III. First-Order Systems of Ordinary Differential Equations
2. Linear Systems: General Methods and Theory

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2. Linear Systems: General Methods and Theory

The n-dimensional first-order system 1.1 is called *linear* when it has the form

$$
\begin{aligned}
\frac{\mathrm{d}x_1}{\mathrm{d}t} &= a_{11}(t)x_1 + a_{12}(t)x_2 + \cdots + a_{1n}(t)x_n + f_1(t)\,, \\
\frac{\mathrm{d}x_2}{\mathrm{d}t} &= a_{21}(t)x_1 + a_{22}(t)x_2 + \cdots + a_{2n}(t)x_n + f_2(t)\,, \\
&\vdots \\
\frac{\mathrm{d}x_n}{\mathrm{d}t} &= a_{n1}(t)x_1 + a_{n2}(t)x_2 + \cdots + a_{nn}(t)x_n + f_n(t)\,.
\end{aligned} \tag{2.1}
$$

The functions $a_{jk}(t)$ are called *coefficients* while the functions $f_j(t)$ are called *forcings*.

We can use *matrix notation* to write the linear system (2.1) compactly as

$$
\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} = \mathbf{A}(t)\mathbf{x} + \mathbf{f}(t)\,, \tag{2.2}
$$

where $\mathbf{x}$ and $\mathbf{f}(t)$ are the n-dimensional column vectors

$$
\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}, \qquad \mathbf{f}(t) = \begin{pmatrix} f_1(t) \\ f_2(t) \\ \vdots \\ f_n(t) \end{pmatrix},
$$

while $\mathbf{A}(t)$ is the $n{\times}n$ matrix

$$
\mathbf{A}(t) = \begin{pmatrix} a_{11}(t) & a_{12}(t) & \cdots & a_{1n}(t) \\ a_{21}(t) & a_{22}(t) & \cdots & a_{2n}(t) \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1}(t) & a_{n2}(t) & \cdots & a_{nn}(t) \end{pmatrix}.
$$

We call $\mathbf{A}(t)$ the *coefficient matrix* and $\mathbf{f}(t)$ the *forcing vector*. System (2.2) is said to be *homogeneous* if $\mathbf{f}(t) = 0$ and *nonhomogeneous* otherwise.

The product $\mathbf{A}(t)\mathbf{x}$ appearing in system (2.2) denotes the column vector that results from the *matrix multiplication* of the matrix $\mathbf{A}(t)$ with the column vector $\mathbf{x}$. The sum appearing in (2.2) denotes the column vector that results from the *matrix addition* of the column vector $\mathbf{A}(t)\mathbf{x}$ with the column vector $\mathbf{f}(t)$. These matrix operations are presented in the next chapter. You should familiarize yourself with them before going further in this chapter.

Remark. Any n^{th}-order linear equation in the normal form

$$
\frac{\mathrm{d}^n y}{\mathrm{d}t^n} + a_1(t)\frac{\mathrm{d}^{n-1}y}{\mathrm{d}t^{n-1}} + \cdots + a_{n-1}(t)\frac{\mathrm{d}y}{\mathrm{d}t} + a_n(t)y = f(t)\,,
$$

can be recast as the n-dimensional first-order linear system

$$\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} = \mathbf{A}(t)\mathbf{x} + \mathbf{f}(t)\,, \qquad \text{where} \quad \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} y \\ y' \\ \vdots \\ y^{(n-1)} \end{pmatrix}\,,$$

with

$$\mathbf{A}(t) = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & 0 & 1 & 0 \\ 0 & \cdots & 0 & 0 & 1 \\ -a_n(t) & \cdots & -a_3(t) & -a_2(t) & -a_1(t) \end{pmatrix}\,, \qquad \mathbf{f}(t) = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ f(t) \end{pmatrix}\,.$$

Therefore the study of first-order linear systems contains the study of higher-order linear equations. Later we will see that solving a first-order linear system can be reduced to solving a higher-order linear equation.

2.1. **Initial-Value Problems.** We will consider linear initial-value problems in the form

$$\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} = \mathbf{A}(t)\mathbf{x} + \mathbf{f}(t)\,, \qquad \mathbf{x}(t_I) = \mathbf{x}^I\,, \tag{2.3}$$

where $\mathbf{x}^I$ is called the *vector of initial values*, or simply the *initial vector*.

A major theme of this chapter is that for every fact that we studied about higher-order linear equations there is an analogous fact about linear first-order systems. For example, the basic existence and uniqueness theorem is the following.

> **Theorem 2.1.** If $\mathbf{A}(t)$ and $\mathbf{f}(t)$ are continuous over the time interval (t_L, t_R) then for every initial time t_I in (t_L, t_R) and every initial vector $\mathbf{x}^I$ the initial-value problem (2.3) has a unique solution $\mathbf{x}(t)$ that is continuously differentiable over (t_L, t_R). Moreover, if $\mathbf{A}(t)$ and $\mathbf{f}(t)$ are k-times continuously differentiable over the time interval (t_L, t_R) then $\mathbf{x}(t)$ will be $(k+1)$-times continuously differentiable over (t_L, t_R).

You should be able to use the Basic Existence and Uniqueness Theorem to identify the interval of definition for solutions of the initial-value problem (2.3). This is done very much like the way you identified intervals of definition for solutions of higher-order linear equations. Specifically, if $\mathbf{x}(t)$ is the solution of the initial-value problem (2.3) then its interval of definition will be (t_L, t_R) whenever:

- the initial time t_I is in (t_L, t_R),
- all the entries of the coefficient matrix $\mathbf{A}(t)$ and of the forcing vector $\mathbf{f}(t)$ are continuous over (t_L, t_R),
- at least one entry of either the coefficient matrix $\mathbf{A}(t)$ or the forcing vector $\mathbf{f}(t)$ is undefined at each of $t = t_L$ and $t = t_R$.

You can do this because the first two bullets along with the Basic Existence and Uniqueness Theorem imply that the interval of definition will be at least (t_L, t_R), while the last bullet along with our definition of solution imply that the interval of definition can be no bigger than (t_L, t_R) because the equation breaks down at $t = t_L$ and $t = t_R$. This argument works when $t_L = -\infty$ or $t_R = \infty$.

2.2. **Homogeneous Systems.** Just as with higher-order linear equations, the key to solving a first-order linear system (2.2) is understanding how to solve its associated homogeneous system

$$\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} = \mathbf{A}(t)\mathbf{x}\,. \tag{2.4}$$

We will assume throughout this chapter that the coefficient matrix $\mathbf{A}(t)$ is continuous over an interval (t_L, t_R), so that Therorem 1.1 can be applied. We will exploit the following property of homogeneous systems.

Theorem 2.2. (Linear Superposition). If $\mathbf{x}_1(t)$ and $\mathbf{x}_2(t)$ are solutions of system (2.4) then so is $c_1\mathbf{x}_1(t) + c_2\mathbf{x}_2(t)$ for any values of the constants c_1 and c_2. More generally, if $\mathbf{x}_1(t)$, $\mathbf{x}_2(t)$, $\cdots$, $\mathbf{x}_m(t)$ are m solutions of system (2.4) then so is the linear combination

$$\mathbf{x}(t) = c_1\mathbf{x}_1(t) + c_2\mathbf{x}_2(t) + \cdots + c_m\mathbf{x}_m(t)\,, \tag{2.5}$$

for any values of the constants c_1, c_2, $\cdots$, c_m.

