III. First-Order Systems of Ordinary Differential Equations
5. Linear Systems: Eigen Methods

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Contents

Return to Main Webpage

5. LINEAR SYSTEMS: EIGEN METHODS

This chapter developes a class of methods that plays a central role in the study of large first-order linear systems with constant coefficients. We will illustrate these methods in the setting of small systems. Their foundation is the construction of solutions to $\mathbf{x}' = \mathbf{A}\mathbf{x}$ from eigenpairs of $\mathbf{A}$. We begin by explaining eigenpairs.

5.1. **Eigenpairs.** Let $\mathbf{A}$ be a real $n \times n$ matrix. A number λ (possibly complex) is an *eigenvalue* of $\mathbf{A}$ if there exists a nonzero vector $\mathbf{v}$ (possibly complex) such that

$$\mathbf{A}\mathbf{v} = \lambda\mathbf{v}\,. \tag{5.1}$$

Each such vector is an *eigenvector* associated with λ, and $(\lambda, \mathbf{v})$ is an *eigenpair* of $\mathbf{A}$.

> **Fact 1.** If $(\lambda, \mathbf{v})$ is an eigenpair of $\mathbf{A}$ then so is $(\lambda, \alpha\mathbf{v})$ for every complex $\alpha \neq 0$. In other words, if $\mathbf{v}$ is an eigenvector associated with an eigenvalue λ of $\mathbf{A}$ then so is $\alpha\mathbf{v}$ for every complex $\alpha \neq 0$. In particular, eigenvectors are not unique.

Reason. Because $(\lambda, \mathbf{v})$ is an eigenpair of $\mathbf{A}$ we know that (5.1) holds. It follows that

$$\mathbf{A}(\alpha\mathbf{v}) = \alpha\mathbf{A}\mathbf{v} = \alpha\lambda\mathbf{v} = \lambda(\alpha\mathbf{v})\,.$$

Because the scalar α and vector $\mathbf{v}$ are nonzero, the vector $\alpha\mathbf{v}$ is also nonzero. Therefore $(\lambda, \alpha\mathbf{v})$ is also an eigenpair of $\mathbf{A}$. $\square$

5.1.1. *Finding Eigenvalues.* Recall that the characteristic polynomial of $\mathbf{A}$ is given by

$$p_{\mathbf{A}}(z) = \det(z\mathbf{I} - \mathbf{A})\,.$$

It has the form

$$p_{\mathbf{A}}(z) = z^n + p_1 z^{n-1} + p_2 z^{n-2} + \cdots + p_{n-1} z + p_n\,,$$

where the coefficients p_1, p_2, $\cdots$, p_n are real. In other words, it is a real monic polynomial of degree n. It can be shown that in general

$$p_1 = -\mathrm{tr}(\mathbf{A})\,, \qquad p_n = (-1)^n \det(\mathbf{A})\,,$$

where $\mathrm{tr}(\mathbf{A})$ is the sum of the diagonal entries of $\mathbf{A}$. In particular, when $n = 2$ we have

$$p_{\mathbf{A}}(z) = z^2 - \mathrm{tr}(\mathbf{A})z + \det(\mathbf{A})\,.$$

> **Fact 2.** A number λ is an eigenvalue of $\mathbf{A}$ if and only if $p_{\mathbf{A}}(\lambda) = 0$. In other words, the eigenvalues of $\mathbf{A}$ are the roots of $p_{\mathbf{A}}(z)$.

Reason. If λ is an eigenvalue of $\mathbf{A}$ then by (5.1) there exists a nonzero vector $\mathbf{v}$ such that

$$(\lambda\mathbf{I} - \mathbf{A})\mathbf{v} = \lambda\mathbf{v} - \mathbf{A}\mathbf{v} = \mathbf{0}\,.$$

It follows that $p_{\mathbf{A}}(\lambda) = \det(\lambda\mathbf{I} - \mathbf{A}) = 0$. Conversely, if $p_{\mathbf{A}}(\lambda) = \det(\lambda\mathbf{I} - \mathbf{A}) = 0$ then there exists a nonzero vector $\mathbf{v}$ such that $(\lambda\mathbf{I} - \mathbf{A})\mathbf{v} = \mathbf{0}$. It follows that

$$\lambda\mathbf{v} - \mathbf{A}\mathbf{v} = (\lambda\mathbf{I} - \mathbf{A})\mathbf{v} = \mathbf{0}\,,$$

whereby λ and $\mathbf{v}$ satisfy (5.1), which implies λ is an eigenvalue of $\mathbf{A}$. $\square$

Fact 2 shows that the eigenvalues of a $n \times n$ matrix $\mathbf{A}$ can be found if we can find all the roots of the characteristic polynomial of $\mathbf{A}$. Because the degree of this characteristic polynomial is n, and because every polynomial of degree n has exactly n roots counting multiplicity, the $n{\times}n$ matrix $\mathbf{A}$ therefore must have at least one eigenvalue and can have at most n eigenvalues.

Example. Find the eigenvalues of $\mathbf{A} = \begin{pmatrix} 3 & 2 \\ 2 & 3 \end{pmatrix}$.

Solution. The characteristic polynomial of $\mathbf{A}$ is

$$p_{\mathbf{A}}(z) = z^2 - 6z + 5 = (z-1)(z-5)\,.$$

By Fact 2 the eigenvalues of $\mathbf{A}$ are 1 and 5.

Example. Find the eigenvalues of $\mathbf{B} = \begin{pmatrix} 4 & -1 \\ 1 & 2 \end{pmatrix}$.

Solution. The characteristic polynomial of $\mathbf{A}$ is

$$p_{\mathbf{B}}(z) = z^2 - 6z + 9 = (z-3)^2\,.$$

By Fact 2 the only eigenvalue of $\mathbf{B}$ is 3.

Example. Find the eigenvalues of $\mathbf{C} = \begin{pmatrix} 3 & 2 \\ -2 & 3 \end{pmatrix}$.

Solution. The characteristic polynomial of $\mathbf{A}$ is

$$p_{\mathbf{C}}(z) = z^2 - 6z + 13 = (z-3)^2 + 4 = (z-3)^2 + 2^2\,.$$

By Fact 2 the eigenvalues of $\mathbf{C}$ are $3 + i2$ and $3 - i2$.

Remark. The above examples show all the possibilities that can arise for 2×2 matrices — namely, a 2×2 matrix can have either, two real eigenvalues, one real eigenvalue, or a conjugate pair of eigenvalues.

The next example shows that finding eigenvalues this way gets harder for larger matrices.

