II. Higher-Order Linear Ordinary Differential Equations
9. Laplace Transform Method

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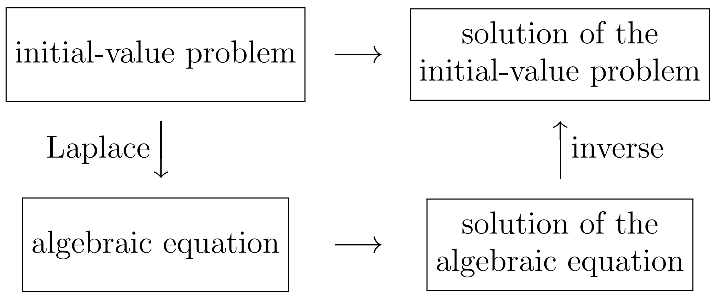
9. Laplace Transform Method

The Laplace transform allows us to transform an initial-value problem for a linear ordinary differential equation *with constant coefficients* into a linear algebraic equation that can be easily solved. The solution of the initial-value problem can be obtained from the solution of the algebraic equation by taking the so-called inverse Laplace transform. This so-called *Laplace transform method* is illustrated below.



We will begin by defining the Laplace transform and developing its properties. We will then show how to transform an initial-value problem into an algebraic equation and how to take an inverse Laplace transform. We will apply the method to find new ways to compute Green functions and natural fundamental sets of solutions. Finally, we will see that the method can efficiently treat a broader class of forcings than either Key Identity Evaluations or Undetermined Coefficients.

9.1. **Definition of the Transform.**

The Laplace transform of a function $f(t)$ defined over $t \geq 0$ is another function $\mathcal{L}[f](s)$ that is formally defined by

$$\mathcal{L}[f](s) = \int_0^\infty e^{-st} f(t) \, \mathrm{d}t \,. \tag{9.1}$$

You should recall from calculus that the above definite integral is *improper* because its upper endpoint is ∞. Because improper definite integrals are defined by limits, the correct definition of the Laplace transform is

$$\mathcal{L}[f](s) = \lim_{T\to\infty} \int_0^T e^{-st} f(t) \, \mathrm{d}t \,, \tag{9.2}$$

provided that the definite integrals over $[0, T]$ appearing in the above limit are proper. The Laplace transform $\mathcal{L}[f](s)$ is defined only at those s for which this limit exists.

Example. Use definition (9.2) to compute $\mathcal{L}[e^{at}](s)$ for any real a.

Solution. From (9.2) we see that for any $s \neq a$ we have

$$\begin{aligned}
\mathcal{L}[e^{at}](s) &= \lim_{T\to\infty} \int_0^T e^{-st} e^{at} \, \mathrm{d}t = \lim_{T\to\infty} \int_0^T e^{-(s-a)t} \, \mathrm{d}t \\
&= \lim_{T\to\infty} -\frac{e^{-(s-a)t}}{s-a}\bigg|_{t=0}^{T} = \lim_{T\to\infty} \left[\frac{1}{s-a} - \frac{e^{-(s-a)T}}{s-a}\right] = \begin{cases} \dfrac{1}{s-a} & \text{for } s > a \,, \\ \infty & \text{for } s < a \,, \end{cases}
\end{aligned}$$

while for $s = a$ we have

$$\mathcal{L}[e^{at}](s) = \lim_{T\to\infty} \int_0^T e^{-(s-a)t}\,\mathrm{d}t = \lim_{T\to\infty} \int_0^T \mathrm{d}t = \lim_{T\to\infty} T = \infty\,.$$

Therefore $\mathcal{L}[e^{at}](s)$ is only defined for $s > a$ with

$$\mathcal{L}[e^{at}](s) = \frac{1}{s-a} \qquad \text{for } s > a\,.$$

Example. Use definition (9.2) to compute $\mathcal{L}[t\,e^{at}](s)$ for any real a.

Solution. From (9.2) we see that for any $s \neq a$ we have

$$\begin{aligned}
\mathcal{L}[t\,e^{at}](s) &= \lim_{T\to\infty} \int_0^T t\,e^{-st}e^{at}\,\mathrm{d}t = \lim_{T\to\infty} \int_0^T t\,e^{-(s-a)t}\,\mathrm{d}t \\
&= \lim_{T\to\infty} \left(-\frac{t}{s-a} - \frac{1}{(s-a)^2} \right) e^{-(s-a)t} \bigg|_{t=0}^{T} \\
&= \lim_{T\to\infty} \left[\frac{1}{(s-a)^2} - \left(\frac{T}{s-a} + \frac{1}{(s-a)^2} \right) e^{-(s-a)T} \right] = \begin{cases} \dfrac{1}{(s-a)^2} & \text{for } s > a\,, \\ \infty & \text{for } s < a\,, \end{cases}
\end{aligned}$$

while for $s = a$ we have

$$\mathcal{L}[t\,e^{at}](s) = \lim_{T\to\infty} \int_0^T t\,e^{-(s-a)t}\,\mathrm{d}t = \lim_{T\to\infty} \int_0^T t\,\mathrm{d}t = \lim_{T\to\infty} \tfrac{1}{2}T^2 = \infty\,.$$

Therefore $\mathcal{L}[t\,e^{at}](s)$ is only defined for $s > a$ with

$$\mathcal{L}[t\,e^{at}](s) = \frac{1}{(s-a)^2} \qquad \text{for } s > a\,.$$

Example. Use definition (9.2) to compute $\mathcal{L}[e^{ibt}](s)$ for any real b.

Solution. The case $b \neq 0$ is similar to our first example with $a \neq 0$. We simply replace a in that calculation by ib. Indeed, for $b \neq 0$ we see from (9.2) that for any real s we have

$$\begin{aligned}
\mathcal{L}[e^{ibt}](s) &= \lim_{T\to\infty} \int_0^T e^{-st}e^{ibt}\,\mathrm{d}t = \lim_{T\to\infty} \int_0^T e^{-(s-ib)t}\,\mathrm{d}t = \lim_{T\to\infty} \left(-\frac{e^{-(s-ib)t}}{s-ib} \right) \bigg|_{t=0}^{T} \\
&= \lim_{T\to\infty} \left[\frac{1}{s-ib} - \frac{e^{-(s-ib)T}}{s-ib} \right] = \begin{cases} \dfrac{1}{s-ib} & \text{for } s > 0\,, \\ \text{undefined} & \text{for } s \leq 0\,. \end{cases}
\end{aligned}$$

The case $b = 0$ is identical to our first example with $a = 0$. In every case $\mathcal{L}[e^{ibt}](s)$ is only defined for $s > 0$ with

$$\mathcal{L}[e^{ibt}](s) = \frac{1}{s-ib} \qquad \text{for } s > 0\,.$$

9.2. **Properties of the Transform.** If we always had to return to the definition of the Laplace transform everytime we wanted to apply it, it would not be easy to use. Rather, we will use the definition to compute the Laplace transform for a few basic functions and to establish some general properties that will allow us to build formulas for more complicated functions.

9.2.1. *Linearity.* The most important property of the Laplace transform $\mathcal{L}$ is that it is a *linear operator.*

Theorem 9.1. If $\mathcal{L}[f](s)$ and $\mathcal{L}[g](s)$ exist for some s then so does $\mathcal{L}[f+g](s)$ and $\mathcal{L}[cf](s)$ for every constant c with

$$\mathcal{L}[f+g](s) = \mathcal{L}[f](s) + \mathcal{L}[g](s)\,, \qquad \mathcal{L}[cf](s) = c\mathcal{L}[f](s)\,. \tag{9.3}$$

Proof. This follows directly from definition (9.2) and the facts that definite integrals and limits depend linearly on their arguments. Specifically, we see that

$$\begin{aligned}
\mathcal{L}[f+g](s) &= \lim_{T\to\infty}\int_0^T e^{-st}\big(f(t)+g(t)\big)\,\mathrm{d}t \\
&= \lim_{T\to\infty}\int_0^T e^{-st}f(t)\,\mathrm{d}t + \lim_{T\to\infty}\int_0^T e^{-st}g(t)\,\mathrm{d}t = \mathcal{L}[f](s) + \mathcal{L}[g](s)\,, \\
\mathcal{L}[cf](s) &= \lim_{T\to\infty}\int_0^T e^{-st}cf(t)\,\mathrm{d}t = c\lim_{T\to\infty}\int_0^T e^{-st}f(t)\,\mathrm{d}t = c\mathcal{L}[f](s)\,.
\end{aligned}$$

□

Example. Compute $\mathcal{L}[\cos(bt)](s)$ and $\mathcal{L}[\sin(bt)](s)$ for any real $b \neq 0$.

Solution. This can be done by using the Euler Formula $e^{ibt} = \cos(bt) + i\sin(bt)$ and the linearity (9.3) of $\mathcal{L}$. The Euler Formula implies that

$$\cos(bt) = \frac{e^{ibt}+e^{-ibt}}{2}\,, \qquad \sin(bt) = \frac{e^{ibt}-e^{-ibt}}{i2}\,.$$

Then the linearity of $\mathcal{L}$ and the formula for $\mathcal{L}[e^{ibt}](s)$ derived in the last example of the previous section shows that for every $s > 0$ we have

$$\begin{aligned}
\mathcal{L}[\cos(bt)](s) &= \tfrac{1}{2}\mathcal{L}[e^{ibt}](s) + \tfrac{1}{2}\mathcal{L}[e^{-ibt}](s) \\
&= \tfrac{1}{2}\,\frac{1}{s-ib} + \tfrac{1}{2}\,\frac{1}{s+ib} = \tfrac{1}{2}\,\frac{s+ib}{s^2+b^2} + \tfrac{1}{2}\,\frac{s-ib}{s^2+b^2} = \frac{s}{s^2+b^2}\,, \\
\mathcal{L}[\sin(bt)](s) &= \tfrac{1}{i2}\mathcal{L}[e^{ibt}](s) - \tfrac{1}{i2}\mathcal{L}[e^{-ibt}](s) \\
&= \tfrac{1}{i2}\,\frac{1}{s-ib} - \tfrac{1}{i2}\,\frac{1}{s+ib} = \tfrac{1}{i2}\,\frac{s+ib}{s^2+b^2} - \tfrac{1}{i2}\,\frac{s-ib}{s^2+b^2} = \frac{b}{s^2+b^2}\,.
\end{aligned}$$

Therefore we have derived the formulas

$$\mathcal{L}[\cos(bt)](s) = \frac{s}{s^2+b^2}\,, \qquad \mathcal{L}[\sin(bt)](s) = \frac{b}{s^2+b^2}\,, \qquad \text{for } s > 0\,.$$

Remark. These formulas are the real and imaginary parts of the relation

$$\mathcal{L}[\cos(bt)](s) + i\mathcal{L}[\sin(bt)](s) = \mathcal{L}[e^{ibt}](s) = \frac{1}{s - ib} = \frac{s + ib}{s^2 + b^2} \qquad \text{for } s > 0\,.$$

Remark. The approach of deriving these formulas by using the Euler Formula and linearity is much more efficient than deriving them directly from the definitions

$$\mathcal{L}[\cos(bt)](s) = \lim_{T\to\infty} \int_0^T e^{-st}\cos(bt)\,\mathrm{d}t\,, \qquad \mathcal{L}[\sin(bt)](s) = \lim_{T\to\infty} \int_0^T e^{-st}\cos(bt)\,\mathrm{d}t\,.$$

You might recall from calculus that such integrals can be evaluated by a recurrence trick that involves two integrations by parts. By this route it takes four integrations by parts to derive the two formulas that we derived with no integration by parts. You will find the more laborious route taken by many source materials!

Example. Compute $\mathcal{L}[\cosh(bt)](s)$ and $\mathcal{L}[\sinh(bt)](s)$ for any real $b \neq 0$.

Solution. These hyperbolic functions are defined by

$$\cosh(bt) = \frac{e^{bt} + e^{-bt}}{2}\,, \qquad \sinh(bt) = \frac{e^{bt} - e^{-bt}}{2}\,.$$

Then the linearity of $\mathcal{L}$ and the formula for $\mathcal{L}[e^{at}](s)$ that we derived in the first example of the previous section applied with $a = \pm b$ show that for every $s > \max\{b, -b\} = |b|$ we have

$$\begin{aligned}
\mathcal{L}[\cosh(bt)](s) &= \tfrac{1}{2}\mathcal{L}[e^{bt}](s) + \tfrac{1}{2}\mathcal{L}[e^{-bt}](s) \\
&= \tfrac{1}{2}\frac{1}{s-b} + \tfrac{1}{2}\frac{1}{s+b} = \tfrac{1}{2}\frac{s+b}{s^2-b^2} + \tfrac{1}{2}\frac{s-b}{s^2-b^2} = \frac{s}{s^2-b^2}\,, \\
\mathcal{L}[\sinh(bt)](s) &= \tfrac{1}{2}\mathcal{L}[e^{bt}](s) - \tfrac{1}{2}\mathcal{L}[e^{-bt}](s) \\
&= \tfrac{1}{2}\frac{1}{s-b} - \tfrac{1}{2}\frac{1}{s+b} = \tfrac{1}{2}\frac{s+b}{s^2-b^2} - \tfrac{1}{2}\frac{s-b}{s^2-b^2} = \frac{b}{s^2-b^2}\,.
\end{aligned}$$

Therefore we have derived the formulas

$$\mathcal{L}[\cosh(bt)](s) = \frac{s}{s^2 - b^2}\,, \qquad \mathcal{L}[\sinh(bt)](s) = \frac{b}{s^2 - b^2}\,, \qquad \text{for } s > |b|\,.$$

9.2.2. *Exponentials and Translations.* Another property of the Laplace transform $\mathcal{L}$ is that it turns multiplication by an *exponential* in t into a *translation* of s.

Theorem 9.2. If $\mathcal{L}[f](s)$ exists for every $s > \alpha$ and if a is any real number then $\mathcal{L}[e^{at}f(t)](s)$ exists for every $s > \alpha + a$ with

$$\mathcal{L}[e^{at}f(t)](s) = \mathcal{L}[f](s - a)\,. \tag{9.4}$$

Proof. This follows directly from definition (9.2). Specifically, we see that

$$\mathcal{L}[e^{at}f](s) = \lim_{T\to\infty} \int_0^T e^{-st}e^{at}f(t)\,\mathrm{d}t = \lim_{T\to\infty} \int_0^T e^{-(s-a)t}f(t)\,\mathrm{d}t = \mathcal{L}[f](s-a)\,.$$

□

Examples. From our previous examples and the above theorem we see that

$$
\begin{aligned}
\mathcal{L}[e^{(a+ib)t}](s) &= \frac{1}{s-a-ib} && \text{for } s > a\,,\\
\mathcal{L}[e^{at}\cos(bt)](s) &= \frac{s-a}{(s-a)^2+b^2} && \text{for } s > a\,,\\
\mathcal{L}[e^{at}\sin(bt)](s) &= \frac{b}{(s-a)^2+b^2} && \text{for } s > a\,.\\
\mathcal{L}[e^{at}\cosh(bt)](s) &= \frac{s-a}{(s-a)^2-b^2} && \text{for } s > a+|b|\,,\\
\mathcal{L}[e^{at}\sinh(bt)](s) &= \frac{b}{(s-a)^2-b^2} && \text{for } s > a+|b|\,.
\end{aligned}
\tag{9.5}
$$

Similarly, the Laplace transform turns a *translation* of t into multiplication by an *exponential* in s. Notice that $\mathcal{L}[f](s)$ only depends on the values of $f(t)$ over $[0.\infty)$. Therefore before translating f we multiply it by the *unit step* or *Heaviside* function $u(t)$ defined by

$$
u(t) = \begin{cases} 1 & \text{for } t \geq 0\,,\\ 0 & \text{for } t < 0\,. \end{cases}
\tag{9.6}
$$

It should be clear from this definition that the function uf is given by

$$
u(t)f(t) = \begin{cases} f(t) & \text{for } t \geq 0\,,\\ 0 & \text{for } t < 0\,. \end{cases}
$$

Because the functions uf and f agree over $[0,\infty)$, it is clear that $\mathcal{L}[uf](s) = \mathcal{L}[f](s)$.

