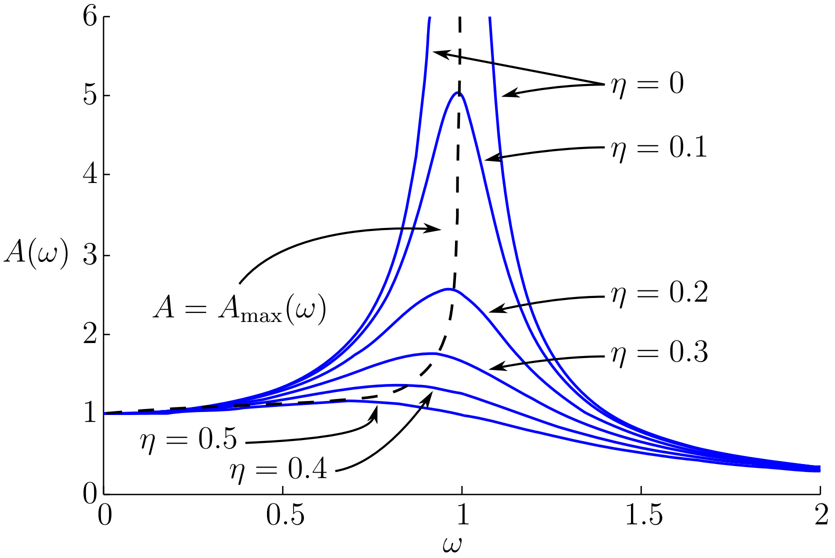

Figure 8.8. Amplitude $A(\omega)$ plotted versus driving frequency ω for various values of η when the forcing has $a = 1$ and $\omega_o = 1$. Notice how the amplitude blows up at the resonant frequency as $\eta \to 0$. The dotted line is the curve $A = A_{\max}(\omega)$ on which the points $(\omega_{\max}, A(\omega_{\max}))$ lie.

8.6. **Oscillations and Phasors.** In the previous section we saw that oscillations play an important role in forced damped systems. In particular, we saw that when such a system is driven by a simple harmonic oscillatory external forcing that the response is a simple harmonic oscillatory steady-state with the same frequency. Here we want to focus on such oscillations.

Consider the forced damped system

$$\ddot{h} + 2\eta\dot{h} + \omega_o^2 h = f(t)\,, \tag{8.26}$$

where $\eta > 0$ is the damping rate, $\omega_o > 0$ is the natural frequency, and $f(t)$ is the external forcing. We consider simple harmonic external forcings in the form

$$f(t) = \alpha\cos(\omega t) + \beta\sin(\omega t)\,, \tag{8.27}$$

where $\omega > 0$ is the forcing frequency. This forcing is nonzero provided either $\alpha \neq 0$ or $\beta \neq 0$. In that case it can be put into the *amplitude-phase form*

$$f(t) = a\cos(\omega t - \phi)\,, \tag{8.28}$$

where $a > 0$ is the forcing amplitude, and ϕ is the forcing phase. We can assume that $0 \leq \phi < 2\pi$. Here (a, ϕ) are the polar coordinates of the point in the plane whose Cartesian coordinates are (α, β).

The forcing $f(t)$ can also be expressed the *phasor form*

$$f(t) = \mathrm{Re}\big(\gamma e^{i\omega t}\big)\,, \tag{8.29}$$

where the *phasor* is the complex number γ, which can be expressed as

$$\gamma = \alpha - i\beta = a e^{-i\phi}\,. \tag{8.30}$$

We call the first expression for γ its *Cartesian form* and the second its *polar form.*

When the Cartesian form of the phasor γ is used in (8.29) we call it the *Cartesian phasor form* of $f(t)$. It becomes

$$\begin{aligned} f(t) = \mathrm{Re}\big(\gamma e^{i\omega t}\big) &= \mathrm{Re}\big((\alpha - i\beta)\big(\cos(\omega t) + i\sin(\omega t)\big)\big) \\ &= \alpha\cos(\omega t) + \beta\sin(\omega t)\,, \end{aligned}$$

which recovers the original form (8.27).