Example: Find the eigenvalues of

$$\mathbf{A} = \begin{pmatrix} 4 & 0 & -3 \\ 0 & 5 & 4 \\ 2 & 2 & 1 \end{pmatrix}\,.$$

Solution. It can be shown that the characteristic polynomial of $\mathbf{A}$ is

$$\begin{aligned} p_{\mathbf{A}}(z) = \det(z\mathbf{I} - \mathbf{A}) &= \det\begin{pmatrix} z-4 & 0 & 3 \\ 0 & z-5 & -4 \\ -2 & -2 & z-1 \end{pmatrix} \\ &= (z-4)(z-5)(z-1) - 8(z-4) + 6(z-5) = z^3 - 10z^2 + 27z - 18\,, \end{aligned}$$

where we are not showing some steps used to get the final equality. It is fairly easy to see that $z = 1$ is a root of $p_{\mathbf{A}}(z)$ — i.e. that $p_{\mathbf{A}}(1) = 0$. Hence, $(z-1)$ is a factor of $p_{\mathbf{A}}(z)$. Upon dividing $p_{\mathbf{A}}(z)$ by $(z-1)$ we see that

$$p_{\mathbf{A}}(z) = z^3 - 10z^2 + 27z - 18 = (z-1)(z^2 - 9z^2 + 18) = (z-1)(z-3)(z-6)\,.$$

By Fact 2 the eigenvalues of $\mathbf{A}$ are 1, 3, and 6.

5.1.2. *Finding Eigenvectors.* Once we have found the eigenvalues of a matrix, we can find all the eigenvectors associated with each eigenvalue λ by finding a general nonzero solution of (5.1), which is equivalent to

$$(\mathbf{A} - \lambda \mathbf{I})\mathbf{v} = \mathbf{0}\,. \tag{5.2}$$

For an $n\times n$ matrix with n distinct eigenvalues this means solving n homogeneous linear algebraic systems, which might take some time. We illustrate this with a single example.

Example. Given that 3 is an eigenvalue of the matrix

$$\mathbf{A} = \begin{pmatrix} 4 & 0 & -3 \\ 0 & 5 & 4 \\ 2 & 2 & 1 \end{pmatrix},$$

find all the eigenvectors of $\mathbf{A}$ associated with 3.

Solution. The eigenvectors of $\mathbf{A}$ associated with 3 are all nonzero vectors $\mathbf{v}$ such that $\mathbf{A}\mathbf{v} = 3\mathbf{v}$. Equivalently, they are all nonzero vectors $\mathbf{v}$ such that $(\mathbf{A} - 3\mathbf{I})\mathbf{v} = \mathbf{0}$, which is

$$\begin{pmatrix} 1 & 0 & -3 \\ 0 & 2 & 4 \\ 2 & 2 & -2 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

The entries of $\mathbf{v}$ thereby satisfy the homogeneous linear algebraic system

$$\begin{aligned} v_1 \qquad\quad - 3v_3 &= 0\,, \\ 2v_2 + 4v_3 &= 0\,, \\ 2v_1 + 2v_2 - 2v_3 &= 0\,. \end{aligned}$$

This system may be solved either by elimination or by row reduction. By either method its general solution is found to be

$$v_1 = 3\alpha\,, \quad v_2 = -2\alpha\,, \quad v_3 = \alpha\,, \qquad \text{for any constant } \alpha\,.$$

Therefore the eigenvectors of $\mathbf{A}$ associated with 3 each have the form

$$\alpha \begin{pmatrix} 3 \\ -2 \\ 1 \end{pmatrix} \qquad \text{for some constant } \alpha \neq 0\,.$$

Remark. Earlier we showed that $\mathbf{A}$ in the example above has eigenvalues 1, 3, and 6. This example shows that an eigenpair associated with the eigenvalue 3 is given by

$$\left(3\,, \begin{pmatrix} 3 \\ -2 \\ 1 \end{pmatrix} \right).$$

If we also wanted to find eigenpairs associated with the eigenvalues 1 and 6 we would have to do two more similar calculations.

5.1.3. *Finding Eigenvectors for* $2{\times}2$ *Matrices.* There is a method that allows us to quickly find the eigenvectors associated with each eigenvalue of any $2{\times}2$ matrix $\mathbf{A}$ without solving any linear systems. The method is based on the Cayley-Hamilton Theorem, which states that $p_{\mathbf{A}}(\mathbf{A}) = \mathbf{0}$. The eigenvalues λ_1 and λ_2 are the (possibly complex) roots of $p_{\mathbf{A}}(z)$, so that $p_{\mathbf{A}}(z)$ factors as

$$p_{\mathbf{A}}(z) = (z - \lambda_1)(z - \lambda_2)\,.$$

Hence, by the Cayley-Hamilton Theorem

$$\mathbf{0} = p_{\mathbf{A}}(\mathbf{A}) = (\mathbf{A} - \lambda_1\mathbf{I})(\mathbf{A} - \lambda_2\mathbf{I}) = (\mathbf{A} - \lambda_2\mathbf{I})(\mathbf{A} - \lambda_1\mathbf{I})\,. \tag{5.3}$$

This shows that:

- every nonzero column of $\mathbf{A} - \lambda_2\mathbf{I}$ is an eigenvector associated with λ_1;
- every nonzero column of $\mathbf{A} - \lambda_1\mathbf{I}$ is an eigenvector associated with λ_2.

This observation even works in the case where $\lambda_1 = \lambda_2$ and $\mathbf{A} - \lambda_1\mathbf{I} \neq \mathbf{0}$. Of course, if $\mathbf{A} - \lambda_1\mathbf{I} = \mathbf{0}$ then every nonzero vector is an eigenvector.

Example. Find an eigenpair for each of the eigenvalues of $\mathbf{A} = \begin{pmatrix} 3 & 2 \\ 2 & 3 \end{pmatrix}$.

Solution. We have already shown that the eigenvalues of $\mathbf{A}$ are 1 and 5. Compute

$$\mathbf{A} - \mathbf{I} = \begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix}, \qquad \mathbf{A} - 5\mathbf{I} = \begin{pmatrix} -2 & 2 \\ 2 & -2 \end{pmatrix},$$

Every column of $\mathbf{A} - 5\mathbf{I}$ has the form

$$\alpha \begin{pmatrix} 1 \\ -1 \end{pmatrix} \quad \text{for some } \alpha \neq 0\,,$$

while every column of $\mathbf{A} - \mathbf{I}$ has the form

$$\beta \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad \text{for some } \beta \neq 0\,.$$

It follows from (5.3) that eigenpairs of $\mathbf{A}$ are

$$\left(1\,, \begin{pmatrix} 1 \\ -1 \end{pmatrix}\right), \qquad \left(5\,, \begin{pmatrix} 1 \\ 1 \end{pmatrix}\right).$$

Example. Find an eigenpair for each of the eigenvalues of $\mathbf{B} = \begin{pmatrix} 4 & -1 \\ 1 & 2 \end{pmatrix}$.

Solution. We have already shown that the only eigenvalue of $\mathbf{B}$ is 3. Compute

$$\mathbf{B} - 3\mathbf{I} = \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix}.$$

Every column of $\mathbf{B} - 3\mathbf{I}$ has the form

$$\alpha \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad \text{for some } \alpha \neq 0\,.$$

It follows from (5.3) that an eigenpair of $\mathbf{B}$ is

$$\left(3\,,\ \begin{pmatrix}1\\1\end{pmatrix}\right)\,.$$

Example. Find an eigenpair for each of the eigenvalues of $\mathbf{C} = \begin{pmatrix}3 & 2\\-2 & 3\end{pmatrix}$.