We now consider the Laplace transform of $u(t-c)f(t-c)$ for every $c > 0$. Because $c > 0$ this translation will shift the graph of uf to the right. We have

$$
u(t-c)f(t-c) = \begin{cases} f(t-c) & \text{for } t \geq c\,,\\ 0 & \text{for } t < c\,. \end{cases}
$$

It has a Laplace transform that is related to the Laplace transform of f as follows.

Theorem 9.3. If $\mathcal{L}[f](s)$ exists for every $s > \alpha$ and if c is any positive number then $\mathcal{L}[u(t-c)f(t-c)](s)$ exists for every $s > \alpha$ with

$$
\mathcal{L}[u(t-c)f(t-c)](s) = e^{-cs}\mathcal{L}[f](s) \qquad \text{for } s > \alpha\,.
\tag{9.7}
$$

Proof. For every $T > c$ we have

$$
\begin{aligned}
\int_0^T e^{-st}u(t-c)f(t-c)\,\mathrm{d}t &= \int_c^T e^{-st}f(t-c)\,\mathrm{d}t = e^{-cs}\int_c^T e^{-s(t-c)}f(t-c)\,\mathrm{d}t\\
&= e^{-cs}\int_0^{T-c} e^{-st'}f(t')\,\mathrm{d}t'\,,
\end{aligned}
$$

where the substitution $t' = t - c$ was used in the last step. We thereby see that

$$\begin{aligned}\mathcal{L}[u(t-c)f(t-c)](s) &= \lim_{T\to\infty} \int_0^T e^{-st} u(t-c) f(t-c)\,\mathrm{d}t \\ &= e^{-cs} \lim_{T\to\infty} \int_0^{T-c} e^{-st} f(t)\,\mathrm{d}t = e^{-cs}\mathcal{L}[f](s) \qquad \text{for } s > \alpha\,.\end{aligned}$$

□

Remark. Formula (9.7) plays a leading role later in this chapter.

9.3. **Existence and Differentiablity of the Transform.** In each of the above examples the definite integrals over $[0, T]$ that appear in the limit (9.2) were proper. Indeed, we were able to evaluate the definite integrals analytically and determine the limit (9.2) for every sufficiently large real number s. In this section we identify two properties that when possessed by a function $f(t)$ insure that its Laplace transform exists for every sufficiently large real number s. The first property insures that the definite integrals over $[0, T]$ that appear in the limit (9.2) are all proper. The second property insures that the limit (9.2) of these proper definite integrals exists for every s larger than a certain value. We then use these properties to argue that the Laplace transform $F(s)$ of such an $f(t)$ has derivatives in s of all orders. Moreover, we show that the k^{th} derivative of $F(s)$ is related to the Laplace transform of $t^k f(t)$.

9.3.1. *Piecewise Continuity.* We know from calculus that a definite integral over $[0, T]$ is proper whenever its integrand is:

- bounded over $[0, T]$,
- continuous at all but a finite number of points in $[0, T]$.

Such an integrand is said to be *piecewise continuous* over $[0, T]$. Because e^{-st} is a continuous (and therefore bounded) function of t over every $[0, T]$ for each real s, the definite integrals over $[0, T]$ that appear in the limit (9.2) will be proper whenever $f(t)$ is *piecewise continuous* over every $[0, T]$.

Example. Consider the function

$$f(t) = u(t-\pi)\cos(t) = \begin{cases} 0 & \text{for } 0 \le t < \pi\,, \\ \cos(t) & \text{for } t \ge \pi\,. \end{cases}$$

It is clearly bounded over $[0, \infty)$ and its only discontinuity is at the point $t = \pi$. Therefore it is piecewise continuous over every $[0, T]$.

Example. Consider the so-called *sawtooth* function

$$f(t) = t - k \qquad \text{for } k \le t < k+1 \text{ where } k = 0, 1, 2, 3, \cdots\,.$$

It is clearly bounded over $[0, \infty)$ and has discontinuities at the points $t = 1, 2, 3, \cdots$, only a finite number of which lie in each $[0, T]$. Therefore it is piecewise continuous over every $[0, T]$.

9.3.2. *Exponential Order.* Even if $f(t)$ is piecewise continuous over every $[0,T]$, we still have to give a condition under which the limit (9.2) will exist for certain s. Such a condition is provided by the following definition.

Definition. A function $f(t)$ defined over $[0,\infty)$ that is piecewise continuous over every $[0,T]$ is said to be of *exponential order* α *as* $t\to\infty$ provided for every $\sigma>\alpha$ we have

$$\lim_{t\to\infty} e^{-\sigma t}|f(t)| = 0\,. \tag{9.8}$$

This definition need not be memorized. Rather, we are going to use it to build an understanding of what it means through examples. Roughly speaking, it says that a function is of exponential order α as $t\to\infty$ if its absolute value grows slower than $e^{\sigma t}$ as $t\to\infty$ for every $\sigma>\alpha$.

Example. The function e^{at} is of exponential order a as $t\to\infty$ because for every $\sigma>a$ we have

$$\lim_{t\to\infty} e^{-\sigma t}e^{at} = \lim_{t\to\infty} e^{-(\sigma-a)t} = 0\,.$$

Therefore (9.8) holds for every $\sigma>a$.

Example. The functions $\cos(bt)$ and $\sin(bt)$ are of exponential order 0 as $t\to\infty$ because for every $\sigma>0$ we have

$$\begin{aligned} 0 &\le \lim_{t\to\infty} e^{-\sigma t}|\cos(bt)| \le \lim_{t\to\infty} e^{-\sigma t} = 0\,,\\ 0 &\le \lim_{t\to\infty} e^{-\sigma t}|\sin(bt)| \le \lim_{t\to\infty} e^{-\sigma t} = 0\,. \end{aligned}$$

Therefore (9.8) holds for every $\sigma>0$.

Example. For every $c>0$ the function $u(t-c)$ is of exponential order 0 as $t\to\infty$ because for every $\sigma>0$ we have

$$\lim_{t\to\infty} e^{-\sigma t}u(t-c) = 0\,.$$

Therefore (9.8) holds for every $\sigma>0$.

Example. For every $p>0$ the function t^p is of exponential order 0 as $t\to\infty$. This can be seen by first using methods from calculus to show that for every $\eta>0$ we have

$$e^{-\eta t}t^p \le \left(\frac{p}{e\eta}\right)^{\frac{1}{p}} \quad \text{for every } t\ge 0\,.$$

Then for every $\sigma>0$ we set $\eta=\frac{1}{2}\sigma$ and we have

$$\lim_{t\to\infty} e^{-\sigma t}t^p = \lim_{t\to\infty} e^{-2\eta t}t^p \le \left(\frac{p}{e\eta}\right)^{\frac{1}{p}} \lim_{t\to\infty} e^{-\eta t} = 0\,.$$

Therefore (9.8) holds for every $\sigma>0$.

Remark. This shows that for every $p>0$ the power function t^p is of exponential order 0 as $t\to\infty$ even though $t^p\to\infty$ as $t\to\infty$. This reflects something that you might recall from calculus — namely, the fact that growing exponential functions grow faster than power functions as $t\to\infty$.

Now that we have understood the exponential order as $t \to \infty$ of the functions e^{at}, $\cos(bt)$, $\sin(bt)$, $u(t-c)$, and t^p, let us consider the exponential order of combinations of these functions.

Fact. If functions f and g are of exponential orders α and β respectively as $t \to \infty$ then

- the function $f + g$ is of exponential order $\max\{\alpha, \beta\}$ as $t \to \infty$,
- the function fg is of exponential order $\alpha + \beta$ as $t \to \infty$.

We will not prove these properties. They can be recalled by thinking of the case when f and g are both exponential functions, say $f(t) = e^{\alpha t}$ and $g(t) = e^{\beta t}$. They are easily applied.

Example. For every real b the function $e^{bt} + e^{-bt}$ is of exponential order $|b|$ as $t \to \infty$. This is because the functions e^{bt} and e^{-bt} are exponential orders b and $-b$ respectively as $t \to \infty$, and because $|b| = \max\{b, -b\}$.

Example. For every positive c and p and real a and b the function $u(t-c)t^p e^{at}\cos(bt)$ is of exponential order a as $t \to \infty$. This is because the functions $u(t-c)$, t^p, e^{at}, and $\cos(bt)$ are of exponential orders 0, 0, a, and 0 respectively as $t \to \infty$.

9.3.3. *Existence and Differentiablity.* The fact you should know about the existence of the Laplace transform for certain s is the following.

Theorem 9.4. Let $f(t)$ be defined over $[0, \infty)$ such that:

- it is piecewise continuous over every $[0, T]$;
- it is of exponential order α as $t \to \infty$.

Then

1. for every positive integer k the function $t^k f(t)$ has these same properties;
2. the function $F(s) = \mathcal{L}[f](s)$ is defined for every $s > \alpha$ and has derivatives of all orders with its k^{th} derivative satisfying

$$\mathcal{L}[t^k f(t)](s) = (-1)^k F^{(k)}(s) \qquad \text{for } s > \alpha\,. \tag{9.9}$$

Proof. Formula (9.9) can be derived by formally differentiating the integrands:

$$\begin{aligned}
F(s) &= \int_0^\infty e^{-st} f(t)\,\mathrm{d}t\,,\\
F'(s) &= -\int_0^\infty t\,e^{-st} f(t)\,\mathrm{d}t\,,\\
F''(s) &= \int_0^\infty t^2 e^{-st} f(t)\,\mathrm{d}t\,,\\
&\vdots\\
F^{(k)}(s) &= (-1)^k \int_0^\infty t^k e^{-st} f(t)\,\mathrm{d}t\,.
\end{aligned}$$

□

Remark. This theorem shows that if $f(t)$ is of exponential order α as $t \to \infty$ then $\mathcal{L}[f](s)$ will be defined at least for every $s > \alpha$. In general it might be defined over a larger set, but that is not the case for the functions we have studied so far.

Remark. A correct proof would require a justification of taking the derivatives inside the above imporper integrals. Such a proof is beyond the scope of this course.

We now use formula (9.9) to extend our list of Laplace transform formulas.

Example. Because for every real a and b we have

$$\mathcal{L}[e^{(a+ib)t}](s) = \frac{1}{s-a-ib} \qquad \text{for } s > a\,,$$

it follows from the above theorem that for every nonnegative integer k

$$\mathcal{L}[t^k e^{(a+ib)t}](s) = (-1)^k \frac{\mathrm{d}^k}{\mathrm{d}s^k} \frac{1}{s-a-ib} = \frac{k!}{(s-a-ib)^{k+1}} \qquad \text{for } s > a\,.$$

This formula implies that for every real a and b and every nonnegative integer k

$$\begin{aligned}
\mathcal{L}[t^k](s) &= \frac{k!}{s^{k+1}} && \text{for } s > 0\,,\\
\mathcal{L}[t^k e^{at}](s) &= \frac{k!}{(s-a)^{k+1}} && \text{for } s > a\,.\\
\mathcal{L}[t^k e^{at}\cos(bt)](s) &= \mathrm{Re}\left(\frac{k!}{(s-a-ib)^{k+1}}\right) && \text{for } s > a\,,\\
\mathcal{L}[t^k e^{at}\sin(bt)](s) &= \mathrm{Im}\left(\frac{k!}{(s-a-ib)^{k+1}}\right) && \text{for } s > a\,.
\end{aligned} \tag{9.10}$$

9.4. **Transform of Derivatives.** Theorem 9.4 shows that the Laplace transform turns a multiplication by t into a derivative with respect to s. The next result shows that the Laplace transform turns a derivative with respect to t into a multiplication by s.

Theorem 9.5. Let $f(t)$ be continuous over $[0,\infty)$ such that

- $f(t)$ is of exponential order α as $t \to \infty$,
- $f'(t)$ is piecewise continuous over every $[0,T]$.

Then $\mathcal{L}[f'](s)$ is defined for every $s > \alpha$ with

$$\mathcal{L}[f'](s) = s\,\mathcal{L}[f](s) - f(0)\,. \tag{9.11}$$

Proof. Let $s > \alpha$. By definition (9.2), an integration by parts, the fact that $f(t)$ is of exponential order α as $t \to \infty$, and the fact that $\mathcal{L}[f](s)$ exists, we see that

$$\begin{aligned}
\mathcal{L}[f'](s) &= \lim_{T\to\infty} \int_0^T e^{-st} f'(t)\,\mathrm{d}t = \lim_{T\to\infty} \left[e^{-st} f(t)\Big|_{t=0}^T + s \int_0^T e^{-st} f(t)\,\mathrm{d}t \right]\\
&= \lim_{T\to\infty} e^{-sT} f(T) - f(0) + s \lim_{T\to\infty} \int_0^T e^{-st} f(t)\,\mathrm{d}t = -f(0) + s\,\mathcal{L}[f](s)\,.
\end{aligned}$$

□

If $f(t)$ is sufficiently differentiable then we can apply formula (9.11) repeatedly. For example, if $f(t)$ is twice differentiable then

$$\begin{aligned}\mathcal{L}[f''](s) = s\,\mathcal{L}[f'](s) - f'(0) &= s\big(s\,\mathcal{L}[f](s) - f(0)\big) - f'(0)\\ &= s^2\mathcal{L}[f](s) - s\,f(0) - f'(0)\,.\end{aligned}$$

If $f(t)$ is thrice differentiable then

$$\begin{aligned}\mathcal{L}[f'''](s) = s\,\mathcal{L}[f''](s) - f''(0) &= s\big(s^2\mathcal{L}[f](s) - s\,f(0) - f'(0)\big) - f''(0)\\ &= s^3\mathcal{L}[f](s) - s^2 f(0) - s\,f'(0) - f''(0)\,.\end{aligned}$$

Proceeding in this way we can use induction to prove the following.

Theorem 9.6. Let $f(t)$ be $(n-1)$-times continuously differentiable over $[0,\infty)$ such that

- $f(t)$, $f'(t)$, $\cdots$, $f^{(n-1)}(t)$ are of exponential order α as $t\to\infty$,
- $f^{(n)}(t)$ is piecewise continuous over every interval $[0,T]$.

Then $\mathcal{L}[f^{(n)}](s)$ is defined for every $s>\alpha$ with

$$\mathcal{L}[f^{(n)}](s) = s^n\mathcal{L}[f](s) - s^{n-1}f(0) - s^{n-2}f'(0) - \cdots - s\,f^{(n-2)}(0) - f^{(n-1)}(0)\,. \tag{9.12}$$

Rather than memorize (9.12), it is simpler to remember (9.11) and just apply it as often as needed. For example, suppose we know that

- $y(t)$ is $(n-1)$-times continuously differentiable over $[0,\infty)$,
- $y(t)$ and its first $n-1$ derivatives are of exponential order as $t\to\infty$,
- $y^{(n)}(t)$ is piecewise continuous over every $[0,T]$.