When the polar form of the phasor γ is used in (8.29) we call it the *polar phasor form* of $f(t)$. It becomes

$$f(t) = \mathrm{Re}\big(\gamma e^{i\omega t}\big) = \mathrm{Re}\big(a e^{-i\phi} e^{i\omega t}\big) = a\,\mathrm{Re}\big(e^{i(\omega t - \phi)}\big) = a\,\cos(\omega t - \phi)\,,$$

which recovers the amplitude-phase form (8.28).

Phasors have properties that make them very useful. For example, the sum of simple harmonic oscillations that each have the *same* frequency is also a simple harmonic oscillation with that frequency and *its phasor is the sum of the constituent phasors.* Indeed, the sum of n simple harmonic oscillations with phasors $\gamma_1, \cdots, \gamma_n$, and common frequency ω is

$$\sum_{k=1}^{n} \mathrm{Re}\big(\gamma_k e^{i\omega t}\big) = \mathrm{Re}\left(\sum_{k=1}^{n}\big(\gamma_k e^{i\omega t}\big)\right) = \mathrm{Re}\left(\left(\sum_{k=1}^{n}\gamma_k\right)e^{i\omega t}\right) = \mathrm{Re}\big(\gamma e^{i\omega t}\big)\,,$$

where the phasor γ of the sum is the sum of the constituent phasors

$$\gamma = \sum_{k=1}^{n}\gamma_k\,.$$

Remark. The word *phasor* is a portmanteau for *phase vector.* It reflects the fact shown above that phasors contain phase information and have vector-like properties.

Phasors also give a way to understand how a simple harmonic forcing produces a simple harmonic steady-state solution of a differential equation. Notice that the phasor form (8.29) is a special case of the way characteristic forcings with degree $d = 0$ and a complex characteristic $\mu + i\nu$ were represented when we apply Key Identity Evaluations. Indeed, the simple harmonic external forcing (8.27) has degree $d = 0$ and characteristic $\mu + i\nu = i\omega$. The characteristic polynomial is

$$p(z) = z^2 + 2\eta z + \omega_o^2\,.$$

Because $\mu + i\nu = i\omega$ and

$$p(i\omega) = -\omega^2 + i2\eta\omega + \omega_o^2 \neq 0\,,$$

we see that $m = 0$. Here we have used the fact that ω_o and η are positive.

Because $d = 0$ we can use either Key Identity Evaluations or the Zero Degree Formula to obtain a particular solution of (8.26). Below we will show that both methods yield the particular solution

$$h_P(t) = \mathrm{Re}\left(\frac{\gamma e^{i\omega t}}{p(i\omega)}\right) . \tag{8.31}$$

This is a periodic function of time with frequency ω. Because every solution $h_H(t)$ of the associated homogeneous equation decays to zero as $t \to \infty$, every solution $h(t)$ of equation (8.26) will behave more like $h_P(t)$ as t increases.

Notice that $h_P(t)$ given by (8.31) has the phasor form

$$h_P(t) = \mathrm{Re}\big(\Gamma e^{i\omega t}\big) , \quad \text{where} \quad \Gamma = \frac{\gamma}{p(i\omega)} .$$

This says that the phasor Γ of the response $h_P(t)$ is simply the phasor γ of the forcing $f(t)$ multiplied by the factor $1/p(i\omega)$. If we express γ in its polar form $\gamma = ae^{-i\phi}$ and $p(i\omega)$ in the polar form

$$p(i\omega) = |p(i\omega)|e^{i\delta(\omega)} ,$$

then the polar form of Γ is

$$\Gamma = \frac{\gamma}{p(i\omega)} = \frac{ae^{-i\phi}}{|p(i\omega)|e^{i\delta(\omega)}} = \frac{a}{|p(i\omega)|}\, e^{-i(\phi+\delta(\omega))} .$$

The simple harmonic motion $h_P(t)$ then has the amplitude-phase form

$$h_P(t) = \frac{a}{|p(i\omega)|}\, \cos\big(\omega t - \phi - \delta(\omega)\big) .$$

In other words, the response of the system is that the forcing will be multiplied by a factor of $1/|p(i\omega)|$ and its phase will shifted by $\delta(\omega)$ radians. This simple statement of how a damped system responds to an oscillatory forcing shows the usefulness of the polar phasor representation.