Solution. We have already shown that the eigenvalues of $\mathbf{C}$ are $3 + i2$ and $3 - i2$. Compute

$$\mathbf{C} - (3+i2)\mathbf{I} = \begin{pmatrix}-i2 & 2\\-2 & -i2\end{pmatrix}\,, \qquad \mathbf{C} - (3-i2)\mathbf{I} = \begin{pmatrix}i2 & 2\\-2 & i2\end{pmatrix}\,.$$

Every column of $\mathbf{C} - (3 - i2)\mathbf{I}$ has the form

$$\alpha\begin{pmatrix}1\\i\end{pmatrix} \quad \text{for some } \alpha \neq 0\,,$$

while every column of $\mathbf{C} - (3 + i2)\mathbf{I}$ has the form

$$\beta\begin{pmatrix}1\\-i\end{pmatrix} \quad \text{for some } \beta \neq 0\,.$$

It follows from (5.3) that eigenpairs of $\mathbf{C}$ are

$$\left(3+i2\,,\ \begin{pmatrix}1\\i\end{pmatrix}\right)\,, \qquad \left(3-i2\,,\ \begin{pmatrix}1\\-i\end{pmatrix}\right)\,.$$

5.1.4. *Eigenpairs of Real Matrices.* Notice that in the foregoing examples we found real eigenvectors for real eigenvalues and in the last example we found eigenvectors that are complex conjugates of each other for the conjuagate pair of eigenvalues $3 \pm i2$. These examples illustrate some general facts about eigenpairs of real matrices. We begin with the following fact.

Fact 3. If $(\lambda, \mathbf{v})$ is an eigenpair of the real matrix $\mathbf{A}$ then so is $(\overline{\lambda}, \overline{\mathbf{v}})$.

Reason. Because $(\lambda, \mathbf{v})$ is an eigenpair of $\mathbf{A}$ we know by (5.1) that $\mathbf{A}\mathbf{v} = \lambda\mathbf{v}$. By taking the complex conjugate of this equation we obtain

$$\overline{\mathbf{A}\mathbf{v}} = \overline{\lambda\mathbf{v}}\,.$$

Because $\mathbf{A}$ is real we then see that

$$\mathbf{A}\overline{\mathbf{v}} = \overline{\mathbf{A}\mathbf{v}} = \overline{\lambda\mathbf{v}} = \overline{\lambda}\,\overline{\mathbf{v}}\,,$$

where $\overline{\mathbf{v}}$ is nonzero because $\mathbf{v}$ is nonzero. It follows that $(\overline{\lambda}, \overline{\mathbf{v}})$ is an eigenpair of $\mathbf{A}$. ☐

This fact leads to the following facts about eigenvectors of real matrices.

Fact 4. Let λ be an eigenvalue of a real matrix $\mathbf{A}$.

- If λ is real then it has a real eigenvector.
- If λ is not real then none of its eigenvectors are real.

Reason. Let $\mathbf{v}$ be any eigenvector associated with λ, so that $(\lambda, \mathbf{v})$ is an eigenpair of $\mathbf{A}$. Let $\lambda = \mu + i\nu$ and $\mathbf{v} = \mathbf{u} + i\mathbf{w}$ where μ and ν are real numbers and $\mathbf{u}$ and $\mathbf{w}$ are real vectors. We then have

$$\mathbf{Au} + i\mathbf{Aw} = \mathbf{Av} = \lambda\mathbf{v} = (\mu + i\nu)(\mathbf{u} + i\mathbf{w}) = (\mu\mathbf{u} - \nu\mathbf{w}) + i(\mu\mathbf{w} + \nu\mathbf{u}),$$

which is equivalent to

$$\mathbf{Au} - \mu\mathbf{u} = -\nu\mathbf{w}, \qquad \text{and} \qquad \mathbf{Aw} - \mu\mathbf{w} = \nu\mathbf{u}. \tag{5.4}$$

The two assertions of Fact 4 are then argued as follows.

- If λ is real then $\nu = 0$ and $\lambda = \mu$. Then the equations (5.4) show that either $\mathbf{u}$ or $\mathbf{w}$ will be a real eigenvector associated with λ whenever it is nonzero. But at least one of $\mathbf{u}$ and $\mathbf{w}$ must be nonzero because $\mathbf{v} = \mathbf{u} + i\mathbf{w}$ is nonzero. Therefore λ has a real eigenvector associated with it.
- If λ is not real then $\nu \neq 0$. If $\mathbf{v}$ is a real eigenvector then $\mathbf{w} = \mathbf{0}$ and $\mathbf{v} = \mathbf{u}$. Because $\mathbf{w} = \mathbf{0}$ and $\nu \neq 0$, the second equation in (5.4) implies that $\mathbf{u} = \mathbf{0}$ too. But this contradicts the fact that the eigenvector $\mathbf{v} = \mathbf{u}$ must be nonzero. Therefore, λ has no real eigenvector associated with it. ☐

5.2. **Constructing Solutions of First-Order Systems.** Eigenpairs can be used to construct real solutions of first-order differential systems with a constant coefficient matrix. The system we study is

$$\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} = \mathbf{Ax}, \tag{5.5}$$

where $\mathbf{x}(t)$ is a vector and $\mathbf{A}$ is a real $n \times n$ matrix. We begin with the following fact.

Fact 5. If $(\lambda, \mathbf{v})$ is an eigenpair of $\mathbf{A}$ then a solution of system (5.5) is

$$\mathbf{x}(t) = e^{\lambda t}\mathbf{v}. \tag{5.6}$$

Reason. Because $\lambda\mathbf{v} = \mathbf{Av}$, a direct calculation shows that

$$\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}\big(e^{\lambda t}\mathbf{v}\big) = e^{\lambda t}\lambda\mathbf{v} = e^{\lambda t}\mathbf{Av} = \mathbf{A}\big(e^{\lambda t}\mathbf{v}\big) = \mathbf{Ax},$$

whereby $\mathbf{x}(t)$ given by (5.6) solves system (5.5). ☐

Solutions constructed by recipe (5.6) are called *eigensolutions* of system (5.5). If $(\lambda, \mathbf{v})$ is a real eigenpair of $\mathbf{A}$ then the associated eigensolution is real. But if λ is an eigenvalue of $\mathbf{A}$ that is not real then no associated eigensolution is real. However, if we also use the eigensolution associated with the conjugate eigenpair $(\overline{\lambda}, \overline{\mathbf{v}})$ then we can construct two real solutions of system (5.5).

Fact 6. Let $(\lambda, \mathbf{v})$ be an eigenpair of $\mathbf{A}$ such that λ is not real. Then two real solutions of system (5.5) are

$$\mathbf{x}_1(t) = \mathrm{Re}\big(e^{\lambda t}\mathbf{v}\big), \qquad \mathbf{x}_2(t) = \mathrm{Im}\big(e^{\lambda t}\mathbf{v}\big). \tag{5.7}$$

These are the real and imaginary parts of the associated eigensolution.