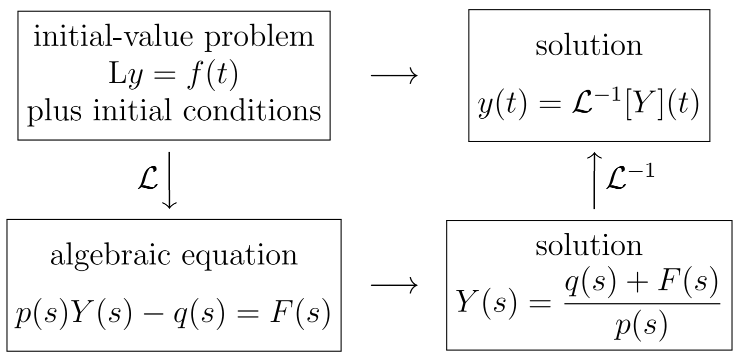
If $Y(s)=\mathcal{L}[y](s)$ then an n-fold application of (9.11) gives the table

$$\begin{aligned}\mathcal{L}[y](s) &= Y(s)\,,\\ \mathcal{L}[y'](s) &= s\,\mathcal{L}[y](s) - y(0) = s\,Y(s) - y(0)\,,\\ \mathcal{L}[y''](s) &= s\,\mathcal{L}[y'](s) - y'(0) = s^2Y(s) - s\,y(0) - y'(0)\,,\\ \mathcal{L}[y'''](s) &= s\,\mathcal{L}[y''](s) - y''(0) = s^3Y(s) - s^2y(0) - s\,y'(0) - y''(0)\,,\\ &\;\vdots\\ \mathcal{L}[y^{(n)}](s) &= s\,\mathcal{L}[y^{(n-1)}](s) - y^{(n-1)}(0)\\ &= s^nY(s) - s^{n-1}y(0) - s^{n-2}y'(0) - \cdots - s\,y^{(n-2)}(0) - y^{(n-1)}(0)\,.\end{aligned} \tag{9.13}$$

Notice that the right-hand side of each line after the first line is just s times the right-hand side of the line above it minus the appropriate initial value.

9.5. **Application to Initial-Value Problems.** Because the Laplace transform turns derivatives with respect to t into multiplications by s, it transforms an initial-value problem for $y(t)$ into a linear algebraic equation for $Y(s) = \mathcal{L}[y](s)$. This linear algebraic equation is easily solved to find $Y(s)$. We will see that $y(t)$ can be determined from $Y(s)$ by a process called the *inverse Laplace transform*, which is denoted $y(t) = \mathcal{L}^{-1}[Y](t)$.

This so-called *Laplace transform method* for solving an initial-value problem can be visualized with the aid of the following diagram.



In this section we will see that $p(s)$, $q(s)$, and $F(s)$ are known:

- $p(s)$ is the characteristic polynomial of L;
- $q(s)$ is a polynomial determined by the initial data;
- $F(s)$ is the Laplace transform of the forcing $f(t)$.

They determine $Y(s)$. In Section 9.6 we will see how to determine $y(t)$ from $Y(s)$.

Theorem. If $f(t)$ is piecewise continuous over every $[0,T]$ and is of exponential order as $t \to \infty$ then the initial-value problem

$$\begin{gathered} y^{(n)} + a_1 y^{(n-1)} + \cdots + a_{n-1}y' + a_n y = f(t)\,, \\ y(0) = y_0\,, \qquad y'(0) = y_1\,, \qquad \cdots \qquad y^{(n-1)}(0) = y_{n-1}\,. \end{gathered} \tag{9.14}$$

has a unique solution $y(t)$ such that

- $y(t)$ has $n-1$ continuous derivatives over $[0,\infty)$,
- $y^{(n-1)}(t)$ is of exponential order as $t \to \infty$,
- $y^{(n)}(t)$ is piecewise continuous over every $[0,T]$.

We thereby can use the Laplace transform to find $Y(s) = \mathcal{L}[y](s)$ in terms of the inital data y_0, y_1, $\cdots$, y_{n-1}, and the Laplace transform of the forcing, $F(s) = \mathcal{L}[f](s)$.

Remark. When $f(t)$ is continuous over $[0,\infty)$ then $y^{(n)}(t)$ will also be continuous over $[0,\infty)$ and $y(t)$ will solve the initial-value problem (9.14) in the classical sense that we defined in Chapter 1. However, when $f(t)$ is not continuous over $[0,\infty)$, but is piecewise continuous over every $[0,T]$, then $y(t)$ will solve the initial-value problem (9.14) in a weaker sense than we defined in Chapter 1. We will not describe this weaker sense here.

The computation of $Y(s) = \mathcal{L}[y](s)$ from the initial-value problem has five steps.

1. By the linearity of $\mathcal{L}$ the Laplace transform of the differential equation is
$$\mathcal{L}[y^{(n)}] + a_1\mathcal{L}[y^{(n-1)}] + \cdots + a_{n-1}\mathcal{L}[y'] + a_n\mathcal{L}[y] = \mathcal{L}[f]\,.$$
2. An n-fold application of (9.11) using the initial conditions gives the table
$$\begin{aligned}\mathcal{L}[y](s) &= Y(s)\,,\\ \mathcal{L}[y'](s) &= s\,\mathcal{L}[y](s) - y(0) = s\,Y(s) - y_0\,,\\ \mathcal{L}[y''](s) &= s\,\mathcal{L}[y'](s) - y'(0) = s^2Y(s) - s\,y_0 - y_1\,,\\ &\;\vdots\\ \mathcal{L}[y^{(n)}](s) &= s\,\mathcal{L}[y^{(n-1)}](s) - y^{(n-1)}(0)\\ &= s^nY(s) - s^{n-1}y_0 - s^{n-2}y_1 - \cdots - s\,y_{n-2} - y_{n-1}\,.\end{aligned}$$
3. Compute $F(s) = \mathcal{L}[f](s)$. We will come back to this step later.
4. Place the results of the second and third steps into the relation obtained in the first step. We thereby see that $Y(s)$ satisfies the linear algebraic equation
$$p(s)Y(s) - q(s) = F(s)\,,$$
where $p(s)$ is the characteristic polynomial
$$p(s) = s^n + a_1s^{n-1} + \cdots + a_{n-1}s + a_n\,.$$
and $q(s)$ is the polynomial given in terms of the initial data by
$$\begin{aligned}q(s) &= \big(s^{n-1} + a_1s^{n-2} + \cdots + a_{n-2}s + a_{n-1}\big)y_0\\ &\quad + \big(s^{n-2} + a_1s^{n-3} + \cdots + a_{n-3}s + a_{n-2}\big)y_1\\ &\quad + \cdots + \big(s^2 + a_1s + a_2\big)y_{n-3} + \big(s + a_1\big)y_{n-2} + y_{n-1}\,.\end{aligned}$$
5. Solve the linear algebraic equation for $Y(s)$ to obtain

$$Y(s) = \frac{q(s) + F(s)}{p(s)}\,. \tag{9.15}$$

Example. Find $X(s) = \mathcal{L}[x](s)$ where $x(t)$ solves the initial-value problem
$$x'' - 2x' - 8x = 0\,, \qquad x(0) = 3\,, \quad x'(0) = 7\,.$$

Solution. The Laplace transform of the differential equation is
$$\mathcal{L}[x''](s) - 2\mathcal{L}[x'](s) - 8\mathcal{L}[x](s) = 0\,.$$
Two applications of (9.11) using the initial conditions gives the table
$$\begin{aligned}\mathcal{L}[x](s) &= X(s)\,,\\ \mathcal{L}[x'](s) &= s\,\mathcal{L}[x](s) - x(0) = s\,X(s) - 3\,,\\ \mathcal{L}[x''](s) &= s\,\mathcal{L}[x'](s) - x'(0) = s^2X(s) - 3s - 7\,.\end{aligned}$$

Because $f(t) = 0$, we have $F(s) = 0$. The Laplace transform of the initial-value problem thereby becomes

$$[s^2 X(s) - 3s - 7] - 2[s\,X(s) - 3] - 8X(s) = (s^2 - 2s - 8)X(s) - 3s - 1 = 0\,,$$

whereby

$$X(s) = \frac{3s + 1}{s^2 - 2s - 8}\,.$$

The hardest of these steps can be the third — namely, computing $F(s) = \mathcal{L}[f](s)$. Often $f(t)$ is a combination of the basic forms whose Laplace transform we have already computed. These basic forms include

$$\begin{aligned}
\mathcal{L}[t^n](s) &= \frac{n!}{s^{n+1}} && \text{for } s > 0\,,\\
\mathcal{L}[\cos(bt)](s) &= \frac{s}{s^2 + b^2} && \text{for } s > 0\,,\\
\mathcal{L}[\sin(bt)](s) &= \frac{b}{s^2 + b^2} && \text{for } s > 0\,,\\
\mathcal{L}[\cosh(bt)](s) &= \frac{s}{s^2 - b^2} && \text{for } s > |b|\,,\\
\mathcal{L}[\sinh(bt)](s) &= \frac{b}{s^2 - b^2} && \text{for } s > |b|\,,\\
\mathcal{L}[e^{at}t^n](s) &= \frac{n!}{(s-a)^{n+1}} && \text{for } s > a\,,\\
\mathcal{L}[e^{at}\cos(bt)](s) &= \frac{s-a}{(s-a)^2 + b^2} && \text{for } s > a\,,\\
\mathcal{L}[e^{at}\sin(bt)](s) &= \frac{b}{(s-a)^2 + b^2} && \text{for } s > a\,,\\
\mathcal{L}[e^{at}\cosh(bt)](s) &= \frac{s-a}{(s-a)^2 - b^2} && \text{for } s > a + |b|\,,\\
\mathcal{L}[e^{at}\sinh(bt)](s) &= \frac{b}{(s-a)^2 - b^2} && \text{for } s > a + |b|\,,\\
\mathcal{L}[t^n j(t)](s) &= (-1)^n J^{(n)}(s) && \text{where } J(s) = \mathcal{L}[j(t)](s)\,,\\
\mathcal{L}[e^{at} j(t)](s) &= J(s-a) && \text{where } J(s) = \mathcal{L}[j(t)](s)\,,\\
\mathcal{L}[u(t-c)j(t-c)](s) &= e^{-cs}J(s) && \text{where } J(s) = \mathcal{L}[j(t)](s)\,.
\end{aligned} \tag{9.16}$$

These can be used to build a longer table like those found in many textbooks. However, this table is all we need. In fact, its first five entries are the second five for $a = 0$. Alternatively, its second five entries follow from the first five and the twelfth. You will be given a similar table on exams, so you do not have to memorize this one. However, you should learn how to use it efficiently.

Example. Find $Y(s) = \mathcal{L}[y](s)$ where $y(t)$ solves the initial-value problem

$$y' - 2y = e^{5t}\,, \qquad y(0) = 3\,.$$

Solution. The Laplace transform of the differential equation is

$$\mathcal{L}[y'](s) - 2\mathcal{L}[y](s) = \mathcal{L}[e^{5t}](s)\,.$$

One application of (9.11) using the initial condition gives the table

$$\begin{aligned}\mathcal{L}[y](s) &= Y(s)\,,\\ \mathcal{L}[y'](s) &= s\,\mathcal{L}[y] - y(0) = s\,Y(s) - 3\,.\end{aligned}$$

By setting $a = 5$ and $n = 0$ in the sixth entry of table (9.16) we see that

$$\mathcal{L}[e^{5t}](s) = \frac{1}{s-5}\,.$$

The Laplace transform of the initial-value problem thereby becomes

$$[s\,Y(s) - 3] - 2Y(s) = (s-2)Y(s) - 3 = \frac{1}{s-5}\,,$$

whereby

$$Y(s) = \frac{1}{(s-2)(s-5)} + \frac{3}{s-2}\,.$$

Example. Find $H(s) = \mathcal{L}[h](s)$ where $h(t)$ solve the initial-value problem

$$h'' + 4h = \sin(3t)\,, \qquad h(0) = h'(0) = 0\,.$$

Solution. The Laplace transform of the differential equation is

$$\mathcal{L}[h''](s) + 4\mathcal{L}[h](s) = \mathcal{L}[\sin(3t)](s)\,.$$

Two applications of (9.11) using the initial conditions gives the table

$$\begin{aligned}\mathcal{L}[h](s) &= H(s)\,,\\ \mathcal{L}[h'](s) &= s\,\mathcal{L}[h](s) - h(0) = s\,H(s)\,,\\ \mathcal{L}[h''](s) &= s\,\mathcal{L}[h'](s) - h'(0) = s^2 H(s)\,.\end{aligned}$$

By setting $b = 3$ in the third entry of table (9.16) we see that

$$\mathcal{L}[\sin(3t)](s) = \frac{3}{s^2+9}\,.$$

The Laplace transform of the initial-value problem thereby becomes

$$s^2 H(s) + 4H(s) = (s^2+4)H(s) = \frac{3}{s^2+9}\,,$$

whereby

$$H(s) = \frac{3}{(s^2+4)(s^2+9)}\,.$$

9.6. **Inverse Transform.** The process of determining $y(t)$ from $Y(s)$ is called taking the inverse Laplace transform. It is important to know that this process has a unique result. Indeed, we will use the following theorem.

Theorem 9.7. Let $f(t)$ and $g(t)$ be functions over $[0, \infty)$ such that for some α

- $f(t)$ and $g(t)$ are piecewise continuous over every $[0, T]$,
- $f(t)$ and $g(t)$ are of exponential order α as $t \to \infty$,
- $\mathcal{L}[f](s) = \mathcal{L}[g](s)$ for every $s > \alpha$.

Then $f(t) = g(t)$ at every t in $[0, \infty)$ where both $f(t)$ and $g(t)$ are continuous.

The proof of this result requires tools from complex variables that are beyond the scope of this course. Fortunately, you do not need to know how to prove this result to use it! Its usefulness stems from the fact that solutions $y(t)$ to the initial-value problems we are considering lie within the class of functions considered above — namely, they are functions that are piecewise continuous over every $[0, T]$ and that are of exponential order as $t \to \infty$. In fact, they are continuous and piecewise differentiable over every $[0, T]$. This means that if we find a function $y(t)$ within this class such that $\mathcal{L}[y](s) = Y(s)$ then it will be the unique solution of the initial-value problem that we seek.

Because the above result states there is a unique $f(t)$ that is piecewise continuous over every $[0, T]$ and is of exponential order as $t \to \infty$ such that $\mathcal{L}[f](s) = F(s)$, we introduce the notation

$$f(t) = \mathcal{L}^{-1}[F](t)\,.$$

The operator $\mathcal{L}^{-1}$ denotes the *inverse Laplace transform*. Because it undoes the Laplace transform $\mathcal{L}$, it inherits many properties from $\mathcal{L}$. For example, it is linear. We can also easily read-off from the first ten entries in table (9.16) of basic forms that

$$\begin{aligned}
&\mathcal{L}^{-1}\left[\frac{n!}{s^{n+1}}\right](t) = t^n\,, &&\mathcal{L}^{-1}\left[\frac{n!}{(s-a)^{n+1}}\right](t) = e^{at}t^n\,,\\
&\mathcal{L}^{-1}\left[\frac{s}{s^2+b^2}\right](t) = \cos(bt)\,, &&\mathcal{L}^{-1}\left[\frac{s-a}{(s-a)^2+b^2}\right](t) = e^{at}\cos(bt)\,,\\
&\mathcal{L}^{-1}\left[\frac{b}{s^2+b^2}\right](t) = \sin(bt)\,, &&\mathcal{L}^{-1}\left[\frac{b}{(s-a)^2+b^2}\right](t) = e^{at}\sin(bt)\,.\\
&\mathcal{L}^{-1}\left[\frac{s}{s^2-b^2}\right](t) = \cosh(bt)\,, &&\mathcal{L}^{-1}\left[\frac{s-a}{(s-a)^2-b^2}\right](t) = e^{at}\cosh(bt)\,,\\
&\mathcal{L}^{-1}\left[\frac{b}{s^2-b^2}\right](t) = \sinh(bt)\,, &&\mathcal{L}^{-1}\left[\frac{b}{(s-a)^2-b^2}\right](t) = e^{at}\sinh(bt)\,.
\end{aligned} \tag{9.17}$$

It is also clear from the last entry of table (9.16) that

$$\mathcal{L}^{-1}[e^{-cs}J(s)](t) = u(t-c)\,j(t-c)\,, \quad \text{where } j(t) = \mathcal{L}^{-1}[J](t)\,. \tag{9.18}$$

For us, the process of computing $y(t) = \mathcal{L}^{-1}[Y](t)$ for a given $Y(s)$ will be one of expressing $Y(s)$ as a sum of terms that will allow us to read off $y(t)$ from the basic forms above. A longer Laplace Transform Table is at the end of this chapter.

To illustrate this process, we will compute the inverse Laplace transform for the Laplace transforms of the solutions to the initial-value problems that were found for the examples given in the previous two sections, thereby completing our solution of those problems.