Remark. Here $\mathbf{x}_1(t)$, $\mathbf{x}_2(t)$, $\cdots$, $\mathbf{x}_m(t)$ denote m different vector-valued solutions of the system (2.4), and should not be confused with $x_1(t)$, $x_2(t)$, $\cdots$, $x_m(t)$, which denote the first m entries of the vector-valued function $\mathbf{x}(t)$.

Reason. Because $\mathbf{x}_1(t)$, $\mathbf{x}_2(t)$ $\cdots$, $\mathbf{x}_m(t)$ solve (2.4), a direct calculation starting from the linear combination (2.5) shows that

$$\begin{aligned}
\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t}(t) &= \frac{\mathrm{d}}{\mathrm{d}t}\big(c_1\mathbf{x}_1(t) + c_2\mathbf{x}_2(t) + \cdots + c_m\mathbf{x}_m(t)\big) \\
&= c_1\frac{\mathrm{d}\mathbf{x}_1}{\mathrm{d}t}(t) + c_2\frac{\mathrm{d}\mathbf{x}_2}{\mathrm{d}t}(t) + \cdots + c_m\frac{\mathrm{d}\mathbf{x}_m}{\mathrm{d}t}(t) \\
&= c_1\mathbf{A}(t)\mathbf{x}_1(t) + c_2\mathbf{A}(t)\mathbf{x}_2(t) + \cdots + c_m\mathbf{A}(t)\mathbf{x}_m(t) \\
&= \mathbf{A}(t)\big(c_1\mathbf{x}_1(t) + c_2\mathbf{x}_2(t) + \cdots + c_m\mathbf{x}_m(t)\big) \\
&= \mathbf{A}(t)\mathbf{x}(t)\,.
\end{aligned}$$

Therefore $\mathbf{x}(t)$ given by the linear combination (2.5) solves system (2.4). $\square$

Remark. This theorem states that any linear combination of solutions of (2.4) is also a solution of (2.4). It thereby provides a way to construct a whole family of solutions from a finite number of them.

Now consider the initial-value problem

$$\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} = \mathbf{A}(t)\mathbf{x}\,, \qquad \mathbf{x}(t_I) = \mathbf{x}^I\,. \tag{2.6}$$

Suppose we know n "different" solutions of (2.4), $\mathbf{x}_1(t)$, $\mathbf{x}_2(t)$, $\cdots$, $\mathbf{x}_n(t)$. It is natural to ask if we can construct the solution of the initial-value problem (2.6) as a linear combination of $\mathbf{x}_1(t)$, $\mathbf{x}_2(t)$, $\cdots$, $\mathbf{x}_n(t)$. Set

$$\mathbf{x}(t) = c_1\mathbf{x}_1(t) + c_2\mathbf{x}_2(t) + \cdots + c_n\mathbf{x}_n(t)\,. \tag{2.7}$$

By the superposition theorem this is a solution of (2.4). We only have to check that values of c_1, c_2, $\cdots$, c_n can be found so that $\mathbf{x}(t)$ will also satisfy the initial conditions in (2.6) — namely, so that

$$\mathbf{x}^I = \mathbf{x}(t_I) = c_1\mathbf{x}_1(t_I) + c_2\mathbf{x}_2(t_I) + \cdots + c_n\mathbf{x}_n(t_I) = \mathbf{\Psi}(t_I)\mathbf{c}\,,$$

where $\mathbf{\Psi}(t_I)$ is the $n\times n$ matrix given by

$$\mathbf{\Psi}(t_I) = \begin{pmatrix} \mathbf{x}_1(t_I) & \mathbf{x}_2(t_I) & \cdots & \mathbf{x}_n(t_I) \end{pmatrix}\,,$$

and $\mathbf{c}$ is the $n\times 1$ column vector given by

$$\mathbf{c} = \begin{pmatrix} c_1 & c_2 & \cdots & c_n \end{pmatrix}^{\mathrm{T}}\,.$$

This notation indicates that the k^{th} column of $\mathbf{\Psi}(t_I)$ is the column vector $\mathbf{x}_k(t_I)$. In this notation the question becomes whether there is a vector $\mathbf{c}$ such that

$$\mathbf{\Psi}(t_I)\mathbf{c} = \mathbf{x}^I\,, \quad \text{for every } \mathbf{x}^I\,.$$

This linear algebraic system will have a solution for every $\mathbf{x}^I$ if and only if the matrix $\mathbf{\Psi}(t_I)$ is invertible, in which case the solution is unique and is given by

$$\mathbf{c} = \mathbf{\Psi}(t_I)^{-1}\mathbf{x}^I\,.$$

Of course, the matrix $\mathbf{\Psi}(t_I)$ is invertible if and only if $\det\big(\mathbf{\Psi}(t_I)\big) \neq 0$.

2.3. **Wronskians and Fundamental Matrices.** Given any set of n solutions $\mathbf{x}_1(t)$, $\mathbf{x}_2(t)$, $\cdots$, $\mathbf{x}_n(t)$ to the homogeneous equation (2.4), we define its *Wronskian* by

$$\mathrm{Wr}[\mathbf{x}_1, \mathbf{x}_2, \cdots, \mathbf{x}_n](t) = \det\begin{pmatrix} \mathbf{x}_1(t) & \mathbf{x}_2(t) & \cdots & \mathbf{x}_n(t) \end{pmatrix}\,. \tag{2.8}$$

The analog of *Abel's Wronskian Theorem* for first-order systems is *Liouville's Wronskian Theorem*. It states that

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{Wr}[\mathbf{x}_1, \mathbf{x}_2, \cdots, \mathbf{x}_n](t) = \mathrm{tr}\big(\mathbf{A}(t)\big)\,\mathrm{Wr}[\mathbf{x}_1, \mathbf{x}_2, \cdots, \mathbf{x}_n](t)\,, \tag{2.9}$$

where $\mathrm{tr}\big(\mathbf{A}(t)\big)$ denotes the *trace* of $\mathbf{A}(t)$, which is given by

$$\mathrm{tr}\big(\mathbf{A}(t)\big) = a_{11}(t) + a_{22}(t) + \cdots + a_{nn}(t)\,. \tag{2.10}$$

Upon integrating the first-order linear equation (2.9) we obtain

$$\mathrm{Wr}[\mathbf{x}_1, \mathbf{x}_2, \cdots, \mathbf{x}_n](t) = \mathrm{Wr}[\mathbf{x}_1, \mathbf{x}_2, \cdots, \mathbf{x}_n](t_I)\,\exp\left(\int_{t_I}^{t} \mathrm{tr}\big(\mathbf{A}(s)\big)\,\mathrm{d}s\right)\,. \tag{2.11}$$

As was the case for higher-order equations, this shows that if the Wronkian is nonzero somewhere then it is nonzero everywhere, and that if it is zero somewhere, it is zero everywhere!

Again analogous to the higher-order equations case, we have the following definitions.

Definition. A set of n solutions $\mathbf{x}_1(t)$, $\mathbf{x}_2(t)$, $\cdots$, $\mathbf{x}_n(t)$ to the n-dimensional homogeneous linear system (2.4) is called *fundamental* if its Wronskian is nonzero. Then the family

$$\mathbf{x}(t) = c_1\mathbf{x}_1(t) + c_2\mathbf{x}_2(t) + \cdots + c_n\mathbf{x}_n(t) \tag{2.12}$$

is called a *general solution* of system (2.4).

However, now we introduce a new concept for first-order systems.