Reason. Because $(\lambda, \mathbf{v})$ is an eigenpair of $\mathbf{A}$, by Fact 3 so is $(\overline{\lambda}, \overline{\mathbf{v}})$. By recipe (5.6) two eigensolutions of (5.5) are $e^{\lambda t}\mathbf{v}$ and $e^{\overline{\lambda} t}\overline{\mathbf{v}}$, which are complex conjugates of each other. Because system (5.5) is linear, by superposition two real solutions of (5.5) are given by

$$\mathbf{x}_1(t) = \frac{e^{\lambda t}\mathbf{v} + e^{\overline{\lambda} t}\overline{\mathbf{v}}}{2} = \mathrm{Re}\big(e^{\lambda t}\mathbf{v}\big) \,, \qquad \mathbf{x}_2(t) = \frac{e^{\lambda t}\mathbf{v} - e^{\overline{\lambda} t}\overline{\mathbf{v}}}{i2} = \mathrm{Im}\big(e^{\lambda t}\mathbf{v}\big) \,,$$

which are the real and imaginary parts of $e^{\lambda t}\mathbf{v}$, yielding (5.7). □

Recall that if we have n linearly independent real solutions $\mathbf{x}_1(t)$, $\mathbf{x}_2(t)$, $\cdots$, $\mathbf{x}_n(t)$, of system (5.5) then we can construct a fundamental matrix $\mathbf{\Psi}(t)$ by

$$\mathbf{\Psi}(t) = \big(\mathbf{x}_1(t) \quad \mathbf{x}_2(t) \quad \cdots \quad \mathbf{x}_n(t)\big) \,. \tag{5.8}$$

Recipes (5.6) and (5.7) will often, but not always, yield enough solutions to do this.

Example. Use eigenpairs to construct real solutions of

$$\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} = \mathbf{A}\mathbf{x} \,, \qquad \text{where} \quad \mathbf{A} = \begin{pmatrix} 3 & 2 \\ 2 & 3 \end{pmatrix} \,.$$

If possible, use these solutions to construct a fundamental matrix $\mathbf{\Psi}(t)$ for this system.

Solution. By a previous example we know that $\mathbf{A}$ has the real eigenpairs

$$\left(1\,, \begin{pmatrix} 1 \\ -1 \end{pmatrix}\right) \,, \qquad \left(5\,, \begin{pmatrix} 1 \\ 1 \end{pmatrix}\right) \,.$$

By recipe (5.6) the system has the real eigensolutions

$$\mathbf{x}_1(t) = e^{t} \begin{pmatrix} 1 \\ -1 \end{pmatrix} \,, \qquad \mathbf{x}_2(t) = e^{5t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \,.$$

These solutions are linearly independent because

$$\mathrm{Wr}[\mathbf{x}_1, \mathbf{x}_2](0) = \det\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} = 2 \neq 0 \,.$$

Therefore by (5.8) a fundamental matrix for the system is given by

$$\mathbf{\Psi}(t) = \big(\mathbf{x}_1(t) \quad \mathbf{x}_2(t)\big) = \begin{pmatrix} e^{t} & e^{5t} \\ -e^{t} & e^{5t} \end{pmatrix} \,.$$

Example. Use eigenpairs to construct real solutions of

$$\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} = \mathbf{B}\mathbf{x} \,, \qquad \text{where} \quad \mathbf{B} = \begin{pmatrix} 4 & -1 \\ 1 & 2 \end{pmatrix} \,.$$

If possible, use these solutions to construct a fundamental matrix $\mathbf{\Psi}(t)$ for this system.

Solution. By a previous example we know that $\mathbf{B}$ has the eigenpair

$$\left(3\,, \begin{pmatrix} 1 \\ 1 \end{pmatrix}\right) \,.$$

By recipe (5.6) the system has the real eigensolution

$$\mathbf{x}(t) = e^{3t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \,.$$

Because we have not constructed two linearly independent solutions, we cannot yet construct a fundamental matrix for this system.

Example. Use eigenpairs to construct real solutions of

$$\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} = \mathbf{C}\mathbf{x}\,, \qquad \text{where} \quad \mathbf{C} = \begin{pmatrix} 3 & 2 \\ -2 & 3 \end{pmatrix} .$$

If possible, use these solutions to construct a fundamental matrix $\boldsymbol{\Psi}(t)$ for this system.

Solution. By a previous example we know that $\mathbf{C}$ has the conjugate eigenpairs

$$\left(3 + i2\,, \begin{pmatrix} 1 \\ i \end{pmatrix}\right) , \qquad \left(3 - i2\,, \begin{pmatrix} 1 \\ -i \end{pmatrix}\right) .$$

Because

$$e^{(3+i2)t}\begin{pmatrix} 1 \\ i \end{pmatrix} = e^{3t}\big(\cos(2t) + i\sin(2t)\big)\begin{pmatrix} 1 \\ i \end{pmatrix} = e^{3t}\begin{pmatrix} \cos(2t) + i\sin(2t) \\ -\sin(2t) + i\cos(2t) \end{pmatrix} ,$$

by recipe (5.7) the system has the real solutions

$$\mathbf{x}_1(t) = e^{3t}\begin{pmatrix} \cos(2t) \\ -\sin(2t) \end{pmatrix} , \qquad \mathbf{x}_2(t) = e^{3t}\begin{pmatrix} \sin(2t) \\ \cos(2t) \end{pmatrix} .$$

These solutions are linearly independent because

$$\mathrm{Wr}[\mathbf{x}_1, \mathbf{x}_2](0) = \det\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = 1 \neq 0\,.$$

Therefore by (5.8) a fundamental matrix for the system is given by

$$\boldsymbol{\Psi}(t) = \big(\mathbf{x}_1(t) \quad \mathbf{x}_2(t)\big) = e^{3t}\begin{pmatrix} \cos(2t) & \sin(2t) \\ -\sin(2t) & \cos(2t) \end{pmatrix} .$$

Remark. Recipes (5.6) and (5.7) will always yield n linearly independent real solutions of system (5.5) whenever $p_{\mathbf{A}}(z)$ only has simple roots, as was the case in the first and third examples above. They will typically fail to do so whenever $p_{\mathbf{A}}(z)$ has a root with multiplicity greater than 1, as was the case in the second example above.

Finally, whenever we can construct a fundamental matrix $\boldsymbol{\Psi}(t)$ for system (5.5) from eigensolutions, we can use the fact that $e^{t\mathbf{A}}$ is the natural fundamental matrix for $t_I = 0$ to compute $e^{t\mathbf{A}}$ as

$$e^{t\mathbf{A}} = \boldsymbol{\Psi}(t)\boldsymbol{\Psi}(0)^{-1}\,. \tag{5.9}$$

It is easy to check that the right-hand side above satisfies the matrix-valued initial-value problem (4.2), so it must be equal to $e^{t\mathbf{A}}$.