Because

$$p(i\omega) = \omega_o^2 - \omega^2 + i2\eta\omega ,$$

we see that

$$\frac{1}{|p(i\omega)|} = \frac{1}{\sqrt{(\omega_o^2 - \omega^2)^2 + 4\eta^2\omega^2}} , \qquad \delta(\omega) = \cos^{-1}\left(\frac{\omega_o^2 - \omega^2}{\sqrt{(\omega_o^2 - \omega^2)^2 + 4\eta^2\omega^2}}\right) .$$

This is in accord with what we found in (8.23) for the case $\phi = 0$.

Remark. This means that all of the subsequent analysis in the previous section about how the phenomenon of resonance is modified by the presence of damping carries over without change. In particular, a resonance-like phenomenon persists when $2\eta^2 < \omega_o^2$.

Remark. With our usual conventions for the inverse cosine function we have

$$0 < \delta(\omega) < \pi .$$

The addition $\phi + \delta(\omega)$ can be carried out "mod 2π". This means that we can set

$$\phi + \delta(\omega) = \begin{cases} \phi + \delta(\omega) & \text{when } \phi + \delta(\omega) < 2\pi , \\ \phi + \delta(\omega) - 2\pi & \text{otherwise} . \end{cases}$$

Remark. To express the phasor Γ in its Cartesian form we first express γ in its Cartesian form $\gamma = \alpha - i\beta$ and $p(i\omega)$ in the Cartesian form

$$p(i\omega) = q(\omega) + ir(\omega)\,,$$

where $q(\omega)$ and $r(\omega)$ are real. Then

$$\begin{aligned}\Gamma &= \frac{\gamma}{p(i\omega)} = \frac{\alpha - i\beta}{q(\omega) + ir(\omega)} = \frac{\alpha - i\beta}{q(\omega) + ir(\omega)}\,\frac{q(\omega) - ir(\omega)}{q(\omega) - ir(\omega)} \\ &= \frac{(\alpha - i\beta)(q(\omega) - ir(\omega))}{q(\omega)^2 + r(\omega)^2} = \frac{(\alpha q(\omega) - \beta r(\omega)) - i(\alpha r(\omega) + \beta q(\omega))}{q(\omega)^2 + r(\omega)^2}\,.\end{aligned}$$

The Cartesian form not only required more work, but hides the much useful information. This is why the polar phasor form is prefered.

We now give the promised derivations of the particular solution $h_P(t)$ given by (8.31).

Key Identity Evaluations. Because $d = 0$ and $m = 0$, we need to evaluate the Key Identity at $z = i\omega$. The Key Identity is

$$\mathrm{L}(e^{zt}) = p(z)e^{zt}\,.$$

By evaluating the Key Identity at $z = i\omega$ we obtain

$$\mathrm{L}(e^{i\omega t}) = p(i\omega)e^{i\omega t}\,.$$

Because the external forcing is given by the phasor representaion (8.29), we multiply by γ and divide by $p(i\omega)$ to obtain

$$\mathrm{L}\left(\frac{\gamma e^{i\omega t}}{p(i\omega)}\right) = \gamma e^{i\omega t}\,.$$

By taking the real part of this equation we see that a particular solution is given by

$$h_p(t) = \mathrm{Re}\left(\frac{\gamma e^{i\omega t}}{p(i\omega)}\right)\,,$$

which is what was claimed in (8.31).

Zero Degree Formula. For a forcing in the form

$$f(t) = \alpha e^{\mu t}\cos(\nu t) + \beta e^{\mu t}\sin(\nu t)\,,$$

the Zero Degree Formula gives the particular solution

$$h_P(t) = t^m e^{\mu t}\mathrm{Re}\left(\frac{(\alpha - i\beta)e^{i\nu t}}{p^{(m)}(\mu + i\nu)}\right)\,.$$

Because the external forcing is given by the phasor representaion (8.29), we see that $\mu + i\nu = i\omega$, $\alpha - i\beta = \gamma$, and $m = 0$, whereby

$$h_p(t) = \mathrm{Re}\left(\frac{\gamma e^{i\omega t}}{p(i\omega)}\right)\,,$$

which is what was claimed in (8.31).