Example. Find $x(t) = \mathcal{L}^{-1}[X](t)$ for

$$X(s) = \frac{3s+1}{s^2 - 2s - 8}\,.$$

Solution. By the partial fraction identity

$$\frac{3s+1}{s^2-2s-8} = \frac{3s+1}{(s-4)(s+2)} = \frac{\frac{13}{6}}{s-4} + \frac{\frac{5}{6}}{s+2}\,,$$

we can express $X(s)$ as

$$X(s) = \tfrac{13}{6}\frac{1}{s-4} + \tfrac{5}{6}\frac{1}{s+2}\,.$$

The top right entry of table (9.17) with $a = 4$ and $a = -2$ then yields

$$\begin{aligned} x(t) = \mathcal{L}^{-1}[X(s)](t) &= \tfrac{13}{6}\mathcal{L}^{-1}\left[\frac{1}{s-4}\right](t) + \tfrac{5}{6}\mathcal{L}^{-1}\left[\frac{1}{s+2}\right](t) \\ &= \tfrac{13}{6}e^{4t} + \tfrac{5}{6}e^{-2t}\,. \end{aligned}$$

Example. Find $y(t) = \mathcal{L}^{-1}[Y](t)$ for

$$Y(s) = \frac{1}{(s-2)(s-5)} + \frac{3}{s-2}\,.$$

Solution. By the partial fraction identity

$$\frac{1}{(s-2)(s-5)} = \frac{\frac{1}{3}}{s-5} + \frac{-\frac{1}{3}}{s-2}\,,$$

we can express $Y(s)$ as

$$Y(s) = \tfrac{1}{3}\frac{1}{s-5} + \tfrac{8}{3}\frac{1}{s-2}\,.$$

The top right entry of table (9.17) with $a = 5$ and $a = 2$ then yields

$$\begin{aligned} y(t) = \mathcal{L}^{-1}[Y(s)](t) &= \tfrac{1}{3}\mathcal{L}^{-1}\left[\frac{1}{s-5}\right](t) + \tfrac{8}{3}\mathcal{L}^{-1}\left[\frac{1}{s-2}\right](t) \\ &= \tfrac{1}{3}e^{5t} + \tfrac{8}{3}e^{2t}\,. \end{aligned}$$

Example. Find $h(t) = \mathcal{L}^{-1}[H](t)$ for

$$H(s) = \frac{3}{(s^2+4)(s^2+9)}\,.$$

Solution. By the partial fraction identity

$$\frac{3}{(z+4)(z+9)} = \frac{\frac{3}{5}}{(z+4)} + \frac{-\frac{3}{5}}{(z+9)}\,,$$

we can express $H(s)$ as

$$H(s) = \frac{\frac{3}{5}}{s^2+4} - \frac{\frac{3}{5}}{s^2+9} = \tfrac{3}{10}\frac{2}{s^2+2^2} - \tfrac{1}{5}\frac{3}{s^2+3^2}\,.$$

The third left entry of table (9.17) with $b = 2$ and $b = 3$ then yields

$$\begin{aligned} h(t) = \mathcal{L}^{-1}[H(s)](t) &= \tfrac{3}{10}\mathcal{L}^{-1}\bigg[\frac{2}{s^2+2^2}\bigg](t) - \tfrac{1}{5}\mathcal{L}^{-1}\bigg[\frac{3}{s^2+3^2}\bigg](t) \\ &= \tfrac{3}{10}\sin(2t) - \tfrac{1}{5}\sin(3t)\,. \end{aligned}$$

9.7. **Computing Green Functions.** The Laplace transform can be used to efficiently compute Green functions for differential operators with constant coefficients. Recall that for the n^{th}-order differential operator L with constant coefficients that is given by

$$\mathrm{L} = \mathrm{D}^n + a_1\mathrm{D}^{n-1} + \cdots + a_{n-1}\mathrm{D} + a_n\,,$$

the Green function $g(t)$ for the operator L is the solution of the initial-value problem

$$\begin{gathered} g^{(n)} + a_1 g^{(n-1)} + \cdots + a_{n-1}g' + a_n g = 0\,, \\ g(0) = 0\,, \quad g'(0) = 0\,, \quad \cdots \quad g^{(n-2)}(0) = 0\,, \quad g^{(n-1)}(0) = 1\,. \end{gathered}$$

The Laplace transform of the differential equation is

$$\mathcal{L}\big[g^{(n)}\big](s) + a_1\mathcal{L}\big[g^{(n-1)}\big](s) + \cdots + a_{n-1}\mathcal{L}[g'](s) + \mathcal{L}[g](s) = 0\,.$$

An n-fold application of (9.11) using the initial conditions gives the table

$$\begin{aligned} \mathcal{L}[g](s) &= G(s)\,, \\ \mathcal{L}[g'](s) &= s\,\mathcal{L}[g](s) - g(0) = s\,G(s)\,, \\ \mathcal{L}[g''](s) &= s\,\mathcal{L}[g'](s) - g'(0) = s^2G(s)\,, \\ &\;\;\vdots \\ \mathcal{L}\big[g^{(n-1)}\big](s) &= s\,\mathcal{L}\big[g^{(n-2)}\big](s) - g^{(n-2)}(0) = s^{n-1}G(s)\,, \\ \mathcal{L}\big[g^{(n)}\big](s) &= s\,\mathcal{L}\big[g^{(n-1)}\big](s) - g^{(n-1)}(0) = s^nG(s) - 1\,. \end{aligned}$$

We thereby see that $G(s)$ satisfies

$$p(s)G(s) - 1 = 0\,,$$

where $p(s)$ is the characteristic polynomial of L, which is given by

$$p(s) = s^n + a_1s^{n-1} + \cdots + a_{n-1}s + a_n\,.$$

Therefore

$$G(s) = \frac{1}{p(s)}\,. \tag{9.19}$$

In other words, the Laplace transform of the Green function for L is the reciprocal of the characteristic polynomial of L.

The problem of computing a Green function is thereby reduced to the problem of finding an inverse Laplace transform. This can often be done quickly.

Example. Find the Green function $g(t)$ for the operator $\mathrm{L} = \mathrm{D}^2 + 6\mathrm{D} + 13$.

Solution. Because $p(s) = s^2 + 6s + 13 = (s+3)^2 + 2^2$, the third right entry of table (9.17) with $a = -3$ and $b = 2$ shows that the Green function is given by

$$g(t) = \mathcal{L}^{-1}\left[\frac{1}{p(s)}\right] = \tfrac{1}{2}\mathcal{L}^{-1}\left[\frac{2}{(s+3)^2 + 2^2}\right] = \tfrac{1}{2}e^{-3t}\sin(2t)\,.$$

Example. Find the Green function $g(t)$ for the operator $\mathrm{L} = \mathrm{D}^2 + 2\mathrm{D} - 15$.

Solution. Because $p(s) = s^2 + 2s - 15 = (s-3)(s+5)$, we use the partial fraction identity

$$\frac{1}{(s-3)(s+5)} = \frac{\frac{1}{8}}{s-3} - \frac{\frac{1}{8}}{s+5}\,.$$

The top right entry of table (9.17) with $a = 3$ and $n = 0$ and with $a = -5$ and $n = 0$ shows that the Green function is given by

$$g(t) = \mathcal{L}^{-1}\left[\frac{1}{p(s)}\right] = \tfrac{1}{8}\mathcal{L}^{-1}\left[\frac{1}{s-3}\right] - \tfrac{1}{8}\mathcal{L}^{-1}\left[\frac{1}{s+5}\right] = \frac{e^{3t} - e^{-5t}}{8}\,.$$

Example. Find the Green function $g(t)$ for the operator $\mathrm{L} = \mathrm{D}^4 + 13\mathrm{D}^2 + 36$.

Solution. Because $p(s) = s^4 + 13s^2 + 36 = (s^2+4)(s^2+9)$ only depends on s^2, we can use the partial fraction identity

$$\frac{1}{(z+4)(z+9)} = \frac{\frac{1}{5}}{z+4} - \frac{\frac{1}{5}}{z+9} \quad \text{at } z = s^2\,.$$

The third left entry of table (9.17) with $b = 2$ and with $b = 3$ shows that the Green function is given by

$$g(t) = \mathcal{L}^{-1}\left[\frac{1}{p(s)}\right] = \tfrac{1}{10}\mathcal{L}^{-1}\left[\frac{2}{s^2 + 2^2}\right] - \tfrac{1}{15}\mathcal{L}^{-1}\left[\frac{3}{s^2 + 3^2}\right] = \tfrac{1}{10}\sin(2t) - \tfrac{1}{15}\sin(3t)\,.$$

Example. Find the Green function $g(t)$ for the operator $\mathrm{L} = \mathrm{D}^3 + 49\mathrm{D}$.

Solution. Because $p(s) = s^3 + 49s = s(s^2 + 49)$, we use the partial fraction identity

$$\frac{1}{s(s^2+49)} = \frac{\frac{1}{49}}{s} - \frac{\frac{1}{49}\,s}{s^2+49}\,.$$

The top left entry of table (9.17) with $n = 0$ and the second left entry with $b = 7$ show that the Green function is given by

$$g(t) = \mathcal{L}^{-1}\left[\frac{1}{p(s)}\right] = \tfrac{1}{49}\mathcal{L}^{-1}\left[\frac{1}{s}\right] - \tfrac{1}{49}\mathcal{L}^{-1}\left[\frac{s}{s^2 + 7^2}\right] = \tfrac{1}{49}\big(1 - \cos(7t)\big)\,.$$

9.8. **Computing Natural Fundamental Sets.** Let L be the n^{th}-order linear differential operator L with constant coefficients that is given by

$$\mathrm{L} = \mathrm{D}^n + a_1 \mathrm{D}^{n-1} + \cdots + a_{n-1}\mathrm{D} + a_n \,.$$

The natural fundamental set of solutions for L associated with the initial time $t = 0$ can be computed by solving the general initial-value problem

$$\begin{gathered} y^{(n)} + a_1 y^{(n-1)} + \cdots + a_{n-1} y' + a_n y = 0 \,, \\ y(0) = y_0 \,, \quad y'(0) = y_1 \,, \quad \cdots \quad y^{(n-2)}(0) = y_{n-2} \,, \quad y^{(n-1)}(0) = y_{n-1} \,, \end{gathered} \tag{9.20}$$

and then expressing its solution as

$$y(t) = y_0 N_0(t) + y_1 N_1(t) + \cdots + y_{n-2} N_{n-2}(t) + y_{n-1} N_{n-1}(t) \,.$$

We can then read off that $N_0(t)$, $N_1(t)$, $\cdots$, $N_{n-1}(t)$ is the natural fundamental set of solutions for L that is associated with the initial time $t = 0$.

Remark. Recall that $\mathrm{Wr}[N_0, N_1, \cdots, N_{n-1}](0) = 1$. The Abel Theorem then implies that $\mathrm{Wr}[N_0, N_1, \cdots, N_{n-1}](t) = e^{-a_1 t}$.

The Laplace transform method gives us a new way to attack the general initial-value problem (9.20). Notice that if $y(t)$ is the solution of (9.20) and $\mathcal{L}[y](s) = Y(s)$ then $Y(s)$ is given by (9.15) with $F(s) = 0$. Specifically,

$$Y(s) = \frac{q(s)}{p(s)} \,,$$

where $p(s)$ is the characteristic polynomial of L and $q(s)$ is given by

$$\begin{aligned} q(s) = {} & \big(s^{n-1} + a_1 s^{n-2} + \cdots + a_{n-3}s^2 + a_{n-2}s + a_{n-1}\big) y_0 \\ & + \big(s^{n-2} + a_1 s^{n-3} + \cdots + a_{n-3}s + a_{n-2}\big) y_1 \\ & \vdots \\ & + \big(s^2 + a_1 s + a_2\big) y_{n-3} + \big(s + a_1\big) y_{n-2} + y_{n-1} \,. \end{aligned}$$

We can read off from this that

$$\begin{aligned} \mathcal{L}[N_0](s) &= \frac{s^{n-1} + a_1 s^{n-2} + \cdots + a_{n-3}s^2 + a_{n-2}s + a_{n-1}}{p(s)} \,, \\ \mathcal{L}[N_1](s) &= \frac{s^{n-2} + a_1 s^{n-3} + \cdots + a_{n-3}s + a_{n-2}}{p(s)} \,, \\ & \vdots \\ \mathcal{L}[N_{n-3}](s) &= \frac{s^2 + a_1 s + a_2}{p(s)} \,, \\ \mathcal{L}[N_{n-2}](s) &= \frac{s + a_1}{p(s)} \,, \\ \mathcal{L}[N_{n-1}](s) &= \frac{1}{p(s)} \,. \end{aligned}$$

Therefore the natural fundamental set is given by

$$
(9.21)\qquad
\begin{aligned}
N_0(t) &= \mathcal{L}^{-1}\left[\frac{s^{n-1} + a_1 s^{n-2} + \cdots + a_{n-3}s^2 + a_{n-2}s + a_{n-1}}{p(s)}\right](t)\,,\\
N_1(t) &= \mathcal{L}^{-1}\left[\frac{s^{n-2} + a_1 s^{n-3} + \cdots + a_{n-3}s + a_{n-2}}{p(s)}\right](t)\,,\\
&\vdots\\
N_{n-3}(t) &= \mathcal{L}^{-1}\left[\frac{s^2 + a_1 s + a_2}{p(s)}\right](t)\,,\\
N_{n-2}(t) &= \mathcal{L}^{-1}\left[\frac{s + a_1}{p(s)}\right](t)\,,\\
N_{n-1}(s) &= \mathcal{L}^{-1}\left[\frac{1}{p(s)}\right](t)\,.
\end{aligned}
$$

Because $\mathcal{L}[g](s) = G(s) = 1/p(s)$, and because

$$
\mathrm{D}^k g(t) = \mathcal{L}^{-1}[s^k G(s)](t) = \mathcal{L}^{-1}\left[\frac{s^k}{p(s)}\right](t)\,, \qquad \text{for every } k = 0,\, 2,\, \cdots,\, n-1\,,
$$

formula (9.21) shows that the *natural fundamental set of solutions* for the operator L can be expressed in terms of its Green function g by

$$
\begin{aligned}
N_{n-1}(t) &= g(t)\,,\\
N_{n-2}(t) &= \big(\mathrm{D} + a_1\big) g(t)\,,\\
N_{n-3}(t) &= \Big(\mathrm{D}^2 + a_1\mathrm{D} + a_2\Big) g(t)\,,\\
&\vdots\\
N_1(t) &= \Big(\mathrm{D}^{n-2} + a_1\mathrm{D}^{n-3} + \cdots + a_{n-3}\mathrm{D} + a_{n-2}\Big) g(t)\,,\\
N_0(t) &= \Big(\mathrm{D}^{n-1} + a_1\mathrm{D}^{n-2} + \cdots + a_{n-3}\mathrm{D}^2 + a_{n-2}\mathrm{D} + a_{n-1}\Big) g(t)\,.
\end{aligned}
$$

This calculation can be organized more efficiently by the recipe

$$
(9.22)\qquad
\begin{aligned}
N_{n-1}(t) &= g(t)\,,\\
N_{n-2}(t) &= \mathrm{D}N_{n-1}(t) + a_1 g(t)\,,\\
N_{n-3}(t) &= \mathrm{D}N_{n-2}(t) + a_2 g(t)\,,\\
&\vdots\\
N_1(t) &= \mathrm{D}N_2(t) + a_{n-2} g(t)\,,\\
N_0(t) &= \mathrm{D}N_1(t) + a_{n-1} g(t)\,,
\end{aligned}
$$

where the $a_1,\, a_2,\, \cdots,\, a_{n-2},\, a_{n-1}$ are the coefficients of the characteristic polynomial

$$
p(s) = s^n + a_1 s^{n-1} + a_2 s^{n-2} + \cdots + a_{n-2}t^2 + a_{n-1}t + a_n\,.
$$

Remark. Notice that the right-hand side of each line in (9.22) after the first line is simply the derivative of the right-hand side of the line above it plus the multiple of $g(t)$ determined by the appropriate coefficient of the characteristic polynomial. Notice that all the coefficints of characteristic polynomial enter into recipe (9.22) except for a_n.