Example. Compute $e^{t\mathbf{A}}$ for $\mathbf{A} = \begin{pmatrix} 3 & 2 \\ 2 & 3 \end{pmatrix}$.

Solution. In the first example given above we constructed a fundamental matrix $\boldsymbol{\Psi}(t)$ for the associated linear system. We obtained

$$\boldsymbol{\Psi}(t) = \begin{pmatrix} e^{t} & e^{5t} \\ -e^{t} & e^{5t} \end{pmatrix} .$$

Because

$$\mathbf{\Psi}(0) = \begin{pmatrix} e^0 & e^0 \\ -e^0 & e^0 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} ,$$

and $\det(\mathbf{\Psi}(0)) = 1 - (-1) = 2$, we see that

$$\mathbf{\Psi}(0)^{-1} = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}^{-1} = \frac{1}{2}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} ,$$

whereby formula (5.9) yields

$$e^{t\mathbf{A}} = \mathbf{\Psi}(t)\mathbf{\Psi}(0)^{-1} = \begin{pmatrix} e^t & e^{5t} \\ -e^t & e^{5t} \end{pmatrix} \frac{1}{2}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} e^t + e^{5t} & -e^t + e^{5t} \\ -e^t + e^{5t} & e^t + e^{5t} \end{pmatrix} .$$

Example. Compute $e^{t\mathbf{C}}$ for $\mathbf{C} = \begin{pmatrix} 3 & 2 \\ -2 & 3 \end{pmatrix}$.

Solution. In the third example given above we constructed a fundamental matrix $\mathbf{\Psi}(t)$ for the associated linear system. We obtained

$$\mathbf{\Psi}(t) = e^{3t}\begin{pmatrix} \cos(2t) & \sin(2t) \\ -\sin(2t) & \cos(2t) \end{pmatrix} .$$

Because

$$\mathbf{\Psi}(0) = e^0\begin{pmatrix} \cos(0) & \sin(0) \\ -\sin(0) & \cos(0) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \mathbf{I} ,$$

we see that $\mathbf{\Psi}(0)^{-1} = \mathbf{I}$, whereby formula (5.9) yields

$$e^{t\mathbf{C}} = \mathbf{\Psi}(t)\mathbf{\Psi}(0)^{-1} = \mathbf{\Psi}(t)\mathbf{I} = \mathbf{\Psi}(t) = e^{3t}\begin{pmatrix} \cos(2t) & \sin(2t) \\ -\sin(2t) & \cos(2t) \end{pmatrix} .$$

5.3. **Two-by-Two Fundamental Matrices.** Recipes (5.6) and (5.7) will always yield n linearly independent real solutions of system (5.5) with which we can construct a real fundamental matrix whenever the characteristic polynomial $p_{\mathbf{A}}(z)$ only has simple roots. For 2×2 matrices they will fail to do so whenever $p_{\mathbf{A}}(z) = (z-\mu)^2$ and $\mathbf{A} - \mu\mathbf{I} \neq \mathbf{0}$. In that case μ is the only eigenvalue of $\mathbf{A}$ and recipe (5.6) yields the real eigensolution

$$\mathbf{x}_1(t) = e^{\mu t}\mathbf{v} , \tag{5.10a}$$

where the vector $\mathbf{v}$ is proportional to any nonzero column of $\mathbf{A} - \mu\mathbf{I}$. In that case we can construct a second solution from any vector $\mathbf{w}$ that is *not* proportional to $\mathbf{v}$ by the recipe

$$\mathbf{x}_2(t) = e^{\mu t}\mathbf{w} + t\, e^{\mu t}(\mathbf{A} - \mu\mathbf{I})\mathbf{w} . \tag{5.10b}$$

This is just $e^{t\mathbf{A}}\mathbf{w}$ where $e^{t\mathbf{A}}$ is given by formula (4.19c). We can simplify this calculation by picking a $\mathbf{w}$ with a zero entry. After we have constructed $\mathbf{x}_1(t)$ and $\mathbf{x}_2(t)$ then we can construct the real fundamental matrix

$$\mathbf{\Psi}(t) = \begin{pmatrix} \mathbf{x}_1(t) & \mathbf{x}_2(t) \end{pmatrix} . \tag{5.11}$$

With this method we can construct a fundamental matrix of system (5.5) for any 2×2 matrix for which recipes (5.6) and (5.7) fail to do so.

Example. Construct a real fundamental matrix for the system

$$\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} = \mathbf{B}\mathbf{x}\,, \qquad \text{where} \quad \mathbf{B} = \begin{pmatrix} 4 & -1 \\ 1 & 2 \end{pmatrix} .$$

Solution. The characteristic polynomial of $\mathbf{B}$ is

$$p_{\mathbf{B}}(z) = z^2 - \mathrm{tr}(\mathbf{B})z + \det(\mathbf{B}) = z^2 - 6z + 9 = (z-3)^2 .$$

Because

$$\mathbf{B} - 3\mathbf{I} = \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix} ,$$

we see that $\mathbf{B}$ has the eigenpair

$$\left(3\,, \begin{pmatrix} 1 \\ 1 \end{pmatrix}\right) .$$

Recipe (5.10a) yields the real eigensolution

$$\mathbf{x}_1(t) = e^{3t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} .$$

Recipe (5.10b) with $\mathbf{w} = \begin{pmatrix} 1 & 0 \end{pmatrix}^{\mathrm{T}}$ yields the second real solution

$$\begin{aligned} \mathbf{x}_2(t) &= e^{3t} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + t\, e^{3t} \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ &= e^{3t} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + t\, e^{3t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = e^{3t} \begin{pmatrix} 1+t \\ t \end{pmatrix} . \end{aligned}$$

Therefore by (5.11) a real fundamental matrix for the system is given by

$$\mathbf{\Psi}(t) = e^{3t} \begin{pmatrix} 1 & 1+t \\ 1 & t \end{pmatrix} .$$

Because this method allows us to construct a fundamental matrix $\mathbf{\Psi}(t)$ for any system $\mathbf{x}' = \mathbf{A}\mathbf{x}$ with a 2×2 coefficient matrix $\mathbf{A}$, it also allows us to again use the fact that $e^{t\mathbf{A}}$ is the natural fundamental matrix for $t_I = 0$ to compute $e^{t\mathbf{A}}$ as (5.9),

(5.12) $$e^{t\mathbf{A}} = \mathbf{\Psi}(t)\mathbf{\Psi}(0)^{-1} .$$

Recall that while there are many fundamental matrices for every system $\mathbf{x}' = \mathbf{A}\mathbf{x}$, there is a unique matrix exponential $e^{t\mathbf{A}}$.

Example. Compute $e^{t\mathbf{B}}$ for $\mathbf{B} = \begin{pmatrix} 4 & -1 \\ 1 & 2 \end{pmatrix}$.