EXERCISES ON MECHANICAL VIBRATIONS

Set up and solve the following spring problems.

(1) When a mass of 5 grams is hung vertically from a spring, at rest it stretches the spring 2.45 cm. (Gravitational acceleration is $g = 980 cm/sec^2$.) At $t = 0$ the mass is moved from its equilibrium position with an initial velocity of 2 cm/sec. Find a solution to this initial value problem for relatively small values of t where we will assume the effect of dampening is negligible. Write your solution as $A\cos(\omega_0 t - \delta)$ where w_0 is the natural frequency, and δ is the phase shift.

Short Answer
Solution

(2) When a mass of 4 grams is hung vertically from a spring, at rest it stretches the spring 9.8 cm. (Gravitational acceleration is $g = 980 cm/sec^2$.) At $t = 0$ the mass is stretched 2 cm from its equilibrium position and released with no initial velocity. Find a solution to this initial value problem for relatively small values of t where we will assume the effect of dampening is negligible. Write your solution as $A\cos(\omega_0 t - \delta)$ where w_0 is the natural frequency, and δ is the phase shift.

Short Answer
Solution

(3) Assume a 8 lb weight stretches a spring 6 inches. Recall that the gravity constant for standard units is $32 ft/s^2$ (also recall that a pound is a unit of force (weight) not mass). Find an expression which represents the position of the mass with respect to its equilibrium position if we assume the effect of dampening is negligible and there are no external forces acting on the mass, where the mass is initially stretched an addition 3 in beyond the equilibrium position an is released with an initial velocity of 2 ft/s.

Short Answer
Solution

(4) Assume a 4 lb weight stretches a spring 2 feet. Recall that the gravity constant for standard units is $32 ft/s^2$ (also recall that a pound is a unit of force (weight) not mass) Find an expression which represents the position of the mass with respect to its equilibrium position if we assume the effect of dampening is negligible and there are no external forces acting on the mass, where the mass is initially stretched an addition $1/\sqrt{3}$ ft beyond the equilibrium position an is released with an initial velocity of 4 ft/s.

Short Answer
Solution

Suppose each of the following equations model the motion of a mass on spring. Determine if the spring is critically damped, overdamped, or underdamped.

(5) $4z'' + 2z' + z = 0$

Short Answer
Solution

(6) $2u'' + 10u' + 3u = 0$

Short Answer
Solution

(7) $2w'' + 5w' + 2w = 0$

Short Answer
Solution

(8) $3u'' + 12u' + 12u = 0$

Short Answer
Solution

(9) $2v'' + 20v' + 50v = 0$

Short Answer
Solution

(10) $3u'' + 2u' + 4u = 0$

Short Answer
Solution

Solve the following problems

(11) Assume a 4 lb weight stretches a spring 3 inches. Recall that the gravity constant for standard units is $32ft/s^2$ (also recall that a pound is a unit of force (weight) not mass). Suppose we also that when the mass is moving $2ft/sec$ the medium in which the mass moves imparts of force of 6 lbs. Is this system overdamped, critically damped, or underdamped? Justify with the appropriate computations. (You do not need to find a solution; Please just classify the system).

Short Answer
Solution

(12) Assume a 4 lb weight stretches a spring 3 inches. Recall that the gravity constant for standard units is $32ft/s^2$ (also recall that a pound is a unit of force (weight) not mass). Suppose we also that when the mass is moving $2ft/sec$ the medium in which the mass moves imparts of force of 6 lbs. Find a function $u(t)$ that measures the position of the spring if it is initially moved 2 inches from its equilibrium position and released with no initial velocity.

Short Answer
Solution

(13) Assume a 8 lb weight stretches a spring 6 inches. Recall that the gravity constant for standard units is $32ft/s^2$ (also recall that a pound is a unit of force (weight) not mass). Suppose we also consider the effect of dampening, where when the mass is moving $2ft/sec$ the medium in which the mass moves imparts of force of 8 lbs. Is this system overdamped, critically damped, or underdamped? Justify

with the appropriate computations. (You do not need to find a solution; Please just classify the system).