Remark. The advantage of this recipe is that $g(t)$ is easier to find than the solution of the general initial-value problem (9.20).

Remark. Because the operator L has constant coefficients, if the natural fundamental set of solutions for L associated with the initial time $t = 0$ is $N_0(t)$, $N_1(t)$, $\cdots$, $N_n(t)$, then the natural fundamental set of solutions for L associated with any other initial time $t = t_I$ is simply

$$N_0(t - t_I)\,, \quad N_1(t - t_I)\,, \quad \cdots\,, N_n(t - t_I)\,.$$

Therefore it suffices to compute the natural fundamental set of solutions for L associated with the initial time $t = 0$.

Example. Compute the natural fundamental set of solutions for $\mathrm{L} = \mathrm{D}^2 + 6\mathrm{D} + 13$.

Solution. In Section 9.7 we showed that the Green function for L is

$$g(t) = \tfrac{1}{2} e^{-3t} \sin(2t)\,.$$

Because the characteristic polynomial is

$$p(z) = z^2 + 6z + 13\,,$$

by recipe (9.22) the natural fundamental set is given by

$$\begin{aligned} N_1(t) &= g(t) = \tfrac{1}{2} e^{-3t} \sin(2t)\,,\\ N_0(t) &= \mathrm{D}N_1(t) + 6g(t)\\ &= e^{-3t}\cos(2t) - \tfrac{3}{2} e^{-3t} \sin(2t) + \tfrac{6}{2} e^{-3t} \sin(2t)\\ &= e^{-3t}\cos(2t) + \tfrac{3}{2} e^{-3t} \sin(2t)\,. \end{aligned}$$

Example. Compute the natural fundamental set of solutions for $\mathrm{L} = \mathrm{D}^2 + 2\mathrm{D} - 15$.

Solution. In Section 9.7 we showed that the Green function for L is

$$g(t) = \tfrac{1}{8} e^{3t} - \tfrac{1}{8} e^{-5t}\,.$$

Because the characteristic polynomial is

$$p(z) = z^2 + 2z - 15\,,$$

by recipe (9.22) the natural fundamental set is given by

$$\begin{aligned} N_1(t) &= g(t) = \tfrac{1}{8} e^{3t} - \tfrac{1}{8} e^{-5t}\,,\\ N_0(t) &= \mathrm{D}N_1(t) + 2g(t)\\ &= \tfrac{3}{8} e^{3t} + \tfrac{5}{8} e^{-5t} + \tfrac{2}{8} e^{3t} - \tfrac{2}{8} e^{-5t}\\ &= \tfrac{5}{8} e^{3t} + \tfrac{3}{8} e^{-5t}\,. \end{aligned}$$

Example. Compute the natural fundamental set of solutions for $\mathrm{L} = \mathrm{D}^4 + 13\mathrm{D}^2 + 36$.

Solution. In Section 9.7 we showed that the Green function for L is

$$g(t) = \tfrac{1}{10}\sin(2t) - \tfrac{1}{15}\sin(3t)\,.$$

Because the characteristic polynomial is $p(z) = z^4 + 13z^2 + 36$, which we see as

$$p(z) = z^4 + 0z^3 + 13z^2 + 0z + 36\,,$$

by recipe (9.22) the natural fundamental set is given by

$$\begin{aligned}
N_3(t) &= g(t) = \tfrac{1}{10}\sin(2t) - \tfrac{1}{15}\sin(3t)\,,\\
N_2(t) &= \mathrm{D}N_3(t) + 0g(t) = \tfrac{1}{5}\cos(2t) - \tfrac{1}{5}\cos(3t)\,,\\
N_1(t) &= \mathrm{D}N_2(t) + 13g(t)\\
&= -\tfrac{2}{5}\sin(2t) + \tfrac{3}{5}\sin(3t) + \tfrac{13}{10}\sin(2t) - \tfrac{13}{15}\sin(3t) = \tfrac{9}{10}\sin(2t) - \tfrac{4}{15}\sin(3t)\,,\\
N_0(t) &= \mathrm{D}N_1(t) + 0g(t) = \tfrac{9}{5}\cos(2t) - \tfrac{4}{5}\sin(3t)\,.
\end{aligned}$$

Example. Compute the natural fundamental set of solutions for $\mathrm{L} = \mathrm{D}^3 + 49\mathrm{D}$.

Solution. In Section 9.7 we showed that the Green function for L is

$$g(t) = \tfrac{1}{49}\big(1 - \cos(7t)\big)\,.$$

Because the characteristic polynomial is $p(z) = z^3 + 49z$, which we see as

$$p(z) = z^3 + 0z^2 + 49z + 0\,,$$

by recipe (9.22) the natural fundamental set is given by

$$\begin{aligned}
N_2(t) &= g(t) = \tfrac{1}{49}\big(1 - \cos(7t)\big)\,,\\
N_1(t) &= \mathrm{D}N_2(t) + 0g(t) = \tfrac{1}{7}\sin(7t)\,,\\
N_0(t) &= \mathrm{D}N_1(t) + 49g(t) = \cos(7t) + \big(1 - \cos(7t)\big) = 1\,.
\end{aligned}$$

9.9. **Piecewise-Defined Forcing.** The Laplace transform method can be used to solve initial-value problems of the form

$$\begin{gathered}
y^{(n)} + a_1 y^{(n-1)} + \cdots + a_{n-1}y' + a_n y = f(t)\,,\\
y(0) = y_0\,, \qquad y'(0) = y_1\,, \qquad \cdots \qquad y^{(n-1)}(0) = y_{n-1}\,,
\end{gathered} \tag{9.23a}$$

where the forcing $f(t)$ is piecewise-defined over $[0,\infty)$ by a list of cases in the form

$$f(t) = \begin{cases}
f_0(t) & \text{for } 0 \le t < c_1\,,\\
f_1(t) & \text{for } c_1 \le t < c_2\,,\\
\vdots & \qquad\vdots\\
f_{m-1}(t) & \text{for } c_{m-1} \le t < c_m\,,\\
f_m(t) & \text{for } c_m \le t < \infty\,,
\end{cases} \tag{9.23b}$$

where $0 = c_0 < c_1 < c_2 < \cdots < c_m < \infty$. We assume that for each $k = 0,\ 1,\ \cdots,$ $m-1$ the function f_k is continuous and bounded over $[c_k, c_{k+1})$, while the function f_m is continuous over $[c_m, \infty)$ and is of exponential order as $t \to \infty$.

Why would we want to consider such forcing functions? In fact, such functions arise naturally in engineering and science. They arise when modeling designed systems in which a forcing is changed. For example, a load might be changed on a structure or a voltage might be changed across an electrical circuit. This happens whenever you flip a light switch or tap the screen on your cell phone to make it do something. One reason that the Laplace transform method is widely used in engineering is that it can be easily applied to such functions. Here we show how to compute the Laplace transform $F(s) = \mathcal{L}[f](s)$ for such functions, which will allow us to solve initial-value problems such as (9.23).

9.9.1. *Shifty Step Method for a Special Case.* We begin by treating functions that are in the form $u(t-c)h(t)$ for some $c \geq 0$. We will assume that $h(t)$ is continuous over $[c, \infty)$ and is of exponential order as $t \to \infty$. Notice that

$$u(t-c)h(t) = \begin{cases} 0 & \text{for } t < c\,, \\ h(t) & \text{for } t \geq c\,. \end{cases} \tag{9.24}$$

This shows that $u(t-c)h(t)$ "turns off" $h(t)$ for $t < c$ and "turns on" $h(t)$ for $t \geq c$. This function is a special case of the class of functions given by (9.23b). We will show that every function given by (9.23b) can be written as a sum of functions in the form (9.24).

We take the Laplace transform of $u(t-c)h(t)$ by using the last entry in table (9.16). That entry states that

$$\mathcal{L}[u(t-c)j(t-c)](s) = e^{-cs}\mathcal{L}[j](s)\,. \tag{9.25}$$

In order to use this formula we must find a function $j(t)$ such that $h(t) = j(t-c)$ for every $t \geq c$. After we find such a $j(t)$, formula (9.25) then gives

$$\mathcal{L}[u(t-c)h(t)](s) = \mathcal{L}[u(t-c)j(t-c)](s) = e^{-cs}\mathcal{L}[j](s)\,.$$

The key to finding such a $j(t)$ is the observation that the following are equivalent

$$\begin{aligned} j(t-c) &= h(t) && \text{for every } t \geq c\,, \\ j(t') &= h(t'+c) && \text{for every } t' \geq 0\,, \\ j(t) &= h(t+c) && \text{for every } t \geq 0\,. \end{aligned} \tag{9.26}$$

The equivalence of the first and second line is seen with the substitution $t' = t - c$. The equivalence of the second and third line is seen with the substitution $t = t'$. The third line shows that we can find $j(t)$ by simply replacing t by $t + c$ in $h(t)$. We call this the *shifty step method* for finding $j(t)$.

Example. Compute $\mathcal{L}[u(t-3)e^{2t}](s)$.

Solution. By setting $e^{2t} = j(t-3)$ we see from the shifty step method (9.26) that

$$j(t) = e^{2(t+3)} = e^{2t}e^{6}\,.$$

Therefore formula (9.25) shows that

$$\mathcal{L}[u(t-3)e^{2t}](s) = \mathcal{L}[u(t-3)j(t-3)] = e^{-3s}\mathcal{L}[j](s) = e^{-3s}\mathcal{L}[e^{2t}e^{6}](s)\,.$$

Then linearity (9.4) and the sixth entry in table (9.16) with $a = 2$ and $n = 0$ give

$$\mathcal{L}[e^{2t}e^6](s) = e^6\mathcal{L}[e^{2t}](s) = \frac{e^6}{s-2} \quad \text{for } s > 2\,.$$

Therefore

$$\mathcal{L}[u(t-3)e^{2t}](s) = e^{-3s}\mathcal{L}[e^{2t}e^6](s) = e^{6-3s}\,\frac{1}{s-2} \quad \text{for } s > 2\,.$$

Example. Compute $\mathcal{L}[f](s)$ for $f(t) = u(t-4)e^{-3t}\cos(2t)$.

Solution. By setting $e^{-3t}\cos(2t) = j(t-4)$ we see from the shifty step method (9.26) that

$$j(t) = e^{-3(t+4)}\cos(2(t+4)) = e^{-3t}e^{-12}\cos(2t+8)\,.$$

Therefore formula (9.25) shows that

$$\begin{aligned}\mathcal{L}[f](s) &= \mathcal{L}[u(t-4)e^{-3t}\cos(2t)](s)\\ &= \mathcal{L}[u(t-4)j(t-4)] = e^{-4s}\mathcal{L}[j](s) = e^{-4s}\mathcal{L}[e^{-3t}e^{-12}\cos(2t+8)](s)\,.\end{aligned}$$

Then linearity (9.4), the trig identity

$$\cos(2t+8) = \cos(8)\cos(2t) - \sin(8)\sin(2t)\,,$$

and the seventh and eighth entries in table (9.16) with $a = -3$ and $b = 2$ give

$$\begin{aligned}\mathcal{L}[j](s) &= \mathcal{L}[e^{-3t}e^{-12}\cos(2t+8)](s)\\ &= e^{-12}\cos(8)\mathcal{L}[e^{-3t}\cos(2t)](s) - e^{-12}\sin(8)\mathcal{L}[e^{-3t}\sin(2t)](s)\\ &= e^{-12}\cos(8)\,\frac{s+3}{(s+3)^2+2^2}\\ &\qquad - e^{-12}\sin(8)\,\frac{2}{(s+3)^2+2^2} \quad \text{for } s > -3\,.\end{aligned}$$

Therefore

$$\begin{aligned}\mathcal{L}[f](s) &= \mathcal{L}[u(t-4)e^{-3t}\cos(2t)](s)\\ &= e^{-4s}\mathcal{L}[e^{-3t}e^{-12}\cos(2t+8)](s)\\ &= e^{-12-4s}\cos(8)\,\frac{s+3}{(s+3)^2+2^2}\\ &\qquad - e^{-12-4s}\sin(8)\,\frac{2}{(s+3)^2+2^2} \quad \text{for } s > -3\,.\end{aligned}$$

Remark. The above examples could also have been done by first applying the twelfth entry in table (9.16) to respectively obtain

$$\mathcal{L}[u(t-3)e^{2t}](s) = \mathcal{L}[u(t-c)](s-2)\,,$$
$$\mathcal{L}[u(t-4)e^{-3t}\cos(2t)](s) = \mathcal{L}[u(t-4)\cos(2t)](s+3)\,.$$

The last entry in table (9.16) would then be applied to compute $\mathcal{L}[u(t-c)](s)$ or $\mathcal{L}[u(t-4)\cos(2t)](s)$ respectively.

9.9.2. *Reduction to the Special Case.* We now return to the general case given by (9.23b). We consider $f(t)$ that is piecewise-defined over $[0,\infty)$ by a list of cases given in the form

$$f(t) = \begin{cases} f_0(t) & \text{for } 0 \leq t < c_1\,, \\ f_1(t) & \text{for } c_1 \leq t < c_2\,, \\ \vdots & \qquad \vdots \\ f_{m-1}(t) & \text{for } c_{m-1} \leq t < c_m\,, \\ f_m(t) & \text{for } c_m \leq t < \infty\,, \end{cases} \tag{9.27}$$

where $0 = c_0 < c_1 < c_2 < \cdots < c_m < \infty$. We assume that for each $k = 0,\, 1,\, \cdots,\, m-1$ the function f_k is continuous and bounded over $[c_k, c_{k+1})$, while the function f_m is continuous over $[c_m, \infty)$ and is of exponential order as $t \to \infty$. We will show that every such $f(t)$ can be expressed as a sum of functions in the special form (9.24).

The first step is to express $f(t)$ in terms of translations of the unit step $u(t)$. How this is done should become clear once you see that for every $0 \leq c < d$ we have

$$u(t-c) - u(t-d) = \begin{cases} 1 & \text{for } c \leq t < d\,, \\ 0 & \text{otherwise}\,. \end{cases}$$

In other words, the function $u(t-c) - u(t-d)$ is a switch that turns on at $t = c$ and turns off at $t = d$. So for any given function $g(t)$ we have

$$\big(u(t-c) - u(t-d)\big)\, g(t) = \begin{cases} g(t) & \text{for } c \leq t < d\,, \\ 0 & \text{otherwise}\,. \end{cases}$$

This observation allows us to express $f(t)$ given by (9.27) in terms of unit step functions as

$$\begin{aligned} f(t) = \big(u(t) - u(t-c_1)\big)\, f_0(t) + \big(u(t-c_1) - u(t-c_2)\big)\, f_1(t) \\ + \cdots + \big(u(t-c_{m-1}) - u(t-c_m)\big)\, f_{m-1}(t) + u(t-c_m)\, f_m(t)\,. \end{aligned}$$

However, this form of $f(t)$ is not ideal for taking the Laplace transform. Rather, we modify it by grouping terms above that involve the same $u(t-c_k)$, thereby bring $f(t)$ into the form

$$f(t) = f_0(t) + u(t-c_1)\, h_1(t) + \cdots + u(t-c_m)\, h_m(t)\,, \tag{9.28a}$$

where

$$h_k(t) = f_k(t) - f_{k-1}(t) \qquad \text{for } k = 1, 2, \cdots, m\,. \tag{9.28b}$$

You can get to form (9.28) by either

- carrying out the grouping of terms that we did above or
- writing it down directly.

When writing it down directly it helps to recall that each term $u(t-c_k)\, h_k(t)$ appearing in (9.28a) simply changes the forcing from $f_{k-1}(t)$ to $f_k(t)$ at time $t = c_k$ because $h_k(t) = f_k(t) - f_{k-1}(t)$.