Solution. In the previous example we constructed a fundamental matrix $\mathbf{\Psi}(t)$ for the associated linear system. We obtained

$$\mathbf{\Psi}(t) = e^{3t} \begin{pmatrix} 1 & 1+t \\ 1 & t \end{pmatrix} .$$

Because

$$\mathbf{\Psi}(0) = e^{0} \begin{pmatrix} 1 & 1+0 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} ,$$

we see that

$$\mathbf{\Psi}(0)^{-1} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^{-1} = \frac{1}{-1} \begin{pmatrix} 0 & -1 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & -1 \end{pmatrix} .$$

Therefore by formula (5.12) we have

$$e^{t\mathbf{B}} = \mathbf{\Psi}(t)\mathbf{\Psi}(0)^{-1} = e^{3t} \begin{pmatrix} 1 & 1+t \\ 1 & t \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & -1 \end{pmatrix} = e^{3t} \begin{pmatrix} 1+t & -t \\ t & 1-t \end{pmatrix} .$$

Remark. We now have three methods for computing $e^{t\mathbf{A}}$ for any 2×2 matrix $\mathbf{A}$:

(1) the natural fundamental set method given by (4.14) in Section 4.3, which can be applied to any matrix;
(2) the matrix exponential formulas for 2×2 matrices given by (4.19) in Section 4.4, which were derived from the natural fundamental set method;
(3) formula (5.12) using any fundamental matrix $\mathbf{\Psi}(t)$ that is constructed by the eigen methods developed in this chapter.

Which of these methods is best to use depends upon what information is given in a particular problem. The remainder of this chapter will develop more eigen methods that can be used to compute $e^{t\mathbf{A}}$.

5.4. **Diagonalizable Matrices.** If recipe (5.6) yields n linearly independent solutions of the first-order system (5.5) then they can be used to directly construct $e^{t\mathbf{A}}$. The key to this construction is the following fact from linear algebra.

Fact 7. If a real $n\times n$ matrix $\mathbf{A}$ has n eigenpairs, $(\lambda_1, \mathbf{v}_1)$, $(\lambda_2, \mathbf{v}_2)$, $\cdots$, $(\lambda_n, \mathbf{v}_n)$, such that the eigenvectors $\mathbf{v}_1$, $\mathbf{v}_2$, $\cdots$, $\mathbf{v}_n$ are linearly independent then

$$\mathbf{A} = \mathbf{V}\mathbf{D}\mathbf{V}^{-1} \quad , \tag{5.13}$$

where $\mathbf{V}$ is the $n\times n$ matrix whose columns are the vectors $\mathbf{v}_1$, $\mathbf{v}_2$, $\cdots$, $\mathbf{v}_n$ — i.e.

$$\mathbf{V} = \begin{pmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \cdots & \mathbf{v}_n \end{pmatrix} , \tag{5.14a}$$

while $\mathbf{D}$ is the $n\times n$ diagonal matrix

$$\mathbf{D} = \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \lambda_n \end{pmatrix} . \tag{5.14b}$$

Reason. Underlying this result is the fact that

$$\begin{aligned} \mathbf{A}\mathbf{V} = \mathbf{A} \begin{pmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \cdots & \mathbf{v}_n \end{pmatrix} &= \begin{pmatrix} \mathbf{A}\mathbf{v}_1 & \mathbf{A}\mathbf{v}_2 & \cdots & \mathbf{A}\mathbf{v}_n \end{pmatrix} \\ &= \begin{pmatrix} \lambda_1\mathbf{v}_1 & \lambda_2\mathbf{v}_2 & \cdots & \lambda_n\mathbf{v}_n \end{pmatrix} \\ &= \begin{pmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \cdots & \mathbf{v}_n \end{pmatrix} \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \lambda_n \end{pmatrix} = \mathbf{V}\mathbf{D} . \end{aligned} \tag{5.15}$$

Once we show that $\mathbf{V}$ is invertible then (5.13) follows upon multiplying the above relation on the right by $\mathbf{V}^{-1}$.

We claim that $\mathbf{V}$ is invertible because the vectors $\mathbf{v}_1$, $\mathbf{v}_2$, $\cdots$, $\mathbf{v}_n$ are linearly independent. If $\mathbf{V}$ were not invertible then there would exist a nonzero vector $\mathbf{c}$ such that $\mathbf{Vc} = \mathbf{0}$. This would mean that

$$\mathbf{0} = \mathbf{Vc} = \begin{pmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \cdots & \mathbf{v}_n \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_n\mathbf{v}_n \,.$$

Because the vectors $\mathbf{v}_1$, $\mathbf{v}_2$, $\cdots$, $\mathbf{v}_n$ are linearly independent, the above relation implies that $c_1 = c_2 = \cdots = c_n = 0$, which contradicts the fact $\mathbf{c}$ is nonzero. Therefore the matrix $\mathbf{V}$ is invertible. Then (5.13) follows upon multiplying relation (5.15) on the right by $\mathbf{V}^{-1}$. ☐

We call a real $n \times n$ matrix $\mathbf{A}$ *diagonalizable* when there exists an invertible matrix $\mathbf{V}$ and a diagonal matrix $\mathbf{D}$ such that $\mathbf{A} = \mathbf{VDV}^{-1}$. To *diagonalize* $\mathbf{A}$ means to find such a $\mathbf{V}$ and $\mathbf{D}$. Fact 7 states that $\mathbf{A}$ is diagonalizable when it has n linearly independent eigenvectors. The converse of this statement is also true.

Fact 8. If a real $n \times n$ matrix $\mathbf{A}$ is diagonalizable then it has n linearly independent eigenvectors.

Reason. Because $\mathbf{A}$ is diagonalizable it has the form $\mathbf{A} = \mathbf{VDV}^{-1}$ where the matrix $\mathbf{V}$ is invertible and the matrix $\mathbf{D}$ is diagonal.

Let the vectors $\mathbf{v}_1$, $\mathbf{v}_2$, $\cdots$, $\mathbf{v}_n$ be the columns of $\mathbf{V}$. We claim these vectors are linearly independent. Indeed, if $\mathbf{0} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_n\mathbf{v}_n$ then because $\mathbf{V} = \begin{pmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \cdots & \mathbf{v}_n \end{pmatrix}$ we see that

$$\mathbf{0} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_n\mathbf{v}_n = \begin{pmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \cdots & \mathbf{v}_n \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix} = \mathbf{Vc} \,.$$

Because $\mathbf{V}$ is invertible, this implies that $\mathbf{c} = \mathbf{0}$. The vectors $\mathbf{v}_1$, $\mathbf{v}_2$, $\cdots$, $\mathbf{v}_n$ therefore are linearly independent.