Definition. If $\mathbf{x}_1(t)$, $\mathbf{x}_2(t)$, $\cdots$, $\mathbf{x}_n(t)$ is a fundamental set of solutions to system (2.4) then the $n\times n$ matrix-valued function

$$\boldsymbol{\Psi}(t) = \begin{pmatrix} \mathbf{x}_1(t) & \mathbf{x}_2(t) & \cdots & \mathbf{x}_n(t) \end{pmatrix} \tag{2.13}$$

is called a *fundamental matrix* for system (2.4).

Some basic facts about fundamental matrices are as follows.

Fact. Let $\boldsymbol{\Psi}(t)$ be a fundamental matrix for system (2.4). Then

- $\boldsymbol{\Psi}(t)$ satisfies

$$\boldsymbol{\Psi}' = \mathbf{A}(t)\boldsymbol{\Psi}\,, \qquad \det\big(\boldsymbol{\Psi}(t)\big) \neq 0\,; \tag{2.14}$$

- A general solution of system (2.4) is

$$\mathbf{x}(t) = \boldsymbol{\Psi}(t)\mathbf{c}\,;$$

Reason. By (2.13) we see that

$$\begin{aligned}
\boldsymbol{\Psi}'(t) &= \begin{pmatrix} \mathbf{x}_1(t) & \mathbf{x}_2(t) & \cdots & \mathbf{x}_n(t) \end{pmatrix}' = \begin{pmatrix} \mathbf{x}_1'(t) & \mathbf{x}_2'(t) & \cdots & \mathbf{x}_n'(t) \end{pmatrix} \\
&= \begin{pmatrix} \mathbf{A}(t)\mathbf{x}_1(t) & \mathbf{A}(t)\mathbf{x}_2(t) & \cdots & \mathbf{A}(t)\mathbf{x}_n(t) \end{pmatrix} \\
&= \mathbf{A}(t)\begin{pmatrix} \mathbf{x}_1(t) & \mathbf{x}_2(t) & \cdots & \mathbf{x}_n(t) \end{pmatrix} \\
&= \mathbf{A}(t)\boldsymbol{\Psi}(t)\,.
\end{aligned}$$

Also by (2.13) we see that

$$\begin{aligned}
\det\big(\boldsymbol{\Psi}(t)\big) &= \det\begin{pmatrix} \mathbf{x}_1(t) & \mathbf{x}_2(t) & \cdots & \mathbf{x}_n(t) \end{pmatrix} \\
&= \mathrm{Wr}[\mathbf{x}_1, \mathbf{x}_2, \cdots, \mathbf{x}_n](t) \neq 0\,.
\end{aligned}$$

It should be clear from (2.13) that the general solution given by (2.12) can be expressed as $\mathbf{x}(t) = \boldsymbol{\Psi}(t)\mathbf{c}$.

Example. The vector-valued functions

$$\mathbf{x}_1(t) = e^{5t}\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \qquad \mathbf{x}_2(t) = e^{t}\begin{pmatrix} 1 \\ -1 \end{pmatrix},$$

are solutions of the differential system

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 3 & 2 \\ 2 & 3 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}.$$

Construct a general solution and a fundamental matrix for this system.

Solution. It is easy to check that $\mathbf{x}_1(t)$ and $\mathbf{x}_2(t)$ are each solutions to the differential system. Because

$$\mathrm{Wr}[\mathbf{x}_1, \mathbf{x}_2](t) = \det \begin{pmatrix} e^{5t} & e^{t} \\ e^{5t} & -e^{t} \end{pmatrix} = -2e^{6t} \,,$$

we see that $\mathbf{x}_1(t)$ and $\mathbf{x}_2(t)$ comprise a fundamental set of solutions to the system. Therefore a fundamental matrix is given by

$$\mathbf{\Psi}(t) = \begin{pmatrix} \mathbf{x}_1(t) & \mathbf{x}_2(t) \end{pmatrix} = \begin{pmatrix} e^{5t} & e^{t} \\ e^{5t} & -e^{t} \end{pmatrix} \,,$$

while a general solution is given by

$$\mathbf{x}(t) = \mathbf{\Psi}(t)\mathbf{c} = \begin{pmatrix} e^{5t} & e^{t} \\ e^{5t} & -e^{t} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} c_1 e^{5t} + c_2 e^{t} \\ c_1 e^{5t} - c_2 e^{t} \end{pmatrix} \,.$$

Alternatively, we can construct a general solution as

$$\mathbf{x}(t) = c_1\mathbf{x}_1(t) + c_2\mathbf{x}_2(t) = c_1 \begin{pmatrix} e^{5t} \\ e^{5t} \end{pmatrix} + c_2 \begin{pmatrix} e^{t} \\ -e^{t} \end{pmatrix} = \begin{pmatrix} c_1 e^{5t} + c_2 e^{t} \\ c_1 e^{5t} - c_2 e^{t} \end{pmatrix} \,.$$

Example. The vector-valued functions

$$\mathbf{x}_1(t) = \begin{pmatrix} 1 + t^3 \\ t \end{pmatrix} \,, \qquad \mathbf{x}_2(t) = \begin{pmatrix} t^2 \\ 1 \end{pmatrix} \,,$$

are solutions of the differential system

$$\frac{\mathrm{d}}{\mathrm{d}t} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} t^2 & 2t - t^4 \\ 1 & -t^2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \,.$$

Construct a general solution and a fundamental matrix for this system.

Solution. It is easy to check that $\mathbf{x}_1(t)$ and $\mathbf{x}_2(t)$ are each solutions to the differential system. Because

$$\mathrm{Wr}[\mathbf{x}_1, \mathbf{x}_2](t) = \det \begin{pmatrix} 1 + t^3 & t^2 \\ t & 1 \end{pmatrix} = 1 \,,$$

we see that $\mathbf{x}_1(t)$ and $\mathbf{x}_2(t)$ comprise a fundamental set of solutions to the system. Therefore a fundamental matrix is given by

$$\mathbf{\Psi}(t) = \begin{pmatrix} \mathbf{x}_1(t) & \mathbf{x}_2(t) \end{pmatrix} = \begin{pmatrix} 1 + t^3 & t^2 \\ t & 1 \end{pmatrix} \,,$$

while a general solution is given by

$$\mathbf{x}(t) = \mathbf{\Psi}(t)\mathbf{c} = \begin{pmatrix} 1 + t^3 & t^2 \\ t & 1 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} c_1(1 + t^3) + c_2 t^2 \\ c_1 t + c_2 \end{pmatrix} \,.$$

Alternatively, we can construct a general solution as

$$\mathbf{x}(t) = c_1\mathbf{x}_1(t) + c_2\mathbf{x}_2(t) = c_1 \begin{pmatrix} 1 + t^3 \\ t \end{pmatrix} + c_2 \begin{pmatrix} t^2 \\ 1 \end{pmatrix} = \begin{pmatrix} c_1(1 + t^3) + c_2 t^2 \\ c_1 t + c_2 \end{pmatrix} \,.$$

Remark. The solutions $\mathbf{x}_1(t)$ and $\mathbf{x}_2(t)$ were given to you in the problems above. Chapters 4 and 5 will present methods by which we can construct a fundamental set of solutions (and therefore a fundamental matrix) for any homogeneous system with a *constant* coefficient matrix. For systems with a *variable* coefficient matrix you will always be given solutions.