Notice that each term of the sum on the right-hand side of (9.28) is in the special form (9.24). We can thereby use the shifty step method (9.26) to compute the Laplace transform of each term. Specifically, we recast (9.28) into the form

$$f(t) = f_0(t) + u(t-c_1)\, j_1(t-c_1) + \cdots + u(t-c_m)\, j_m(t-c_m)\,, \tag{9.29a}$$

where each function $j_k(t)$ is obtained from the $h_k(t)$ appearing in (9.28) by the shifty step method (9.26), which sets

$$j_k(t) = h_k(t+c_k) \qquad \text{for } k = 1, 2, \cdots, m\,. \tag{9.29b}$$

After all the $j_k(t)$ have been found then the final step is to compute

$$F_0(s) = \mathcal{L}[f_0](s)\,, \quad \text{and} \quad J_k(s) = \mathcal{L}[j_k](s) \quad \text{for } k = 1, 2, \cdots, m\,,$$

and use the fact that

$$\mathcal{L}[u(t-c_k)j_k(t-c_k)](s) = e^{-c_k s} J_k(s) \qquad \text{for } k = 1, 2, \cdots, m\,,$$

to compute $F(s) = \mathcal{L}[f](s)$ as

$$F(s) = F_0(s) + e^{-c_1 s} J_1(s) + \cdots + e^{-c_m s} J_m(s)\,.$$

Often we will have to use identities to express $f_0(t)$ and each $j_k(t)$ in forms that allows us to compute their Laplace transforms from table (9.16).

Now let us return to the general initial-value problem 9.23. The Laplace transform $Y(s)$ of its solution $y(t)$ is given by

$$Y(s) = \frac{q(s) + F(s)}{p(s)} = \frac{q(s) + F_0(s)}{p(s)} + e^{-c_1 s}\frac{J_1(s)}{p(s)} + \cdots + e^{-c_m s}\frac{J_m(s)}{p(s)}\,. \tag{9.30}$$

These methods are illustrated in the following example.

Example. Find the Laplace transform $Y(s)$ of the solution $y(t)$ of the initial-value problem

$$y'' + 4y = f(t)\,, \qquad y(0) = 7\,, \quad y'(0) = 5\,,$$

where

$$f(t) = \begin{cases} t^2 & \text{for } 0 \le t < 2\,, \\ 2t & \text{for } 2 \le t < 4\,, \\ 4 & \text{for } 4 \le t\,. \end{cases}$$

Solution. The Laplace transform of the initial-value problem is

$$\mathcal{L}[y''](s) + 4\mathcal{L}[y](s) = F(s)\,,$$

where $F(s) = \mathcal{L}[f](s)$ and where repeated application of (9.11) and the initial conditions give the table

$$\begin{aligned} \mathcal{L}[y](s) &= Y(s)\,, \\ \mathcal{L}[y'](s) &= s\,\mathcal{L}[y](s) - y(0) = sY(s) - 7\,, \\ \mathcal{L}[y''](s) &= s\,\mathcal{L}[y'](s) - y'(0) = s^2Y(s) - 7s - 5\,. \end{aligned}$$

The Laplace transform of the initial-value problem thereby becomes

$$(s^2+4)Y(s) - 7s - 5 = F(s)\,, \qquad \Longrightarrow \qquad Y(s) = \frac{1}{s^2+4}\big(7s + 5 + F(s)\big)\,.$$

All that remains to be done is to compute $F(s)$. The first step is to express $f(t)$ in terms of unit step functions as

$$\begin{aligned} f(t) &= \big(u(t) - u(t-2)\big)\, t^2 + \big(u(t-2) - u(t-4)\big)\, 2t + u(t-4)\, 4 \\ &= t^2 + u(t-2)\,(2t - t^2) + u(t-4)\,(4 - 2t)\,. \end{aligned}$$

The second step is to write

$$f(t) = t^2 + u(t-2)\, j_1(t-2) + u(t-4)\, j_2(t-4)\,,$$

where by the shift step method (9.26) we have

$$\begin{aligned} j_1(t) &= 2(t+2) - (t+2)^2 = 2t + 4 - t^2 - 4t - 4 = -t^2 - 2t\,, \\ j_2(t) &= 4 - 2(t+4) = -2t - 4\,. \end{aligned}$$

Here we obtained $j_1(t)$ by replacing t with $t+2$ in the factor $(2t - t^2)$ and $j_2(t)$ by replacing t with $t+4$ in the factor $(4 - 2t)$. Finally, the above form for $f(t)$ allows us to use the last entry of table (9.16) to compute $F(s) = \mathcal{L}[f](s)$ as

$$\begin{aligned} F(s) &= \mathcal{L}[t^2](s) + \mathcal{L}[u(t-2)\, j_1(t-2)](s) + \mathcal{L}[u(t-4)\, j_2(t-4)](s) \\ &= \mathcal{L}[t^2](s) - e^{-2s}\mathcal{L}[t^2 + 2t](s) - e^{-4s}\mathcal{L}[2t + 4](s) \\ &= \frac{2}{s^3} - e^{-2s}\left(\frac{2}{s^3} + \frac{2}{s^2}\right) - e^{-4s}\left(\frac{2}{s^2} + \frac{4}{s}\right). \end{aligned}$$

It follows that

$$Y(s) = \frac{7s+5}{s^2+4} - \frac{e^{-2s}}{s^2+4}\left(\frac{2}{s^3} + \frac{2}{s^2}\right) - \frac{e^{-4s}}{s^2+4}\left(\frac{2}{s^2} + \frac{4}{s}\right).$$

Recall that the Laplace transform $Y(s)$ of the solution $y(t)$ to to the general initial-value problem (9.23) was given by (9.30). It follows that $y(t)$ is given by

$$y(t) = \mathcal{L}^{-1}\left[\frac{q(s) + F_0(s)}{p(s)}\right] + \mathcal{L}^{-1}\left[e^{-c_1 s}\frac{J_1(s)}{p(s)}\right] + \cdots + \mathcal{L}^{-1}\left[e^{-c_m s}\frac{J_m(s)}{p(s)}\right].$$

The task of finding $y(t)$ thereby requires using formula (9.18), which is

(9.31) $$\mathcal{L}^{-1}\big[e^{-cs}J(s)\big](t) = u(t-c)j(t-c) \qquad \text{where } j(t) = \mathcal{L}^{-1}[J](t)\,.$$

This formula is easy to use once a $j(t)$ is known, so the hard part of finding $y(t)$ is computing $j(t) = \mathcal{L}^{-1}[J](t)$ for each of the $J(s)$ that arise. Therefore it usually pays to regroup the terms in $Y(s)$ to simplify the forms of the $J(s)$ that we have to compute.

Example. Find $y(t) = \mathcal{L}^{-1}[Y](t)$ for

$$Y(s) = \frac{7s+5}{s^2+4} - \frac{e^{-2s}}{s^2+4}\left(\frac{2}{s^3} + \frac{2}{s^2}\right) - \frac{e^{-4s}}{s^2+4}\left(\frac{2}{s^2} + \frac{4}{s}\right).$$

Solution. We first regroup the terms in $Y(s)$ to isolate four relatively simple rational functions $J(s)$ for which we will compute $j(t) = \mathcal{L}^{-1}[J](t)$ as

$$Y(s) = \frac{7s+5}{s^2+4} + \big(1 - e^{-2s}\big)\,\frac{2}{s^3(s^2+4)} - \big(e^{-2s} + e^{-4s}\big)\,\frac{2}{s^2(s^2+4)} - e^{-4s}\,\frac{4}{s\,(s^2+4)}\,.$$

We then derive the four partial fraction identities

$$\frac{7s+5}{s^2+4} = \frac{7s}{s^2+4} + \frac{5}{s^2+4}\,, \qquad \frac{2}{s^2(s^2+4)} = \frac{\frac{1}{2}}{s^2} - \frac{\frac{1}{2}}{s^2+4}\,,$$
$$\frac{2}{s^3(s^2+4)} = \frac{\frac{1}{2}}{s^3} - \frac{\frac{1}{8}}{s} + \frac{\frac{1}{8}s}{s^2+4}\,, \qquad \frac{4}{s\,(s^2+4)} = \frac{1}{s} - \frac{s}{s^2+4}\,.$$

The top left of these is straightforward. The top right identity only involves s^2, so is simply the identity

$$\frac{2}{z(z+4)} = \frac{\frac{1}{2}}{z} - \frac{\frac{1}{2}}{z+4}\,, \quad \text{evaluated at } z = s^2\,.$$

The bottom right identity is simply $2s$ times the top right one. Finally, the bottom left identity is obtained by first dividing the top right one by s and then employing the bottom right one divided by 8 to the last term.

These partial fraction identites allow us to express $Y(s)$ as

$$\begin{aligned} Y(s) &= \frac{7s}{s^2+4} + \frac{5}{s^2+4} + \big(1-e^{-2s}\big)\left(\frac{\frac{1}{2}}{s^3} - \frac{\frac{1}{8}}{s} + \frac{\frac{1}{8}s}{s^2+4}\right) \\ &\quad - \big(e^{-2s}+e^{-4s}\big)\left(\frac{\frac{1}{2}}{s^2} - \frac{\frac{1}{2}}{s^2+4}\right) - e^{-4s}\left(\frac{1}{s} - \frac{s}{s^2+4}\right) \\ &= 7\frac{s}{s^2+2^2} + \frac{5}{2}\frac{2}{s^2+2^2} + \big(1-e^{-2s}\big)\left(\frac{1}{4}\frac{2}{s^3} - \frac{1}{8}\frac{1}{s} + \frac{1}{8}\frac{s}{s^2+2^2}\right) \\ &\quad - \big(e^{-2s}+e^{-4s}\big)\left(\frac{1}{2}\frac{1}{s^2} - \frac{1}{4}\frac{2}{s^2+2^2}\right) - e^{-4s}\left(\frac{1}{s} - \frac{s}{s^2+2^2}\right). \end{aligned}$$

The formulas in the first column of table (9.17) show that

$$\begin{aligned} \mathcal{L}^{-1}\!\left[7\frac{s}{s^2+2^2} + \frac{5}{2}\frac{2}{s^2+2^2}\right](t) &= 7\cos(2t) + \tfrac{5}{2}\sin(2t)\,, \\ \mathcal{L}^{-1}\!\left[\frac{1}{4}\frac{2}{s^3} - \frac{1}{8}\frac{1}{s} + \frac{1}{8}\frac{s}{s^2+2^2}\right](t) &= \tfrac{1}{4}t^2 - \tfrac{1}{8} + \tfrac{1}{8}\cos(2t)\,, \\ \mathcal{L}^{-1}\!\left[\frac{1}{2}\frac{1}{s^2} - \frac{1}{4}\frac{2}{s^2+2^2}\right](t) &= \tfrac{1}{2}t - \tfrac{1}{4}\sin(2t)\,, \\ \mathcal{L}^{-1}\!\left[\frac{1}{s} - \frac{s}{s^2+2^2}\right](t) &= 1 - \cos(2t)\,. \end{aligned}$$

By combining these facts with formula (9.31), it follows that

$$\begin{aligned} y(t) = 7\cos(2t) &+ \tfrac{5}{2}\sin(2t) + \big(\tfrac{1}{4}t^2 - \tfrac{1}{8} + \tfrac{1}{8}\cos(2t)\big) \\ &- u(t-2)\big(\tfrac{1}{4}(t-2)^2 - \tfrac{1}{8} + \tfrac{1}{8}\cos(2(t-2))\big) \\ &- u(t-2)\big(\tfrac{1}{2}(t-2) - \tfrac{1}{4}\sin(2(t-2))\big) \\ &- u(t-4)\big(\tfrac{1}{2}(t-4) - \tfrac{1}{4}\sin(2(t-4))\big) \\ &- u(t-4)\big(1 - \cos(2(t-4))\big)\,. \end{aligned}$$

9.10. **Impulse Forcing.** Now let us consider the family of piecewise-defined forcing functions given by

$$f_\tau(t) = \frac{1}{\tau}\big(u(t) - u(t-\tau)\big)\,, \qquad \text{for some } \tau > 0\,. \tag{9.32}$$

This forcing $f_\tau(t)$ has amplitude $1/\tau$ that turns on at $t = 0$ and turns off at $t = \tau$. The subscript τ indicates this dependence. We want to consider the effect of such a forcing on the solution of the initial-value problem

$$\mathrm{L}y = f_\tau(t)\,, \qquad y(0) = y'(0) = \cdots = y^{(n-1)}(0) = 0\,, \tag{9.33a}$$

where L is the n^{th}-order differential operator with constant coefficients given by

$$\mathrm{L} = \mathrm{D}^n + a_1\mathrm{D}^{n-1} + \cdots + a_{n-1}\mathrm{D} + a_n\,. \tag{9.33b}$$

More specifically, we want to understand the behavior of this solution when τ is very small — that is, when there is a strong forcing of short duration. Such a forcing is called an *impulse*.

The solution of the initial-value problem (9.33) will depend upon τ through $f_\tau(t)$, so we will denote it y_τ. Then the Laplace transform of the initial-value problem is

$$\mathcal{L}[y_\tau^{(n)}](s) + a_1\mathcal{L}[y_\tau^{(n-1)}](s) + \cdots + a_{n-1}\mathcal{L}[y_\tau'](s) + a_n\mathcal{L}[y_\tau](s) = F_\tau(s)\,,$$

where

$$\begin{aligned}
\mathcal{L}[y_\tau](s) &= Y_\tau(s)\,,\\
\mathcal{L}[y_\tau'](s) &= s\,\mathcal{L}[y_\tau](s) - y_\tau(0) = s\,Y_\tau(s)\,,\\
&\;\;\vdots\\
\mathcal{L}[y_\tau^{(n-1)}](s) &= s\,\mathcal{L}[y_\tau^{(n-2)}] - y_\tau^{(n-2)}(0) = s^{n-1}Y_\tau(s)\,,\\
\mathcal{L}[y_\tau^{(n)}](s) &= s\,\mathcal{L}[y_\tau^{(n-1)}] - y_\tau^{(n-1)}(0) = s^{n}Y_\tau(s)\,,
\end{aligned}$$

and

$$F_\tau(s) = \mathcal{L}[f_\tau](s) = \frac{1}{\tau}\big(\mathcal{L}[u](s) - \mathcal{L}[u(t-\tau)](s)\big) = \frac{1 - e^{-\tau s}}{\tau s} \quad \text{for every } s > 0\,.$$

The Laplace transform of the initial-value problem thereby becomes

$$p(s)Y_\tau(s) = \frac{1 - e^{-\tau s}}{\tau s}\,,$$

where $p(s)$ is the characteristic polynomial of L, which is given by

$$p(s) = s^n + a_1 s^{n-1} + \cdots + a_{n-1}s + a_n\,.$$

Therefore the Laplace transform $Y_\tau(s)$ of the solution $y_\tau(t)$ is given in terms of $F_\tau(s)$ as

$$Y_\tau(s) = \frac{1}{p(s)}\,\frac{1 - e^{-\tau s}}{\tau s}\,. \tag{9.34}$$

Now let $Y(s)$ denote the limit of $Y_\tau(s)$ as τ becomes small. We see from (9.34) that

$$Y(s) = \lim_{\tau\to 0} Y_\tau(s) = \frac{1}{p(s)}\,\lim_{\tau\to 0}\frac{1 - e^{-\tau s}}{\tau s} = \frac{1}{p(s)}\,,$$

where the last limit can be evaluated either by the l'Hospital rule or by making a Taylor approximation of the numerator. But $1/p(s)$ is the Laplace transform of the Green function $g(t)$ associated with the differential operator (9.33b). Therefore it seems as if the solution of the initial-value problem (9.33) will behave like the Green function $g(t)$ as τ becomes small.

It is natural to wonder if this result depends upon the particular form (9.32) of the forcing that we considered. To explore this question we now consider the initial-value problem (9.33) for the more general family of forcing functions given by

$$f_\tau(t) = \frac{1}{\tau} f\left(\frac{t}{\tau}\right), \qquad \text{for some } \tau > 0\,, \tag{9.35}$$

where $f(t)$ is any nonnegative piecewise integrable function of exponential order $\alpha < 0$. (The forcing (9.32) has this form with $f(t) = u(t) - u(t-1)$.)