Because $\mathbf{V} = \begin{pmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \cdots & \mathbf{v}_n \end{pmatrix}$ and because $\mathbf{A} = \mathbf{VDV}^{-1}$ where $\mathbf{D}$ has the form (5.14b), we see that

$$\begin{aligned}
\begin{pmatrix} \mathbf{Av}_1 & \mathbf{Av}_2 & \cdots & \mathbf{Av}_n \end{pmatrix} &= \mathbf{A} \begin{pmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \cdots & \mathbf{v}_n \end{pmatrix} \\
&= \mathbf{AV} = \mathbf{VDV}^{-1}\mathbf{V} = \mathbf{VD} \\
&= \begin{pmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \cdots & \mathbf{v}_n \end{pmatrix} \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \lambda_n \end{pmatrix} \\
&= \begin{pmatrix} \lambda_1\mathbf{v}_1 & \lambda_2\mathbf{v}_2 & \cdots & \lambda_n\mathbf{v}_n \end{pmatrix} \,.
\end{aligned}$$

Because the vectors $\mathbf{v}_1$, $\mathbf{v}_2$, $\cdots$, $\mathbf{v}_n$ are linearly independent, they are all nonzero. It then follows from the above relation that $(\lambda_1, \mathbf{v}_1)$, $(\lambda_2, \mathbf{v}_2)$, $\cdots$, $(\lambda_n, \mathbf{v}_n)$ are eigenpairs of $\mathbf{A}$, such that the eigenvectors $\mathbf{v}_1$, $\mathbf{v}_2$, $\cdots$, $\mathbf{v}_n$ are linearly independent. ☐

Example. Show that $\mathbf{A} = \begin{pmatrix} 3 & 2 \\ 2 & 3 \end{pmatrix}$ is diagonalizable, and diagonalize it.

Solution. By a previous example we know that $\mathbf{A}$ has the real eigenpairs

$$\left(1\,, \begin{pmatrix} 1 \\ -1 \end{pmatrix}\right), \qquad \left(5\,, \begin{pmatrix} 1 \\ 1 \end{pmatrix}\right).$$

Because we also know the eigenvectors are linearly independent, $\mathbf{A}$ is diagonalizable. Then (5.14) yields

$$\mathbf{V} = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}, \qquad \mathbf{D} = \begin{pmatrix} 1 & 0 \\ 0 & 5 \end{pmatrix}.$$

Because $\det(\mathbf{V}) = 2$, we have

$$\mathbf{V}^{-1} = \frac{1}{2}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}.$$

It follows from (5.13) that $\mathbf{A}$ is diagonalized as

$$\mathbf{A} = \mathbf{V}\mathbf{D}\mathbf{V}^{-1} = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 5 \end{pmatrix}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}.$$

Example. Show that $\mathbf{C} = \begin{pmatrix} 3 & 2 \\ -2 & 3 \end{pmatrix}$ is diagonalizable, and diagonalize it.

Solution. By a previous example we know that $\mathbf{C}$ has the conjugate eigenpairs

$$\left(3 + i2\,, \begin{pmatrix} 1 \\ i \end{pmatrix}\right), \qquad \left(3 - i2\,, \begin{pmatrix} 1 \\ -i \end{pmatrix}\right).$$

Because we also know the eigenvectors are linearly independent, $\mathbf{A}$ is diagonalizable. Then (5.14) yields

$$\mathbf{V} = \begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix}, \qquad \mathbf{D} = \begin{pmatrix} 3 + i2 & 0 \\ 0 & 3 - i2 \end{pmatrix}.$$

Because $\det(\mathbf{V}) = -i2$, we have

$$\mathbf{V}^{-1} = \frac{1}{-i2}\begin{pmatrix} -i & -1 \\ -i & 1 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix}.$$

It follows from (5.13) that $\mathbf{C}$ is diagonalized as

$$\mathbf{C} = \mathbf{V}\mathbf{D}\mathbf{V}^{-1} = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix}\begin{pmatrix} 3 + i2 & 0 \\ 0 & 3 - i2 \end{pmatrix}\begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix}.$$

Example. Use Fact 8 to show that $\mathbf{B} = \begin{pmatrix} 4 & -1 \\ 1 & 2 \end{pmatrix}$ is not diagonalizable.

Solution. By previous examples we know that $\mathbf{B}$ only has one real eigenvalue 3 and that all eigenvectors associated with 3 have the form

$$\alpha \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad \text{for some } \alpha \neq 0\,.$$

Because $\mathbf{B}$ does not have two linearly independent eigenvectors, it is not diagonalizable.

Remark. While not every matrix is diagonalizable, most matrices are. Here we give four criteria that ensure a real $n \times n$ matrix $\mathbf{A}$ is diagonalizable.

- If $\mathbf{A}$ has n distinct eigenvalues then it is diagonalizable.
- If $\mathbf{A}$ is symmetric ($\mathbf{A}^{\mathrm{T}} = \mathbf{A}$) then all of its eigenvalues are real ($\overline{\lambda_j} = \lambda_j$), and it will have n real eigenvectors $\mathbf{v}_j$ that can be normalized so that $\mathbf{v}_j^{\mathrm{T}} \mathbf{v}_k = \delta_{jk}$. With this normalization $\mathbf{V}^{-1} = \mathbf{V}^{\mathrm{T}}$.
- If $\mathbf{A}$ is skew-symmetric ($\mathbf{A}^{\mathrm{T}} = -\mathbf{A}$) then all of its eigenvalues are imaginary ($\overline{\lambda_j} = -\lambda_j$), and it will have n eigenvectors $\mathbf{v}_j$ that can be normalized so that $\mathbf{v}_j^{\mathrm{H}} \mathbf{v}_k = \delta_{jk}$. With this normalization $\mathbf{V}^{-1} = \mathbf{V}^{\mathrm{H}}$.
- If $\mathbf{A}$ is normal ($\mathbf{A}^{\mathrm{T}}\mathbf{A} = \mathbf{A}\mathbf{A}^{\mathrm{T}}$) then it will have n eigenvectors $\mathbf{v}_j$ that can be normalized so that $\mathbf{v}_j^{\mathrm{H}} \mathbf{v}_k = \delta_{jk}$. With this normalization $\mathbf{V}^{-1} = \mathbf{V}^{\mathrm{H}}$.

Matrices that are either symmetric or skew-symmetric are also normal. There are normal matrices that are neither symmetric nor skew-symmetric. Because the normal criterion is harder to verify than the symmetric and skew-symmetric criteria, it should be checked last. The first two examples that we gave above have distinct eigenvalues. The first example is symmetric. The second is normal, but is neither symmetric nor skew-symmetric.

5.5. **Computing Matrix Exponentials by Diagonalization.** We are now ready to give a direct construction of the matrix exponential $e^{t\mathbf{A}}$ when $\mathbf{A}$ is diagonalizable. We begin with the observation that it is very easy to compute the exponential of a diagonal matrix.