Remark. Any matrix-valued function $\mathbf{\Psi}(t)$ such that $\det\big(\mathbf{\Psi}(t)\big) \neq 0$ over some time interval (t_L, t_R) is a fundamental matrix for the first-order differential system

$$\mathbf{x}' = \mathbf{A}(t)\mathbf{x}\,, \qquad \text{where} \quad \mathbf{A}(t) = \mathbf{\Psi}'(t)\mathbf{\Psi}(t)^{-1}\,.$$

This can be seen by multiplying (2.14) on the left by $\mathbf{\Psi}(t)^{-1}$.

Example. Find a first-order differential system such that the vector-valued functions

$$\mathbf{x}_1(t) = \begin{pmatrix} 1+t^3 \\ t \end{pmatrix}, \qquad \mathbf{x}_2(t) = \begin{pmatrix} t^2 \\ 1 \end{pmatrix},$$

comprise a fundamental set of solutions.

Solution. Set

$$\mathbf{\Psi}(t) = \begin{pmatrix} \mathbf{x}_1(t) & \mathbf{x}_2(t) \end{pmatrix} = \begin{pmatrix} 1+t^3 & t^2 \\ t & 1 \end{pmatrix}.$$

Because $\det\big(\mathbf{\Psi}(t)\big) = 1$, we see that $\mathbf{\Psi}(t)$ is invertible. Set

$$\begin{aligned} \mathbf{A}(t) = \mathbf{\Psi}'(t)\mathbf{\Psi}(t)^{-1} &= \begin{pmatrix} 3t^2 & 2t \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1+t^3 & t^2 \\ t & 1 \end{pmatrix}^{-1} \\ &= \begin{pmatrix} 3t^2 & 2t \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & -t^2 \\ -t & 1+t^3 \end{pmatrix} = \begin{pmatrix} t^2 & 2t-t^4 \\ 1 & -t^2 \end{pmatrix}. \end{aligned}$$

Therefore $\mathbf{x}_1(t)$ and $\mathbf{x}_2(t)$ are a fundamental set of solutions for the differential system

$$\frac{\mathrm{d}}{\mathrm{d}t} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} t^2 & 2t-t^4 \\ 1 & -t^2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}.$$

2.4. **Natural Fundamental Matrices.** Consider the initial-value problem

$$\mathbf{x}' = \mathbf{A}(t)\mathbf{x}(t)\,, \qquad \mathbf{x}(t_I) = \mathbf{x}^I\,. \tag{2.15}$$

Let $\mathbf{\Psi}(t)$ be any fundamental matrix for this system. Then a general solution of the system is given by

$$\mathbf{x}(t) = \mathbf{\Psi}(t)\mathbf{c}\,.$$

By imposing the initial condition from (2.15) we see that

$$\mathbf{x}^I = \mathbf{x}(t_I) = \mathbf{\Psi}(t_I)\mathbf{c}\,.$$

Because $\det\big(\mathbf{\Psi}(t_I)\big) \neq 0$, the matrix $\mathbf{\Psi}(t_I)$ is invertible and we can solve for $\mathbf{c}$ as

$$\mathbf{c} = \mathbf{\Psi}(t_I)^{-1}\mathbf{x}^I\,.$$

Hence, the solution of the initial-value problem is

$$\mathbf{x}(t) = \mathbf{\Psi}(t)\mathbf{\Psi}(t_I)^{-1}\mathbf{x}^I\,. \tag{2.16}$$

Now let $\mathbf{\Phi}(t)$ be the matrix-valued function defined by

$$\mathbf{\Phi}(t) = \mathbf{\Psi}(t)\mathbf{\Psi}(t_I)^{-1}\,. \tag{2.17}$$

If we differentiate $\mathbf{\Phi}(t)$ and use the fact that $\mathbf{\Psi}(t)$ is a fundamental matrix for system (2.15) we see that

$$\mathbf{\Phi}'(t) = \big(\mathbf{\Psi}(t)\mathbf{\Psi}(t_I)^{-1}\big)' = \mathbf{\Psi}'(t)\mathbf{\Psi}(t_I)^{-1} = \mathbf{A}(t)\mathbf{\Psi}(t)\mathbf{\Psi}(t_I)^{-1} = \mathbf{A}(t)\mathbf{\Phi}(t)\,.$$

Moreover, from (2.17) we see that

$$\mathbf{\Phi}(t_I) = \mathbf{\Psi}(t_I)\mathbf{\Psi}(t_I)^{-1} = \mathbf{I}\,.$$

Therefore $\mathbf{\Phi}(t)$ is the solution of the matrix-valued initial-value problem

$$\mathbf{\Phi}' = \mathbf{A}(t)\mathbf{\Phi}\,, \qquad \mathbf{\Phi}(t_I) = \mathbf{I}\,. \tag{2.18}$$

This shows three things.

1. $\mathbf{\Phi}(t)$ as a function of t is a fundamental matrix for system (2.15);
2. $\mathbf{\Phi}(t)$ is uniquely determined by the matrix-valued initial-value problem (2.18);
3. $\mathbf{\Phi}(t)$ is independent of our original choice of fundamental matrix $\mathbf{\Psi}(t)$ that was used to construct it in (2.17).

We call $\mathbf{\Phi}(t)$ the *natural fundamental matrix* associated with the initial time t_I.

Just like it was easy to express the solution of an initial-value problem for a higher-order equation in terms of its associated natural fundamental sets of solutions, we express the solution of the initial-value problem (2.15) in terms of its associated natural fundamental matrix as simply

$$\mathbf{x}(t) = \mathbf{\Phi}(t)\mathbf{x}^I\,. \tag{2.19}$$

Given any fundamental matrix $\mathbf{\Psi}(t)$, we construct the natural fundamental matrix associated with the initial time t_I by formula (2.17).

Example. Constuct the natural fundamental matrix associated with the initial time 0 for the system

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 3 & 2 \\ 2 & 3 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}.$$

Use it to solve the initial-value problem with the initial conditions $x_1(0) = 4$ and $x_2(0) = -2$.

Solution. We have already seen that a fundamental matrix for this system is

$$\mathbf{\Psi}(t) = \begin{pmatrix} e^{5t} & e^t \\ e^{5t} & -e^t \end{pmatrix}.$$

By formula (2.17) the natural fundamental matrix associated with the initial time 0 is

$$\begin{aligned}\mathbf{\Phi}(t) = \mathbf{\Psi}(t)\mathbf{\Psi}(0)^{-1} &= \begin{pmatrix} e^{5t} & e^t \\ e^{5t} & -e^t \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}^{-1} \\ &= \begin{pmatrix} e^{5t} & e^t \\ e^{5t} & -e^t \end{pmatrix}\frac{1}{-2}\begin{pmatrix} -1 & -1 \\ -1 & 1 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} e^{5t}+e^t & e^{5t}-e^t \\ e^{5t}-e^t & e^{5t}+e^t \end{pmatrix}.\end{aligned}$$

Therefore the solution of the initial-value problem is

$$\mathbf{x}(t) = \mathbf{\Phi}(t)\mathbf{x}^I = \frac{1}{2}\begin{pmatrix} e^{5t}+e^t & e^{5t}-e^t \\ e^{5t}-e^t & e^{5t}+e^t \end{pmatrix}\begin{pmatrix} 4 \\ -2 \end{pmatrix} = \begin{pmatrix} e^{5t}+3e^t \\ e^{5t}-3e^t \end{pmatrix}.$$

Example. Constuct the natural fundamental matrix associated with the initial time 1 for the system

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} t^2 & 2t - t^4 \\ 1 & -t^2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}.$$

Use it to solve the initial-value problem with the initial conditions $x_1(1) = 3$ and $x_2(1) = 0$.