Because $\alpha < 0$ and $F(s) = \mathcal{L}[f](s)$ is defined for every $s > \alpha$, this implies that

$$F(0) = \int_0^\infty f(t)\,\mathrm{d}t < \infty\,.$$

Then

$$\begin{aligned} F_\tau(s) = \mathcal{L}[f_\tau](s) &= \int_0^\infty e^{-st} f_\tau(t)\,\mathrm{d}t = \frac{1}{\tau}\int_0^\infty e^{-st} f\left(\frac{t}{\tau}\right)\,\mathrm{d}t \\ &= \int_0^\infty e^{-\tau s t'} f(t')\,\mathrm{d}t' = F(\tau s) \quad \text{for every } s > \frac{\alpha}{\tau}\,. \end{aligned}$$

Notice that this is consistent with formula (9.34) that was derived for our original forcing (9.32).

The Laplace transform $Y_\tau(s)$ of the solution $y_\tau(t)$ to the initial-value problem (9.33) with this general forcing is given in terms of $F_\tau(s)$ as

$$Y_\tau(s) = \frac{1}{p(s)}\,F_\tau(s) = \frac{1}{p(s)}\,F(\tau s)\,. \tag{9.36}$$

Again let $Y(s)$ denote the limit of $Y_\tau(s)$ as τ becomes small. We see from (9.36) that

$$Y(s) = \lim_{\tau\to 0} Y_\tau(s) = \frac{1}{p(s)}\,\lim_{\tau\to 0} F(\tau s) = \frac{1}{p(s)}\,F(0) = \frac{1}{p(s)}\int_0^\infty f(t')\,\mathrm{d}t'\,.$$

Because $1/p(s)$ is the Laplace transform of the Green function $g(t)$, it seems that $y_\tau(t)$ behaves like a multiple of the Green function as τ becomes small. Specifically, it seems that

$$y_\tau(t) \approx g(t)\int_0^\infty f(t')\,\mathrm{d}t' \quad \text{for small } \tau\,.$$

This shows that the details of an impulse do not matter. All that matters is the integral of an impulse forcing.

Therefore for sufficiently small τ every forcing $f_\tau(t)$ given by (9.35) for some $f(t)$ can be modeled by an idealized impulse forcing $M\delta(t)$, where M is the magnitude of the impulse, which is given by

$$M = \int_0^\infty f(t)\,\mathrm{d}t\,,$$

and $\delta(t)$ is commonly called either the *unit impulse* or *Dirac delta* function even though it is *not a function*! For every interval $[a,b]$ such that $0 \in [a,b]$ the unit impulse is treated like it has the property

$$\int_a^b \delta(t)\,\phi(t)\,\mathrm{d}t = \phi(0) \quad \text{for every } \phi \text{ that is continuous over } [a,b]\,. \tag{9.37}$$

This property allows us to compute

$$\mathcal{L}[\delta](s) = \int_0^\infty e^{-st}\delta(t)\,\mathrm{d}t = 1\,,$$
$$\mathcal{L}[\delta(t-c)](s) = \int_0^\infty e^{-st}\delta(t-c)\,\mathrm{d}t = e^{-cs} \quad \text{for every } c > 0\,.$$

Therefore the initial-value problem (9.33) with $f(t) = M\delta(t)$ has solution $y(t) = Mg(t)$.

Remark. The notion of a unit impulse goes back to Oliver Heaviside in 1899. Paul Dirac introduced the delta notation in his work on quantum mechanics during the late 1920s. These early approaches raised as many questions as they addressed. Subsequently Soloman Bochner (1933), Sergei Sobolev (1938), and Kurt Friedrichs (1944), took important steps towards putting the notion on a firm mathematical footing. In the 1950s Laurant Schwartz developed the theory of *distributions*, which is a framework within which the impulse function and other so-called *generalized functions* exist. These theories lie far beyond the scope of this course. Our motivation comes from Heaviside, our notation comes from Dirac, and our property (9.37) comes from Schwartz.

Example. Solve the initial-value problem

$$y''' - 4y'' + 3y' = 5\delta(t-2)\,, \qquad y(0) = y'(0) = 0\,,\ y''(0) = 7\,.$$

Solution. The Laplace transform of the initial-value problem is

$$\mathcal{L}[y'''](s) - 4\mathcal{L}[y''](s) + 3\mathcal{L}[y'](s) = 5\mathcal{L}[\delta(t-2)](s)\,,$$

where

$$\mathcal{L}[y](s) = Y(s)\,,$$
$$\mathcal{L}[y'](s) = s\mathcal{L}[y](s) - y(0) = sY(s)\,,$$
$$\mathcal{L}[y''](s) = s\mathcal{L}[y'](s) - y'(0) = s^2Y(s)\,,$$
$$\mathcal{L}[y'''](s) = s\mathcal{L}[y''](s) - y''(s) = s^3Y(s) - 7\,,$$

and by property (9.37) we have

$$\mathcal{L}[\delta(t-2)](s) = \int_0^\infty e^{-st}\delta(t-2)\,\mathrm{d}t = e^{-2s}\,.$$

The Laplace transform of the initial-value problem thereby becomes

$$(s^3 - 4s^2 + 3s)Y(s) = 7 + 5e^{-2s}\,.$$

Therefore the Laplace transform of the solution is

$$Y(s) = \frac{7 + 5e^{-2s}}{s^3 - 4s^2 + 3s}\,.$$

By the partial fraction identity

$$\frac{1}{s^3 - 4s^2 + 3s} = \frac{1}{s(s-1)(s-3)} = \frac{\frac{1}{3}}{s} - \frac{\frac{1}{2}}{s-1} + \frac{\frac{1}{6}}{s-3}\,,$$

we see that

$$Y(s) = \left(\frac{\frac{7}{3}}{s} - \frac{\frac{7}{2}}{s-1} + \frac{\frac{7}{6}}{s-3}\right) + e^{-2s}\left(\frac{\frac{5}{3}}{s} - \frac{\frac{5}{2}}{s-1} + \frac{\frac{5}{6}}{s-3}\right)\,.$$

By taking the inverse Laplace transform we find that the solution is

$$y(t) = \left(\tfrac{7}{3} - \tfrac{7}{2}e^{t} + \tfrac{7}{6}e^{3t}\right) + u(t-2)\left(\tfrac{5}{3} - \tfrac{5}{2}e^{t-2} + \tfrac{5}{6}e^{3(t-2)}\right)\,.$$

9.11. **Convolutions.** Let $f(t)$ and $g(t)$ be any two functions that are defined over the interval $[0,\infty)$. Their *convolution* is a third function $(f * g)(t)$ that is defined by the formula

$$(f * g)(t) = \int_0^t f(t-\tau)g(\tau)\,\mathrm{d}\tau\,, \tag{9.38}$$

whenever the above integral makes sense for every $t \geq 0$. In particular, the convolution of f and g will be defined whenever both f and g are piecewise continuous over every $[0, T]$.

The convolution can be thought of a some kind of product between two functions. It is easily checked that this so-called convolution product satisfies some of the properties of ordinary multiplication. For example, for any functions f, g, amd h that are piecewise continuous over every $[0, T]$ we have

$$\begin{aligned} g * f &= f * g && \text{commutative law}\,,\\ h * (f + g) &= h * f + h * g && \text{distributive law}\,,\\ h * (g * f) &= (h * g) * f && \text{associative law}\,. \end{aligned}$$

The commutative law is proved by introducing $\tau' = t - \tau$ as a new variable of integration, whereby we see that

$$(g * f)(t) = \int_0^t g(t-\tau)f(\tau)\,\mathrm{d}\tau = \int_0^t g(\tau')f(t-\tau')\,\mathrm{d}\tau' = (f * g)(t)\,.$$

Verification of the distributive and associative laws is left to you.

The convolution differs from ordinary multiplication in some respects too. For example, it is not generally true that $f * 1 = f$ or that $f * f \geq 0$. Indeed, we see that

$$(1 * 1)(t) = \int_0^t 1 \cdot 1\,\mathrm{d}\tau = t \neq 1\,,$$

and that

$$
\begin{aligned}
(\sin * \sin)(t) &= \int_0^t \sin(t-\tau)\sin(\tau)\,\mathrm{d}\tau \\
&= \sin(t)\int_0^t \cos(\tau)\sin(\tau)\,\mathrm{d}\tau + \cos(t)\int_0^t \sin(\tau)^2\,\mathrm{d}\tau \\
&= \tfrac{1}{2}\sin(t)^3 + \tfrac{1}{2}t\cos(t) - \tfrac{1}{2}\sin(t)\cos(t)^2 \not\geq 0 \quad \text{for every } t > 0\,.
\end{aligned}
$$

In fact, we can show that $1 * f = f$ if and only if $f = 0$. However, for every function f that is continuous over $[0, \infty)$ we have

$$
(f * \delta)(t) = \int_0^t f(t-\tau)\,\delta(\tau)\,\mathrm{d}\tau = f(t)\,.
$$

Therefore we see that the unit impulse function δ acts like an identity.

The main result of this section is that the Laplace transform of a convolution of two functions is the ordinary product of their Laplace transforms. In other words, the Laplace transform maps convolutions to multiplication.

Theorem 9.8 (Convolution Theorem). Let $f(t)$ and $g(t)$ be

- piecewise continuous over every $[0, T]$
- of exponential order α as $t \to \infty$.

Then $\mathcal{L}[f * g](s)$ is defined for every $s > \alpha$ with

$$
\mathcal{L}[f * g](s) = F(s)G(s)\,, \quad \text{where } F(s) = \mathcal{L}[f](s) \text{ and } G(s) = \mathcal{L}[g](s)\,. \tag{9.39}
$$

Proof. For every $T > 0$ definition (9.38) of convolution implies that

$$
\int_0^T e^{-st}\,(f * g)(t)\,\mathrm{d}t = \int_0^T e^{-st}\int_0^t f(t-\tau)g(\tau)\,\mathrm{d}\tau\,\mathrm{d}t = \int_0^T\int_0^t e^{-st} f(t-\tau)g(\tau)\,\mathrm{d}\tau\,\mathrm{d}t\,.
$$

We now exchange the order of the definite integrals over τ and t on the right-hand side. As you recall from multivariable Calculus, this should be done carefully because the upper endpoint of the inner integral depends on the variable of integration t of the outer integral. When viewed in the (τ, t)-plane, the domain over which the double integral is being taken is the triangle given by $0 \leq \tau \leq t \leq T$. In general, when the order of definite integrals is exchanged over this domain we have

$$
\int_0^T\int_0^t \bullet \;\mathrm{d}\tau\,\mathrm{d}t = \int_0^T\int_\tau^T \bullet \;\mathrm{d}t\,\mathrm{d}\tau\,,
$$

where $\bullet$ denotes any appropriate integrand. We thereby obtain

$$
\int_0^T e^{-st}\,(f * g)(t)\,\mathrm{d}t = \int_0^T\int_\tau^T e^{-st} f(t-\tau)g(\tau)\,\mathrm{d}t\,\mathrm{d}\tau\,.
$$

We now factor e^{-st} as $e^{-st} = e^{-s(t-\tau)}e^{-s\tau}$, and group the factor $e^{-s(t-\tau)}$ with $f(t-\tau)$ and the factor $e^{-s\tau}$ with $g(\tau)$, whereby

$$\begin{aligned}\int_0^T e^{-st}\,(f*g)(t)\,\mathrm{d}t &= \int_0^T\int_\tau^T e^{-s(t-\tau)}f(t-\tau)\,e^{-s\tau}g(\tau)\,\mathrm{d}t\,\mathrm{d}\tau\\ &= \int_0^T e^{-s\tau}g(\tau)\int_\tau^T e^{-s(t-\tau)}f(t-\tau)\,\mathrm{d}t\,\mathrm{d}\tau\,.\end{aligned}$$

We then make the change of variable $t' = t-\tau$ in the inner definite integral to obtain

$$\int_0^T e^{-st}\,(f*g)(t)\,\mathrm{d}t = \int_0^T e^{-s\tau}g(\tau)\int_0^{T-\tau} e^{-st'}f(t')\,\mathrm{d}t'\,\mathrm{d}\tau\,.$$

Upon letting $T\to\infty$ above, definition (9.2) of the Laplace transform shows that the inner integral converges to $F(s)$, which is independent of τ. The double integral thereby converges to $G(s)F(s)$, yielding (9.39). □

Remark. Because the upper endpoint of the inner integral depends on the variable of integration τ of the outer integral, properly passing to the limit above requires greater care than we took here. The techniques needed are taught in Advanced Calculus courses. The argument given above suits our purposes because it illuminates why (9.39) holds.

The convolution theorem can be used to help evaluate inverse Laplace transforms. For example, suppose that we know for a given $F(s)$ and $G(s)$ that $f(t) = \mathcal{L}^{-1}[F](t)$ and $g(t) = \mathcal{L}^{-1}[G](t)$. Then (9.39) implies that

$$\mathcal{L}^{-1}\big[F(s)G(s)\big](t) = (f*g)(t)\,. \tag{9.40}$$

You can use this fact to express inverse Laplace transforms as convolutions. You may still have to evaluate the convolution integral, but some of you might find that easier than using partial fraction identities to express $F(s)G(s)$ in basic forms.

Example. Compute $y(t) = \mathcal{L}^{-1}[Y](t)$ for

$$Y(s) = \frac{2}{s^2(s^2+4)}\,.$$

Remark. We computed this inverse transform using a partial fraction identity at the end of Section 9.9. Here we take a different approach.

Solution. Because we know from table (9.17) that

$$\mathcal{L}^{-1}\bigg[\frac{1}{s^2}\bigg] = t\,,\qquad \mathcal{L}^{-1}\bigg[\frac{2}{s^2+2^2}\bigg] = \sin(2t)\,,$$

it follows from (9.40) and an integration by parts that

$$\begin{aligned}y(t) = \mathcal{L}^{-1}\bigg[\frac{2}{s^2(s^2+4)}\bigg] &= \mathcal{L}^{-1}\bigg[\frac{1}{s^2}\,\frac{2}{s^2+2^2}\bigg] = \int_0^t (t-\tau)\sin(2\tau)\,\mathrm{d}\tau\\ &= (\tau-t)\frac{\cos(2\tau)}{2}\bigg|_0^t - \int_0^t \frac{\cos(2\tau)}{2}\,\mathrm{d}\tau = \frac{t}{2} - \frac{\sin(2t)}{4}\,.\end{aligned}$$

This is the same result we got using a partial fraction identity.

The convolution theorem gives us another way to understand Green functions. We have used the Green function to construct a particular solution of the nonhomogeneous equation $\mathrm{L}y = p(\mathrm{D}) = f(t)$ by the formula

$$y_P(t) = \int_0^t g(t-\tau) f(\tau)\,\mathrm{d}\tau\,.$$

Notice that the right-hand side above is exactly $(g*f)(t)$. Taking the Laplace transform of this formula, the Convolution Theorem then yields

$$\mathcal{L}[y_P](s) = \mathcal{L}[g*f](s) = G(s)F(s) = \frac{F(s)}{p(s)}\,, \qquad \text{where } F(s) = \mathcal{L}[f](s)\,.$$

But this agrees with formula (9.15). Indeed, because $y_P(t)$ given by the above formula satisfies the initial conditions

$$y_P(0) = 0\,, \quad y_P'(0) = 0\,, \quad \cdots \quad y_P^{(n-2)}(0) = 0\,, \quad y_P^{(n-1)}(0) = 0\,,$$

it follows that the polynomial $q(s)$ appearing in (9.15) vanishes.