Short Answer
Solution

(14) Assume a 8 lb weight stretches a spring 6 inches. Recall that the gravity constant for standard units is $32ft/s^2$ (also recall that a pound is a unit of force (weight) not mass). Suppose we also consider the effect of dampening, where when the mass is moving $2ft/sec$ the medium in which the mass moves imparts of force of 8 lbs. Find a function $u(t)$ that measures the position of the spring if it moved from its equilibrium position with an initial velocity of 2 ft/sec.

Short Answer
Solution

(15) Assume a 4 kg mass stretches a spring 19.6 cm, and that when the mass is moving with a velocity of 5 cm/s the medium in which the mass is moving exerts a viscous force of 3 N. Is this system overdamped, critically damped, or underdamped? Justify with the appropriate computations. (You do not need to find a solution; Please just classify the system).

Short Answer
Solution

(16) Assume a 4 kg mass stretches a spring 19.6 cm, and that when the mass is moving with a velocity of 5 cm/s the medium in which the mass is moving exerts a viscous force of 3 N. Find an expression which represents the position of the mass with respect to its equilibrium position if there are no external forces acting on the mass, and it is released from its equilibrium position with an initial velocity of 5 cm/s.

Short Answer
Solution

(17) Suppose a 2 kg mass stretches a spring 4.9 meters. Suppose we also consider the effect of dampening and that when the mass is moving 1 m/s the the medium in which the mass moves imparts of force of 4 N. Is this system overdamped, critically damped, or underdamped? Justify with the appropriate computations. (You do not need to find a solution; Please just classify the system).

Short Answer
Solution

(18) Suppose a 2 kg mass stretches a spring 4.9 meters. Suppose we also consider the effect of dampening and that when the mass is moving 1 m/s the the medium in which the mass moves imparts of force of 4 N. Suppose this mass is stretched 50 cm from its equilibrium position and pushed back with an initial velocity of 1 m/s (by pushed back we mean that the initial velocity is actually -1m/s for this system). Find a function which models the position of the mass t seconds after the mass is released.

Short Answer
Solution

(19) Suppose that a 2 slug mass stretches a spring 1/2 ft. Also suppose that when the mass is moving at a speed of 0.5 ft/sec the medium in which the mass is moving exerts a viscous force of 16 lbs.

(a) Is this system underdamped, critically damped or overdamped?

(b) Given some set of initial conditions, let $u(t)$ describe the position of the mass with respect to its equilibrium position when no external forces act on the mass, and $U(t)$ describe the position of the mass when an external force of $2\cos(t)$ ft-lbs is exerted. Compare the limiting behavior of $u(t)$ and $U(t)$. (You need not find either $u(t)$ or $U(t)$, but you should say what happens to each of them as $t \to \infty$).

Short Answer
Solution

(20) When a mass of 4 grams is hung vertically from a spring, at rest it stretches the spring 39.2 cm. (Gravitational acceleration is $g = 980cm/sec^2$.) Assume the effect of dampening is negligible and the mass is acted on continuously by the force $44\cos(6t)$ dynes where the mass is initially at rest in its equilibrium position. Solve the initial value problem. Also use the formulas $\cos(x \pm y) = \cos(x)\cos(y) \mp \sin(x)\sin(y)$ to write your answer in the form $A\sin(\frac{\omega_0+\omega}{2})\sin(\frac{\omega_0-\omega}{2})$ and sketch your answer for $0 < t < 6\pi$.

Short Answer
Solution

(21) When a mass of 5 grams is hung vertically from a spring, at rest it stretches the spring 2.45 cm. (Gravitational acceleration is $g = 980cm/sec^2$.) Assume the effect of dampening is negligible and that the mass is acted on continuously by the force $256\cos(12t)$ dynes where the mass is initially at rest in its equilibrium position. Solve the initial value problem. Also use the formulas $\cos(A \pm B) = \cos(A)\cos(B) \mp \sin(A)\sin(B)$ to write your answer in the form $A\sin(\frac{\omega_0+\omega}{2})\sin(\frac{\omega_0-\omega}{2})$ and sketch your answer for $0 < t < \pi/2$.

Short Answer
Solution

(22) Assume a 10 kg mass stretches a spring 19.6 cm, and that when the mass is moving with a velocity of 4 cm/s the medium in which the mass is moving exerts a viscous force of 4 N.