Solution. We have already seen that a fundamental matrix for this system is

$$\mathbf{\Psi}(t) = \begin{pmatrix} 1 + t^3 & t^2 \\ t & 1 \end{pmatrix}.$$

By formula (2.17) the natural fundamental matrix associated with the initial time 1 is

$$\begin{aligned} \mathbf{\Phi}(t) = \mathbf{\Psi}(t)\mathbf{\Psi}(1)^{-1} &= \begin{pmatrix} 1 + t^3 & t^2 \\ t & 1 \end{pmatrix}\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}^{-1} \\ &= \begin{pmatrix} 1 + t^3 & t^2 \\ t & 1 \end{pmatrix}\begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix} = \begin{pmatrix} 1 + t^3 - t^2 & -1 - t^3 + 2t^2 \\ t - 1 & 2 - t \end{pmatrix}. \end{aligned}$$

Therefore the solution of the initial-value problem is

$$\mathbf{x}(t) = \mathbf{\Phi}(t)\mathbf{x}^I = \begin{pmatrix} 1 + t^3 - t^2 & -1 - t^3 + 2t^2 \\ t - 1 & 2 - t \end{pmatrix}\begin{pmatrix} 3 \\ 0 \end{pmatrix} = 3\begin{pmatrix} 1 + t^3 - t^2 \\ t - 1 \end{pmatrix}.$$

2.5. **Nonhomogeneous Systems and Green Matrices.** We now consider the nonhomogenous first-order linear system

$$\mathbf{x}' = \mathbf{A}(t)\mathbf{x} + \mathbf{f}(t)\,. \tag{2.20}$$

If $\mathbf{x}_P(t)$ is a particular solution of this system and $\mathbf{\Psi}(t)$ is a fundamental matrix of the associated homogeneous system then a general solution of system (2.20) is

$$\mathbf{x}(t) = \mathbf{x}_H(t) + \mathbf{x}_P(t)\,,$$

where $\mathbf{x}_H(t)$ is the general solution of the associated homogeneous system given by

$$\mathbf{x}_H(t) = \mathbf{\Psi}(t)\mathbf{c}\,. \tag{2.21}$$

Recall that if we know a fundamental set of solutions to the associated homogeneous differential equation then we can use either Variations of Parameters or general Green Functions to construct solutions to a nonhomogeneous n^{th}-order linear equation in terms of n integrals. Here we show that an analogous fact holds for the nonhomogenous first-order linear system (2.20). Specifically, if we know a fundamental matrix $\mathbf{\Psi}(t)$ for the associated homogeneous system then we can construct solutions to the n dimensional nonhomogeneous linear equation in terms of n intergrals.

We begin with the analog of the method of Variation of Parameters for the nonhomogeneous first-order linear system (2.20). We will assume that $\mathbf{A}(t)$ and $\mathbf{f}(t)$ are continuous over an interval (t_L, t_R). We also will assume that we know a fundamental matrix $\mathbf{\Psi}(t)$ of the associated homogeneous system. This matrix will be continuously differentiable over (t_L, t_R) and satisfy

$$\mathbf{\Psi}'(t) = \mathbf{A}(t)\mathbf{\Psi}(t)\,, \qquad \det\bigl(\mathbf{\Psi}(t)\bigr) \neq 0\,.$$

Because $\mathbf{x}_H(t)$ has the form (2.21), we *seek* solutions in the form

$$\mathbf{x}(t) = \mathbf{\Psi}(t)\mathbf{u}(t)\,, \tag{2.22}$$

where $\mathbf{u}(t)$ is a vector-valued function. By differentiation we see that

$$\begin{aligned} \mathbf{x}'(t) &= \big(\mathbf{\Psi}(t)\mathbf{u}(t)\big)' = \mathbf{\Psi}'(t)\mathbf{u}(t) + \mathbf{\Psi}(t)\mathbf{u}'(t) \\ &= \mathbf{A}(t)\mathbf{\Psi}(t)\mathbf{u}(t) + \mathbf{\Psi}(t)\mathbf{u}'(t) \\ &= \mathbf{A}(t)\mathbf{x}(t) + \mathbf{\Psi}(t)\mathbf{u}'(t)\,. \end{aligned}$$

By comparing the right-hand side of this equation with the right-hand side of equation (2.20), we see that $\mathbf{x}_P(t)$ will solve (2.20) if $\mathbf{u}(t)$ satisfies

$$\mathbf{\Psi}(t)\mathbf{u}'(t) = \mathbf{f}(t)\,.$$

Because $\mathbf{\Psi}(t)$ is invertible, we find that

$$\mathbf{u}'(t) = \mathbf{\Psi}(t)^{-1}\mathbf{f}(t)\,. \tag{2.23}$$

If $\mathbf{u}_P(t)$ is a primitive of the right-hand side of (2.23) then a general solution of this system has the form

$$\mathbf{u}(t) = \mathbf{c} + \mathbf{u}_P(t)\,.$$

When this solution is placed into the form (2.22), we find that a general solution is given by

$$\mathbf{x}(t) = \mathbf{\Psi}(t)\mathbf{c} + \mathbf{\Psi}(t)\mathbf{u}_P(t)\,. \tag{2.24}$$

Now let t_I be any initial time in (t_L, t_R) and consider the initial-value problem

$$\mathbf{x}' = \mathbf{A}(t)\mathbf{x} + \mathbf{f}(t)\,, \qquad \mathbf{x}(t_I) = \mathbf{x}^I\,. \tag{2.25}$$

If we take the particular solution of (2.23) given by

$$\mathbf{u}_P(t) = \int_{t_I}^{t} \mathbf{\Psi}(s)^{-1}\mathbf{f}(s)\,\mathrm{d}s\,,$$

then (2.24) becomes

$$\mathbf{x}(t) = \mathbf{\Psi}(t)\mathbf{c} + \mathbf{\Psi}(t)\int_{t_I}^{t} \mathbf{\Psi}(s)^{-1}\mathbf{f}(s)\,\mathrm{d}s\,. \tag{2.26}$$

The solution of the initial-value problem is then

$$\mathbf{x}(t) = \mathbf{\Psi}(t)\mathbf{\Psi}(t_I)^{-1}\mathbf{x}^I + \mathbf{\Psi}(t)\int_{t_I}^{t} \mathbf{\Psi}(s)^{-1}\mathbf{f}(s)\,\mathrm{d}s\,. \tag{2.27}$$

We define the *Green matrix* $\mathbf{G}(t,s)$ by

$$\mathbf{G}(t,s) = \mathbf{\Psi}(t)\mathbf{\Psi}(s)^{-1}\,. \tag{2.28}$$

Then we can recast (2.27) as

$$\mathbf{x}(t) = \mathbf{G}(t,t_I)\mathbf{x}^I + \int_{t_I}^{t} \mathbf{G}(t,s)\,\mathbf{f}(s)\,\mathrm{d}s\,. \tag{2.29}$$

In particular, the particular solution of (2.27) that satisfies $\mathbf{x}(t_I) = \mathbf{0}$ is given by

$$\mathbf{x}(t) = \int_{t_I}^{t} \mathbf{G}(t,s)\,\mathbf{f}(s)\,\mathrm{d}s\,. \tag{2.30}$$

Remark. It is evident from formulas (2.17) and (2.28) that the Green matrix has the property that for every t_I in (t_L, t_R) the natural fundamental matrix associated with t_I is given by $\mathbf{\Phi}(t) = \mathbf{G}(t, t_I)$. In other words, $\mathbf{G}(t,s)$ is the natural fundamental matrix associated with s. This implies that $\mathbf{G}(t,s)$ is independent of our original choice of fundamental matrix $\mathbf{\Psi}(t)$ that was used to construct it in (2.28).