	$h(t) = \mathcal{L}^{-1}[H](t)$		$H(s) = \mathcal{L}[h](s)$	
1.	t^n	for $n \geq 0$	$\dfrac{n!}{s^{n+1}}$	for $s > 0$
2.	$\cos(bt)$		$\dfrac{s}{s^2 + b^2}$	for $s > 0$
3.	$\sin(bt)$		$\dfrac{b}{s^2 + b^2}$	for $s > 0$
4.	$\cosh(bt)$		$\dfrac{s}{s^2 - b^2}$	for $s > \|b\|$
5.	$\sinh(bt)$		$\dfrac{b}{s^2 - b^2}$	for $s > \|b\|$
6.	$t^n e^{at}$	for $n \geq 0$	$\dfrac{n!}{(s-a)^{n+1}}$	for $s > a$
7.	$e^{at}\cos(bt)$		$\dfrac{s-a}{(s-a)^2 + b^2}$	for $s > a$
8.	$e^{at}\sin(bt)$		$\dfrac{b}{(s-a)^2 + b^2}$	for $s > a$
9.	$e^{at}\cosh(bt)$		$\dfrac{s-a}{(s-a)^2 - b^2}$	for $s > a + \|b\|$
10.	$e^{at}\sinh(bt)$		$\dfrac{b}{(s-a)^2 - b^2}$	for $s > a + \|b\|$
11.	$t^n j(t)$	for $n \geq 0$	$(-1)^n J^{(n)}(s)$	where $J(s) = \mathcal{L}[j](s)$
12.	$j'(t)$		$s\,J(s) - j(0)$	where $J(s) = \mathcal{L}[j](s)$
13.	$e^{at} j(t)$		$J(s-a)$	where $J(s) = \mathcal{L}[j](s)$
14.	$u(t-c)j(t-c)$	for $c \geq 0$	$e^{-cs} J(s)$	where $J(s) = \mathcal{L}[j](s)$
15.	$\delta(t-c)j(t)$	for $c \geq 0$	$e^{-cs} j(c)$	

Here a, b, and c are real numbers; n is an integer; $j(t)$ is any function that is nice enough; $u(t)$ is the unit step (Heaviside) function; $\delta(t)$ is the unit impulse (Dirac delta).

	$h(t) = \mathcal{L}^{-1}[H](t)$		$H(s) = \mathcal{L}[h](s)$	
16.	$e^{(a+ib)t}$		$\dfrac{1}{s-a-ib}$	for $s > a$
17.	$t^n e^{(a+ib)t}$	for $n \geq 0$	$\dfrac{n!}{(s-a-ib)^{n+1}}$	for $s > a$
18.	$t^n e^{at} \cos(bt)$	for $n \geq 0$	$\mathrm{Re}\left(\dfrac{n!}{(s-a-ib)^{n+1}}\right)$	for $s > a$
19.	$t^n e^{at} \sin(bt)$	for $n \geq 0$	$\mathrm{Im}\left(\dfrac{n!}{(s-a-ib)^{n+1}}\right)$	for $s > a$
20.	$t\ \cos(bt)$		$\dfrac{s^2-b^2}{(s^2+b^2)^2}$	for $s > 0$
21.	$t\ \sin(bt)$		$\dfrac{2bs}{(s^2+b^2)^2}$	for $s > 0$
22.	$t^2 \cos(bt)$		$2\,\dfrac{s^3-3b^2s}{(s^2+b^2)^3}$	for $s > 0$
23.	$t^2 \sin(bt)$		$2\,\dfrac{3bs^2-b^3}{(s^2+b^2)^3}$	for $s > 0$
24.	t^p	for $p > -1$	$\dfrac{\Gamma(p+1)}{s^{p+1}}$	for $s > 0$
25.	$\log(t)$		$-\dfrac{\log(s)+\gamma}{s}$	for $s > 0$
26.	$j(at)$	for $a > 0$	$\dfrac{1}{a}\,J\left(\dfrac{s}{a}\right)$	where $J(s) = \mathcal{L}[j](s)$
27.	$\dfrac{1}{b}\,j\left(\dfrac{t}{b}\right)$	for $b > 0$	$J(bs)$	where $J(s) = \mathcal{L}[j](s)$
28.	$\displaystyle\int_0^t j(r)\,\mathrm{d}r$		$\dfrac{1}{s}\,J(s)$	where $J(s) = \mathcal{L}[j](s)$
29.	$\dfrac{1}{t}\,j(t)$		$\displaystyle\int_s^\infty J(r)\,\mathrm{d}r$	where $J(s) = \mathcal{L}[j](s)$
30.	$j(t) = j(t+P)$	for $P > 0$	$\dfrac{1}{1-e^{-Ps}}\displaystyle\int_0^P e^{-st} j(t)\,\mathrm{d}t$	for $s > 0$

Here a, b, and p are real numbers; n is an integer; $j(t)$ is any function that is nice enough; $\Gamma(r)$ is the Gamma function; $\gamma = -\Gamma'(1)$ is the Euler constant; P is a period.

Remark. The Γ function is defined for every $r > 0$ by the improper definite integral

$$\Gamma(r) = \int_0^\infty e^{-x} x^{r-1}\,\mathrm{d}x\,. \tag{9.41}$$

For every $p > 0$ an integration by parts yields the identity

$$\Gamma(p+1) = \int_0^\infty e^{-x} x^{p}\,\mathrm{d}x = -x^p e^{-x}\Big|_0^\infty + p\int_0^\infty e^{-x} x^{p-1}\,\mathrm{d}x = p\,\Gamma(p)\,.$$

Repeated application of this identity shows that for every $p > 0$ and every positive integer n we have

$$\Gamma(p+n) = p(p+1)(p+2)\cdots(p+n-1)\Gamma(p)\,. \tag{9.42}$$

It is easy to show that important special values are

$$\Gamma(1) = \int_0^\infty e^{-x}\,\mathrm{d}x = 1\,, \qquad \Gamma(2) = \int_0^\infty e^{-x} x\,\mathrm{d}x = 1\,. \tag{9.43}$$

By taking $p = 1$ in (9.42) we see that for every positive integer n we have

$$\Gamma(n+1) = \int_0^\infty e^{-x} x^n\,\mathrm{d}x = n!\,.$$

Therefore the Γ function is an extension of the factorial function.

It can be shown that other important special values are

$$\Gamma(\tfrac{1}{2}) = \int_0^\infty e^{-x} x^{-\frac{1}{2}}\,\mathrm{d}x = \sqrt{\pi}\,, \qquad \Gamma(\tfrac{3}{2}) = \int_0^\infty e^{-x} x^{\frac{1}{2}}\,\mathrm{d}x = \tfrac{1}{2}\sqrt{\pi}\,.$$

By taking $p = \frac{1}{2}$ in (9.42) we see that for every positive integer n we have

$$\Gamma(n+\tfrac{1}{2}) = \tfrac{1}{2}\tfrac{3}{2}\cdots(n-\tfrac{3}{2})(n-\tfrac{1}{2})\sqrt{\pi}\,.$$

It can also be shown that $\Gamma(r)$ given by (9.41) is twice differentiable over $r > 0$ with

$$\Gamma'(r) = \int_0^\infty e^{-x} x^{r-1}\log(x)\,\mathrm{d}x\,, \qquad \Gamma''(r) = \int_0^\infty e^{-x} x^{r-1}\big(\log(x)\big)^2\,\mathrm{d}x\,. \tag{9.44}$$

Because $\Gamma''(r) > 0$ for every $r > 0$ we see that $\Gamma(r)$ is strictly convex over $r > 0$. Because (9.43) shows that $\Gamma(1) = \Gamma(2) = 1$, we conclude that $\Gamma(r)$ has a minimizer in the interval $(1, 2)$.

From (9.44) and (9.43) we see that

$$\Gamma'(1) = \int_0^\infty e^{-x}\log(x)\,\mathrm{d}x = -\int_0^\infty e^{-x}\big(x - 1 - \log(x)\big)\,\mathrm{d}x\,,$$
$$\Gamma'(2) = \int_0^\infty e^{-x} x\log(x)\,\mathrm{d}x = \int_0^\infty e^{-x}\big(x\log(x) - x + 1\big)\,\mathrm{d}x\,.$$

Because we have the elementary inequalities

$$\log(x) < x - 1 < x\log(x) \qquad \text{for every } x > 0 \text{ such that } x \neq 1\,,$$

the integrands of both rightmost integrals above are nonnegative. Finally, an integration by parts applied to the last integral shows that the Euler constant $\gamma = -\Gamma'(1)$ statisfies

$$\Gamma'(2) = 1 + \Gamma'(1) = 1 - \gamma\,.$$

Exercises on the Laplace Transform Method

For problems # 1 – 5 show the Laplace Transform for the given function is what the chapter states

(1) Use integration by parts and the definition of the Laplace transform to justify why
$\mathcal{L}\{y'\} = s\mathcal{L}\{y\} - y(0)$ for $s > 0$ whenever $y(t)$ is bounded.

Solution

(2) Show that $\mathcal{L}[x^2 e^x](s) = \frac{2}{(s-1)^3}$.

Solution

(3) Show that $\mathcal{L}[\cos(t)](s) = \frac{s}{s^2+1}$.

Solution

(4) Show that $\mathcal{L}[u(v-2)](s) = \frac{e^{-2s}}{s}$.

Solution

(5) Show that $\mathcal{L}[u(x-3)x](s) = \dfrac{e^{-3s} + 3se^{-3s}}{s^2}$.

Solution

For problems # 6 – 11 find the function which has the given Laplace Transform

(6) $Y(s) = \dfrac{e^{-\pi s}}{s^2 + 2s + 2}$

Solution

(7) $Y(s) = \dfrac{s+1}{s^2 - s - 6}$

Short Answer
Solution

(8) $Y(s) = \dfrac{s}{s^2 + 2s + 2}$

Solution

(9) $Y(s) = \dfrac{8}{s(s^2 - 4)}$

Short Answer
Solution

(10) $Y(s) = \dfrac{e^{-3s}}{s^2 + 6s + 9}$

Solution

(11) $Y(s) = \dfrac{e^s}{s^3}$

Solution

Solve the given initial value problems using the Laplace Transform

(12) $y'' + 4y' - 21y = 0$ where $y(0) = 2$ and $y'(0) = 3$.

Short Answer
Solution

(13) $y'' - y' - 2y = 4xe^x$; $y(0) = 2$; $y'(0) = 0$.

Short Answer
Solution

(14) Use the Laplace transform to solve $y'' - y = f(t)$ where $f(t) = 1$ for $t < 1$ and 0 everywhere else and $y(0) = 2$ and $y'(0) = 3$.

Short Answer
Solution

(15)

$$\text{Let } f(v) = \begin{cases} \sin(v) & \text{if } 0 \le v < 2\pi \\ v - 2\pi & \text{if } v \ge 2\pi \end{cases}$$

and consider the initial-value problem

$$y'' + y' - 6y = f(v);\ y(0) = 1,\ y'(0) = 2.$$

Short Answer
Solution

Use the Laplace Transform to find the Green function for the given differential operator

(16) $\mathrm{D}^2 + 6\mathrm{D} + 9$

Short Answer
Solution

(17) $\mathrm{D}^2 - 4\mathrm{D} - 12$

Short Answer
Solution

(18) $\mathrm{D}^3 - 3\mathrm{D}^2 - 4\mathrm{D} + 12$

Short Answer
Solution

(19) $\mathrm{D}^4 + 3\mathrm{D}^2 - 4$

Short Answer
Solution

(20) Find a solution to the following initial value problem:

$$2v'' + v' + 2v = \delta(t - 10),$$
$$v(0) = 0, v'(0) = 0.$$

Solution

(21) **Remark** : Here we compare and contrast the methods of Laplace transforms with Green functions for obtaining the general solution to a second-order constant coefficient non-homogeneous differential equation, with prescribed initial conditions.

a) Using Green's functions, show that the solution to the initial value problem

$$w'' + 2w' + 2w = f(v), w(0) = 0, w'(0) = 0,$$

is the following:

$$w(v) = \int_0^v e^{-(v-s)} f(s) \sin(v - s) ds.$$

b) Show that if $f(v) = \delta(v - \pi)$, then the solution in part a) becomes $w(v) = u_\pi(v) e^{-(v-\pi)} \sin(v - \pi)$.

c) Now use the method of Laplace transforms to solve the initial value problem prescribed in a) with $f(v) = \delta(v-\pi)$, and show that the solution obtained in this case agrees with the solution derived in b).

Short Answer
Solution

(22) Suppose that $g(u) = \int_0^u f(s)ds$. If $G(s)$ and $F(s)$ are the Laplace transforms of $g(u)$ and $f(u)$ respectively, show that $G(s) = \frac{F(s)}{s}$.

Solution

(23) **Remark** : Here we explore the fact that Laplace transform might not be useful in solving homogeneous equations with non-constant coefficients, especially when the coefficients at play are not linear functions of the independent variable. We explore this observation in the following two examples below.

By taking the Laplace transform of differential equations with prescribed initial conditions below, show that the differential equation for $Y(s) = \mathcal{L}(y(v))$ is of first order in part a), and of second order in part b).

a) $y'' - vy = 0, y(0) = 1, y'(0) = 0;$
b) $(1-v^2)y'' - 2vy' + \alpha(\alpha+1)y = 0, y(0) = 0, y'(0) = 1.$

Solution

(24) **A bit on integral equations**

Consider the following equation:

$$w(t) + \int_0^t h(t-s)w(s)ds = g(t),$$

where $h(t)$ and $g(t)$ are functions known a priori, and $w(t)$ is our function to be determined. This type of equation belongs to the class of integral equations, because the unknown function $w(t)$ also appears in integral form. Take the Laplace transform of the equation above, and obtain a closed form solution for $\mathcal{L}(w(t))$ in terms of the Laplace transforms for $\mathcal{L}(g(t))$ and $\mathcal{L}(h(t))$. The inverse transform of $\mathcal{L}(w(t))$ would then yield the true solution, $w(t)$ of the original integral equation.

Solution

(25) **An application of Laplace transforms and integral equations**

Consider the following integral equation:

$$v(t) + \int_0^t (t-s)v(s)ds = \sin(2s).$$

a) Solve the integral equation using Laplace transform, using the method outlined in the previous problem.

b) Differentiate the differential equation $v(t) + \int_0^t (t-s)v(s)ds = \sin(2s)$ twice and write down the differential equation that you obtain.

Also show that the initial conditions satisfy the following:

$$v(0) = 0, v'(0) = 2.$$

c) Solve the initial value problem in b). How does it compare to the solution you derived in part a)?

Solution

(26) **The connection between gamma functions and the Laplace transforms**

The gamma function $\Gamma(p)$ is defined by the following integral

$$\Gamma(p+1) = \int_0^\infty e^{-x} x^p dx.$$

Let us now study some of its properties.

a) Show that for $p > 0$, $\Gamma(p+1) = p\Gamma(p)$. Subsequently, show that $\Gamma(1) = 1$.

b) Show that $\Gamma(n+1) = n!$, where n is a positive integer.

c) Show that for $p > 0$, $p(p+1)(p+2)...(p+n-1) = \frac{\Gamma(p+n)}{\Gamma(p)}$. It is possible to show that $\Gamma(\frac{1}{2}) = \sqrt{\pi}$. Knowing this fact, find $\Gamma(\frac{3}{2})$ and $\Gamma(\frac{11}{2})$.

Short Answer
Solution

(27) Now that we've defined and explored some of the properties of the gamma function, let us consider the Laplace transform of x^p, for positive p.

a) Show that $\mathcal{L}(x^p) = \int_0^\infty e^{-sx} x^p dx = \dfrac{1}{s^{p+1}} \int_0^\infty e^{-y} y^p dy = \dfrac{\Gamma(p+1)}{s^{p+1}}$, for $p > 0$.

b) Show that $\mathcal{L}(x^n) = \dfrac{n!}{s^{n+1}}$, for n a positive integer and $s > 0$.

c) Show that $\mathcal{L}(x^{-1/2}) = \dfrac{2}{\sqrt{s}} \int_0^\infty e^{-x^2} dx$, where $s > 0$.

d) Finally, show that $\mathcal{L}(x^{1/2}) = \dfrac{\sqrt{\pi}}{2s^{3/2}}$, for $s > 0$.

Short Answer
Solution

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