(a) Find an expression which represents the position of the mass with respect to its equilibrium position if there are no external forces acting on the mass, and it is initially stretching the spring 1 cm beyond the equilibrium position an is released with no initial velocity.

(b) Suppose instead that we impart an external force of $10\cos(t/2)$ N where initially the mass is at rest in its equilibrium position. Find an expression which represents the position of the mass with respect to its equilibrium position.

(c) Compare the behavior of your result in part (a) with your result in part (b) as $t \to \infty$.

Short Answer
Solution

(23) When a mass of 4 grams is hung vertically from a spring, at rest it stretches the spring 39.2 cm. (Gravitational acceleration is $g = 980cm/sec^2$.) Assume the effect of dampening is negligible and the mass is acted on continuously by the force $40\cos(5t)$ dynes where the mass is initially at rest in its equilibrium position. Solve the initial value problem. Discuss the solution as $t \to \infty$

Short Answer
Solution

(24) The vertical displacement of a mass on a spring is given by $w(x) = 4e^{-x}\cos(14x) - 3e^{-x}\sin(14x)$, where the positive displacement is in the upward direction.

Express $w(x)$ in the amplitude-phase form, i.e. in the form $w(x) = Ae^{-x}\cos(\omega x - \delta)$, where $A > 0$ and $\delta \in [0, 2\pi)$. Identify the phase of the oscillation (you can express it as an inverse trig function) and the quasiperiod of the oscillation.

Short Answer
Solution

(25) A stamping machine applies hammering forces on metal sheets through a die attached to the plunger. It describes an up and down motion through a flywheel spinning at a constant speed. The base on which the metal sheet is located has a mass of $4000kg$. The force acting on the base is described by the function $F(t) = 4000\sin(10t)$, where t measures the time in seconds. The base is supported by an elastic foundation with an equivalent spring constant of $4 \times 10^5 \frac{N}{m}$. Knowing that the base is initially depressed down by $0.1m$ in a resting position, write down and answer the following:

a) the differential equation describing the instantaneous position of the base (e.g. $x(t)$);

b) Does the load yield a resonant vibration? How can you determine whether resonance occurs?

c) the instantaneous position of the base (e.g. $x(t)$) that solves the differential equation you computed in a).

Short Answer
Solution

(26) One of the equations below describes a mass-spring system which undergoes resonance. Identify the equation and find its general solution.

a) $w'' + 9w = 14\cos(9t)$;
b) $4w'' + 16w = 7\cos(2t)$;
c) $w'' + 4w' + 4w = 200\sin(2t)$.

Short Answer
Solution

(27) **Some Parameter Analysis**

Consider the mass-spring system described by the following differential equation $2u'' + 3u' + \alpha u = 0$. For what values of α does the system describe an under, over or critically damped oscillation?

Short Answer
Solution

(28) For what values of β does the system described in the following equation undergo resonance?
a) $3z'' + \beta z = -\pi^2 \cos(x)$;
b) $8v'' + \beta v = 28 \sin(6x)$.

Short Answer
Solution

(29) Consider the mass-spring system described by the following differential equation $4v'' + \gamma v' + 36v = 0$. For what values of γ does the system describe an under, over or critically damped oscillation?

Short Answer
Solution

(30) A vibrating system satisfies the following differential equation:

$$z'' + \alpha z' + z = 0.$$

What is the value of α for which the quasiperiod of the damped motion is 50% higher than the period of the undamped corresponding motion?

Short Answer
Solution

(31) The motion of an undamped spring-mass system satisfies the following initial-value problem:

$$v'' + 2v = 0, \ v(0) = 0, \ v'(0) = 2.$$

What is the solution of this initial-value problem? How do $v(x)$ and $v'(x)$ compare to one another?

Short Answer
Solution

(32) The position of a spring-mass system is described by the initial-value problem:

$$3z'' + \beta z = 0, \ z(0) = 2, \ z'(0) = \gamma.$$

If the period of the resulting motion is π and the amplitude of the same motion is 3, determine the values of β and γ such that the conditions are ensured.

Short Answer
Solution

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