Example. Constuct the Green matrix for the system

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 3 & 2 \\ 2 & 3 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \frac{1}{1+e^{-2t}}\begin{pmatrix} 4 \\ 2 \end{pmatrix}.$$

Solution. We have seen that a fundamental matrix of the associated homogeneous system is

$$\mathbf{\Psi}(t) = \begin{pmatrix} e^{5t} & e^{t} \\ e^{5t} & -e^{t} \end{pmatrix}.$$

Then by formula (2.28) the Green matrix is given by

$$\begin{aligned}
\mathbf{G}(t,s) = \mathbf{\Psi}(t)\mathbf{\Psi}(s)^{-1} &= \begin{pmatrix} e^{5t} & e^{t} \\ e^{5t} & -e^{t} \end{pmatrix}\begin{pmatrix} e^{5s} & e^{s} \\ e^{5s} & -e^{s} \end{pmatrix}^{-1} \\
&= \begin{pmatrix} e^{5t} & e^{t} \\ e^{5t} & -e^{t} \end{pmatrix}\frac{1}{-2e^{6s}}\begin{pmatrix} -e^{s} & -e^{s} \\ -e^{5s} & e^{5s} \end{pmatrix} = \frac{1}{2}\begin{pmatrix} e^{5t} & e^{t} \\ e^{5t} & -e^{t} \end{pmatrix}\begin{pmatrix} e^{-5s} & e^{-5s} \\ e^{-s} & -e^{-s} \end{pmatrix} \\
&= \frac{1}{2}\begin{pmatrix} e^{5(t-s)} + e^{t-s} & e^{5(t-s)} - e^{t-s} \\ e^{5(t-s)} - e^{t-s} & e^{5(t-s)} + e^{t-s} \end{pmatrix}.
\end{aligned}$$

Example. Constuct the Green matrix for the system

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} t^2 & 2t - t^4 \\ 1 & -t^2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} e^{t} \\ t \end{pmatrix}.$$

Solution. We have seen that a fundamental matrix of the associated homogeneous system is

$$\mathbf{\Psi}(t) = \begin{pmatrix} 1+t^3 & t^2 \\ t & 1 \end{pmatrix}.$$

Then by formula (2.28) the Green matrix is given by

$$\begin{aligned}
\mathbf{G}(t,s) = \mathbf{\Psi}(t)\mathbf{\Psi}(s)^{-1} &= \begin{pmatrix} 1+t^3 & t^2 \\ t & 1 \end{pmatrix}\begin{pmatrix} 1+s^3 & s^2 \\ s & 1 \end{pmatrix}^{-1} \\
&= \begin{pmatrix} 1+t^3 & t^2 \\ t & 1 \end{pmatrix}\begin{pmatrix} 1 & -s^2 \\ -s & 1+s^3 \end{pmatrix} = \begin{pmatrix} 1+t^3-t^2s & t^2+t^2s^3-s^2-t^3s^2 \\ t-s & 1+s^3-ts^2 \end{pmatrix}.
\end{aligned}$$

2.6. **Triangular Systems.** We have just seen that if we can construct a fundamental matrix $\mathbf{\Psi}(t)$ for the homogeneous linear system

$$\mathbf{x}' = \mathbf{A}(t)\mathbf{x}\,, \tag{2.31}$$

then

$$\mathbf{G}(t,s) = \mathbf{\Psi}(t)\mathbf{\Psi}(s)^{-1} \tag{2.32}$$

is the Green matrix $\mathbf{G}(t,s)$ for the nonhomogeneous linear system

$$\mathbf{x}' = \mathbf{A}(t)\mathbf{x} + \mathbf{f}(t)\,.$$

In that case if $\mathbf{A}(t)$ and $\mathbf{f}(t)$ are continuous over an interval (t_L, t_R) and t_I is in (t_L, t_R) then

$$\mathbf{x}(t) = \mathbf{G}(t,t_I)\mathbf{x}^I + \int_{t_I}^{t} \mathbf{G}(t,s)\mathbf{f}(s)\,\mathrm{d}s \tag{2.33}$$

is the solution of the initial-value problem

$$\mathbf{x}' = \mathbf{A}(t)\mathbf{x} + \mathbf{f}(t)\,, \qquad \mathbf{x}(t_I) = \mathbf{x}^I\,. \tag{2.34}$$

However, when system (2.31) is of dimension two or more there is no general recipe for constructing a fundamental matrix $\mathbf{\Psi}(t)$.

Here we show that a fundamental matrix $\mathbf{\Psi}(t)$ can be constructed for system (2.31) whenever $\mathbf{A}(t)$ is *triangular.* This means that $\mathbf{A}(t)$ has either the lower triangular form

$$\mathbf{A}(t) = \begin{pmatrix} a_{11}(t) & 0 & \cdots & 0 \\ a_{21}(t) & a_{22}(t) & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ a_{n1}(t) & \cdots & a_{n(n-1)}(t) & a_{nn}(t) \end{pmatrix}, \tag{2.35a}$$

or the upper triangular form

$$\mathbf{A}(t) = \begin{pmatrix} a_{11}(t) & a_{12}(t) & \cdots & a_{1n}(t) \\ 0 & a_{22}(t) & \ddots & \vdots \\ \vdots & \ddots & \ddots & a_{(n-1)n}(t) \\ 0 & \cdots & 0 & a_{nn}(t) \end{pmatrix}. \tag{2.35b}$$

When $\mathbf{A}(t)$ has one of these forms the systems (2.31) and (2.34) are called *triangular systems.*

Before describing the general construction, we will illustrate the idea behind it for the case when $\mathbf{A}(t)$ is the 2×2 lower trangular matrix

$$\mathbf{A}(t) = \begin{pmatrix} a(t) & 0 \\ b(t) & c(t) \end{pmatrix}. \tag{2.36}$$

We will assume that $a(t)$, $b(t)$, and $c(t)$ are continuous over an interval $(t_{\mathrm{L}}, t_{\mathrm{R}})$ and that t_I is in $(t_{\mathrm{L}}, t_{\mathrm{R}})$. We will contruct the natrural fundamental matrix $\mathbf{\Phi}(t)$ for the initial time t_I by solving the general initial-value problem

$$\begin{aligned} x' &= a(t)x & x(t_I) &= x_I\,, \\ y' &= b(t)x + c(t)y & y(t_I) &= y_I\,, \end{aligned} \tag{2.37}$$

Because the first equation decouples from y, its solution is

$$x(t) = e^{A(t)}x_I\,, \quad \text{where} \quad A(t) = \int_{t_I}^{t} a(r)\,\mathrm{d}r\,.$$

Beacuse $x(t)$ is now known, the solution of the second equation is

$$y(t) = e^{C(t)}y_I + e^{C(t)}B(t)\,, \quad \text{where} \quad C(t) = \int_{t_I}^{t} c(r)\,\mathrm{d}r\,,$$

and

$$B(t) = \int_{t_I}^{t} e^{-C(r)}b(r)x(r)\,\mathrm{d}r\,.$$

Collecting everything, the solution of the general initial-value problem (2.36) is

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} e^{A(t)}x_I \\ e^{C(t)}y_I + e^{C(t)}\int_{t_I}^{t} e^{A(r)-C(r)}b(r)\,\mathrm{d}r\,x_I \end{pmatrix}. \tag{2.38}$$

Because the natural fundamental matrix $\mathbf{\Phi}(t)$ satisfies

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \mathbf{\Phi}(t)\begin{pmatrix} x_I \\ y_I \end{pmatrix},$$

its columns can be read off as the terms in ([??]) that multiply x_I and y_I respectively. This yields

$$\mathbf{\Phi}(t) = \begin{pmatrix} e^{A(t)} & 0 \\ e^{C(t)}\int_{t_I}^{t} e^{A(r)-C(r)}b(r)\,\mathrm{d}s & e^{C(t)} \end{pmatrix}, \tag{2.39}$$

where $A(t)$ and $C(t)$ are given by

$$A(t) = \int_{t_I}^{t} a(r)\,\mathrm{d}r\,, \qquad C(t) = \int_{t_I}^{t} c(r)\,\mathrm{d}r\,. \tag{2.40}$$

Exercise. Check that the $\mathbf{\Phi}(t)$ given by (2.39) satisfies the matrix initial-value problem

$$\mathbf{\Phi}' = \mathbf{A}(t)\mathbf{\Phi}\,, \qquad \Phi(t_I) = \mathbf{I}\,, \tag{2.41}$$

where $\mathbf{A}(t)$ is the lower triangular matrix given by (2.36).

Rather than memorize formula (2.39), it is simpler to repeat the steps by which it was derived when faced with solving any triangular system.

Example.

We will describe the construction for the case when $\mathbf{A}(t)$ has the lower triangular form (2.35a). In that case we see from (2.31) that the first entry of $\mathbf{x}$ satisfies

$$x_1' = a_{11}(t)x_1\,, \qquad x_1(t_I) = x_1^I\,.$$

This has the solution

$$x_1(t) = x_1^I e^{A_{11}(t)}\big)\,, \quad \text{where} \quad A_{11}(t) = \int_{t_I}^{t} a_{11}(s)\mathrm{d}s\,.$$

We then see from (2.31) that the second entry of $\mathbf{x}$ satisfies

$$x_2' = a_{21}(t)x_1 + a_{22}(t)x_2\,,$$

Because $x_1(t)$ is known this has the solution

$$x_2(t) = x_2^I e^{A_{22}(t)} + e^{A_{22}(t)}B_2(t)\,, \quad \text{where} \quad A_{22}(t) = \int_{t_I}^{t} a_{22}(s)\,ds\,,$$

and

$$B_2(t) = \int_{t_I}^{t} e^{-A_{22}(s)} a_{21}(s)x_1(s)\,\mathrm{d}s\,.$$

In general we see from (2.31) that the k^{th} entry of $\mathbf{x}$ satisfies

$$x_k' = a_{k1}(t)x_1 + \cdots + a_{k(k-1)}(t)x_{k-1} + a_{kk}(t)x_k\,, \qquad x_k(t_I) = x_k^I\,.$$

Because $x_1(t)$, $\cdots$, $x_{k-1}(t)$ are known this has the solution

$$x_k(t) = e^{A_{kk}(t)}x_k^I + e^{A_{kk}(t)}B_k(t)\,, \quad \text{where} \quad A_{kk}(t) = \int_{t_I}^{t} a_{kk}(s)\,ds\,,$$

and

$$B_k(t) = \int_{t_I}^{t} e^{-A_{kk}(s)}\big(a_{k1}(s)x_1(s) + \cdots + a_{k(k-1)}(s)x_{k-1}(s)\big)\,\mathrm{d}s\,.$$

Remark. The construction is similar for the case when $\mathbf{A}(t)$ has the upper triangular form (2.35b). The difference is that we solve for $x_n(t)$ first, $x_{n-1}(t)$ second, and so forth.,

Exercises on Linear Systems

(1) Express

$$\begin{aligned} x_1' &= 2tx_1 + e^t x_2 \\ x_2' &= 3x_1 - 3tx_2 \end{aligned}$$

with $x_1(0) = -5$ and $x_2(0) = 2$ as a vector equation with a vector initial condition.

Solution

(2) Consider the system

$$\begin{aligned} x' &= 2x + y - z \\ y' &= x - 3y + 5z \\ z' &= 4x - 7y + z. \end{aligned}$$

Write this system as a vector equation.

Solution

(3) Consider the vector equation $\mathbf{x}' = \begin{pmatrix} 4t & 6t^2 \\ 2 & t^3 \end{pmatrix} \mathbf{x} + \begin{pmatrix} e^t \\ e^{-t} \end{pmatrix}$. Write this equation as a system of 2 equations.

Solution

For Problems 4- 7 recast the higher order linear differential equations as a linear system of first order equations. Find the coefficient matrix $\mathbf{A}(t)$ and forcing $\mathbf{f}(t)$. If the problem is an initial value problem, then be sure to state the initial condition.

(4) $u'' + 3tu = e^t$

Solution

(5) $y''' + 2y'' - ty' + y = 0$

Solution

(6) $(1 + t^2)y''' + e^{t^2} y'' - \sin(t)y = \cos(t)$

Solution

(7) $y^{(4)} + t^2 y''' + \cos(t)y'' + t^2 \sin(t)y = te^t$

Solution

(8) Consider the second order equation $y'' + p(t)y' + q(t)y = g(t)$ with initial conditions $y'(0) = 1$ and $y(0) = 2$. Let $x_1 = y$ and $x_2 = y'$, and then express this second order equation as a system of two first order equations. Be sure to include the initial condition for your system.

Solution

(9) Consider the nth order equation $y^{(n)} + a_1(t)y^{(n-1)} + a_2(t)y^{(n-2)} + \cdots + a_{n-1}(t)y' + a_n(t)y = g(t)$ with initial conditions $y^{(i)}(0) = b_{i+1}$ for $i = 0, \ldots, n-1$. Express this nth order equation as a system of n first order equations. Be sure to include the initial condition for your system.

Solution

(10) Two hanging pendula of length ℓ and masses m_1 and m_2 are coupled by a spring. Let θ_1 and θ_2 be the angular displacement of each pendulum from its rest position. For small angles, the equations of motion are approximated by the following linear system:

$$m_1\ell\theta_1'' = -m_1 g\theta_1 - k\ell(\theta_1 - \theta_2)$$
$$m_2\ell\theta_2'' = -m_2 g\theta_2 + k\ell(\theta_1 - \theta_2)$$

Write this as a first-order linear system. Find the corresponding coefficient matrix $\mathbf{A}$ and the forcing $\mathbf{f}$.

Solution

For problems 11- 12 determine the largest interval where a unique solution exists for the following initial value problems for first order systems.

(11)

$$\mathbf{x}' = \begin{pmatrix} \log(t+1) & \sqrt{9-t^2} \\ \tan(t) & t^2 \end{pmatrix}\mathbf{x} + \begin{pmatrix} \frac{1}{t-5} \\ \sin(t) \end{pmatrix}, \qquad \mathbf{x}(0) = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

Solution

(12)

$$\mathbf{x}' = \begin{pmatrix} \log(t) & t\log(t) \\ e^{2t} & t+1 \end{pmatrix}\mathbf{x} + \begin{pmatrix} \frac{1}{t-2} \\ \sin(t) \end{pmatrix}, \qquad \mathbf{x}(1) = \begin{pmatrix} 0 \\ 7 \end{pmatrix}$$

Solution

(13) Consider the differential system

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3 & 1 \\ 2 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

(a) Show that $\begin{pmatrix} e^{4t} \\ e^{4t} \end{pmatrix}$ and $\begin{pmatrix} -e^{t} \\ 2e^{t} \end{pmatrix}$ are both solutions to this system.
(b) Give a fundamental matrix for this system.
(c) Give a general solution to this system in vector form.
(d) Compute the natural fundamental matrix for this system associated with $t = 0$.
(e) Solve the initial value problem for this system with $x(0) = -2$ and $y(0) = 3$.

Solution

(14) Consider the differential system

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 & -\cos(t) \\ \cos(t) & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

(a) Show that both $\begin{pmatrix} e^t\cos(\sin(t)) \\ e^t\sin(\sin(t)) \end{pmatrix}$ and $\begin{pmatrix} -e^t\sin(\sin(t)) \\ e^t\cos(\sin(t)) \end{pmatrix}$ are both solutions to this system.
(b) Give a fundamental matrix for this system.
(c) Write a general solution to this system in vector form.
(d) Compute the natural fundamental matrix of this system associated with $t = 0$.
(e) Solve the initial value problem for this system with $x(0) = 1$, $y(0) = 0$.

Solution

(15) Consider the differential system

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2t & -e^{t^2} \\ e^{-t^2} & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

(a) Show that both $\begin{pmatrix} e^{t^2}\cos(t) \\ \sin(t) \end{pmatrix}$ and $\begin{pmatrix} e^{t^2}\sin(t) \\ -\cos(t) \end{pmatrix}$ are solutions to the above system.
(b) Give a fundamental matrix for this sytem.
(c) Write a general solution to this system in vector form.
(d) Compute the natural fundamental matrix of this system associated with $t = 0$.
(e) Solve the initial value problem for this system with $x(0) = -1$, $y(0) = 3$.

Solution

(16) Consider the differential system

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{1+t^2}\begin{pmatrix} (1+t)^2 & 2(1+t^2)^2 \\ 2 & 1+t^2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

(a) Show that both $\begin{pmatrix} (1+t^2)e^{3t} \\ e^{3t} \end{pmatrix}$ and $\begin{pmatrix} (1+t^2)e^{-t} \\ -e^{-t} \end{pmatrix}$ are solutions to the above system.
(b) Give a fundamental matrix for this sytem.
(c) Write a general solution to this system in vector form.
(d) Compute the natural fundamental matrix of this system associated with $t = 0$.
(e) Solve the initial value problem for this system with $x(0) = -4$, $y(0) = 2$.

Solution

For problems 17- 20, show that the following vector valued solutions to a linear system form a fundamental set. Find the linear system that they solve.

(17) $\mathbf{x}_1(t) = \begin{pmatrix} \cos(t) \\ \sin(t)+\cos(t) \end{pmatrix}, \quad \mathbf{x}_2(t) = \begin{pmatrix} -\sin(t) \\ \cos(t)-\sin(t) \end{pmatrix}$

Solution

(18) $\mathbf{x}_1(t) = \begin{pmatrix} e^t \\ e^t \end{pmatrix}, \quad \mathbf{x}_2(t) = \begin{pmatrix} -e^{-2t} \\ 3e^{-2t} \end{pmatrix}$

Solution

(19) $\mathbf{x}_1(t) = \begin{pmatrix} 2te^{2t} \\ 1-te^{2t} \end{pmatrix}, \quad \mathbf{x}_2(t) = \begin{pmatrix} -2e^{-3t} \\ e^{-3t} \end{pmatrix}$

Solution

(20) $\mathbf{x}_1(t) = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} e^{3t}, \quad \mathbf{x}_2(t) = \begin{pmatrix} t+1 \\ 1 \\ 2t+2 \end{pmatrix} e^{3t}, \quad \mathbf{x}_3(t) = \begin{pmatrix} \frac{1}{2}t^2+t+1 \\ t+1 \\ (t+1)^2 \end{pmatrix} e^{3t}.$

Solution

(21) Let $\mathbf{\Phi}(t)$ be the natural fundamental matrix associated to t_I for the system

$$\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} = \mathbf{A}\mathbf{x},$$

where $\mathbf{A}$ is a constant matrix. Show that $\mathbf{\Phi}(t)$ and $\mathbf{A}$ commute, that is

$$\mathbf{\Phi}(t)\mathbf{A} = \mathbf{A}\mathbf{\Phi}(t).$$

Hint: Show that $\mathbf{\Phi}(t)\mathbf{A}$ show that $\mathbf{A}\mathbf{\Phi}(t)$ solve the same initial value problem.

Solution

(22) Let $\mathbf{\Psi}(t)$ be a fundamental matrix for the system

$$\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} = \mathbf{A}(t)\mathbf{x}.$$

For any constant matrix $\mathbf{C}$ such that $\det(\mathbf{C}) \neq 0$, show that $\mathbf{\Psi}\mathbf{C}$ is also a fundamental matrix, but that $\mathbf{C}\mathbf{\Psi}$ may not be. How must $\mathbf{C}$ and $\mathbf{A}(t)$ be related so that $\mathbf{C}\mathbf{\Psi}$ becomes a fundamental matrix?

Solution

(23) Let $\mathbf{\Psi}_1$ and $\mathbf{\Psi}_2$ be two fundamental matrices of

$$\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} = \mathbf{A}(t)\mathbf{x}.$$

Show that there exists a constant $\mathbf{C}$, $\det(\mathbf{C}) \neq 0$, such that $\mathbf{\Psi}_1 = \mathbf{\Psi}_2\mathbf{C}$.

Hint: Show that $(\mathbf{\Psi}_2^{-1}\mathbf{\Psi}_1)' = \mathbf{0}$, and use the product rule for matrices $(\mathbf{A}\mathbf{B})' = \mathbf{A}'\mathbf{B} + \mathbf{A}\mathbf{B}'$ (see the exercises in the supplement on matrices and vectors).

Solution

(24) The previous problem implies that if $\mathbf{\Psi}_1$ and $\mathbf{\Psi}_2$ are fundamental matricies of

$$\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} = \mathbf{A}(t)\mathbf{x},$$

then their Wronskians $W_1(t) = \det(\mathbf{\Psi}_1(t))$, $W_2(t) = \det(\mathbf{\Psi}_2(t))$ differ by a constant multiple $c \neq 0$, since

$$W_1(t) = \det(\mathbf{\Psi}_1(t)) = \det(\mathbf{C})\det(\mathbf{\Psi}_2(t)) = cW_2(t).$$

Use Liouville's Wronskian Theorem to reach this same conclusion.

Solution

(25) Consider the homogeneous n^{th} order linear differential equation in normal form

$$\frac{\mathrm{d}^n y}{\mathrm{d}t} + a_1(t)\frac{\mathrm{d}^{n-1}y}{\mathrm{d}t} + \ldots + a_{n-1}(t)\frac{\mathrm{d}y}{\mathrm{d}t} + a_n(t)y = 0,$$

and its equivalent n-dimensional first-order linear system

$$\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} = \mathbf{A}(t)\mathbf{x}.$$

Show Abel's Wronskian Theorem for the n^{th} order equation using Liouville's Wronskian Theorem for the first order linear system.

Solution

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