Fact 9. If $\mathbf{D}$ is a diagonal matrix then

$$(5.16) \qquad e^{t\mathbf{D}} = \begin{pmatrix} e^{\lambda_1 t} & 0 & \cdots & 0 \\ 0 & e^{\lambda_2 t} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & e^{\lambda_n t} \end{pmatrix}, \qquad \text{where} \quad \mathbf{D} = \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \lambda_n \end{pmatrix}.$$

Reason. Let

$$(5.17) \qquad \mathbf{\Phi}(t) = \begin{pmatrix} e^{\lambda_1 t} & 0 & \cdots & 0 \\ 0 & e^{\lambda_2 t} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & e^{\lambda_n t} \end{pmatrix}.$$

We will show that $\mathbf{\Phi}(t)$ satisfies the matrix-valued initial-value problem

$$\mathbf{\Phi}'(t) = \mathbf{D}\mathbf{\Phi}(t)\,, \qquad \mathbf{\Phi}(0) = \mathbf{I}\,. \tag{5.18}$$

This shows that $\mathbf{\Phi}(t)$ is the natural fundamental matrix associated with $t = 0$, whereby $\mathbf{\Phi}(t) = e^{t\mathbf{D}}$. We see from (5.17) that

$$\begin{aligned}\mathbf{\Phi}'(t) &= \begin{pmatrix} \lambda_1 e^{\lambda_1 t} & 0 & \cdots & 0 \\ 0 & \lambda_2 e^{\lambda_2 t} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \lambda_n e^{\lambda_n t} \end{pmatrix} \\ &= \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \lambda_n \end{pmatrix} \begin{pmatrix} e^{\lambda_1 t} & 0 & \cdots & 0 \\ 0 & e^{\lambda_2 t} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & e^{\lambda_n t} \end{pmatrix} = \mathbf{D}\mathbf{\Phi}(t)\,.\end{aligned}$$

We also see from (5.17) that $\mathbf{\Phi}(0) = \mathbf{I}$. Therefore $\mathbf{\Phi}(t)$ satisfies the initial-value problem (5.18), whereby $\mathbf{\Phi}(t) = e^{t\mathbf{D}}$. □

Fact 9 means that if we can diagonalize a matrix then it is easy to compute its matrix exponential.

Fact 10. If the real $n\times n$ matrix $\mathbf{A}$ has n eigenpairs, $(\lambda_1, \mathbf{v}_1)$, $(\lambda_2, \mathbf{v}_2)$, $\cdots$, $(\lambda_n, \mathbf{v}_n)$, such that the vectors $\mathbf{v}_1$, $\mathbf{v}_2$, $\cdots$, $\mathbf{v}_n$ are linearly independent then

$$e^{t\mathbf{A}} = \mathbf{V}e^{t\mathbf{D}}\mathbf{V}^{-1}\,, \tag{5.19}$$

where $\mathbf{V}$ and $\mathbf{D}$ are the $n\times n$ matrices given by (5.14).

Reason. Set $\mathbf{\Psi}(t) = \mathbf{V}e^{t\mathbf{D}}$. Because $\mathbf{A}$ is diagonalizable, $\mathbf{AV} = \mathbf{VD}$. It follows that

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{\Psi}(t) = \frac{\mathrm{d}}{\mathrm{d}t}\big(\mathbf{V}e^{t\mathbf{D}}\big) = \mathbf{V}\frac{\mathrm{d}}{\mathrm{d}t}e^{t\mathbf{D}} = \mathbf{VD}e^{t\mathbf{D}} = \mathbf{AV}e^{t\mathbf{D}} = \mathbf{A}\mathbf{\Psi}(t)\,,$$

whereby the matrix-valued function $\mathbf{\Psi}(t)$ satisfies

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{\Psi}(t) = \mathbf{A}\mathbf{\Psi}(t)\,.$$

Moreover, because $e^{0\mathbf{D}} = \mathbf{I}$ we see that $\mathbf{\Psi}(t)$ satisfies

$$\mathbf{\Psi}(0) = \mathbf{V}e^{0\mathbf{D}} = \mathbf{VI} = \mathbf{V}\,.$$

Because $\det\big(\mathbf{\Psi}(0)\big) = \det(\mathbf{V}) \neq 0$, we see that $\mathbf{\Psi}(t)$ is a fundamental matrix. Therefore by fact (5.9) we have

$$e^{t\mathbf{A}} = \mathbf{\Psi}(t)\mathbf{\Psi}(0)^{-1} = \big(\mathbf{V}e^{t\mathbf{D}}\big)\mathbf{V}^{-1} = \mathbf{V}e^{t\mathbf{D}}\mathbf{V}^{-1}\,.$$

□

Formula (5.19) provides a method for computing $e^{t\mathbf{A}}$ whenever $\mathbf{A}$ is diagonalizable. Because not every matrix is diagonalizable, it cannot always be applied. When it can be applied, most of the work needed to apply it goes into computing $\mathbf{V}$ and $\mathbf{V}^{-1}$. The matrix $e^{t\mathbf{D}}$ is simply given by formula (5.16). Once we have $\mathbf{V}$, $\mathbf{V}^{-1}$, and $e^{t\mathbf{D}}$, formula (5.19) requires two matrix multiplications.

Example. Compute $e^{t\mathbf{A}}$ for $\mathbf{A} = \begin{pmatrix} 3 & 2 \\ 2 & 3 \end{pmatrix}$.

Solution. By a previous example we know that $\mathbf{A}$ is diagonalizable with $\mathbf{A} = \mathbf{V}\mathbf{D}\mathbf{V}^{-1}$ where

$$\mathbf{V} = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}, \qquad \mathbf{D} = \begin{pmatrix} 1 & 0 \\ 0 & 5 \end{pmatrix}, \qquad \mathbf{V}^{-1} = \frac{1}{2}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}.$$

By formulas (5.16) and (5.19) we have

$$\begin{aligned} e^{t\mathbf{A}} = \mathbf{V}e^{t\mathbf{D}}\mathbf{V}^{-1} &= \frac{1}{2}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} e^t & 0 \\ 0 & e^{5t} \end{pmatrix}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \\ &= \frac{1}{2}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} e^t & -e^t \\ e^{5t} & e^{5t} \end{pmatrix} = \frac{1}{2}\begin{pmatrix} e^{5t} + e^t & e^{5t} - e^t \\ e^{5t} - e^t & e^{5t} + e^t \end{pmatrix}. \end{aligned}$$

Example. Compute $e^{t\mathbf{C}}$ for $\mathbf{C} = \begin{pmatrix} 3 & 2 \\ -2 & 3 \end{pmatrix}$.

Solution. By a previous example we know that $\mathbf{C}$ is diagonalizable with $\mathbf{C} = \mathbf{V}\mathbf{D}\mathbf{V}^{-1}$ where

$$\mathbf{V} = \begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix}, \qquad \mathbf{D} = \begin{pmatrix} 3 + i2 & 0 \\ 0 & 3 - i2 \end{pmatrix}, \qquad \mathbf{V}^{-1} = \frac{1}{2}\begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix}.$$

By formula (5.16) we have

$$e^{t\mathbf{D}} = \begin{pmatrix} e^{(3+i2)t} & 0 \\ 0 & e^{(3-i2)t} \end{pmatrix} = e^{3t}\begin{pmatrix} e^{i2t} & 0 \\ 0 & e^{-i2t} \end{pmatrix}.$$

By formula (5.19) we have

$$\begin{aligned} e^{t\mathbf{C}} = \mathbf{V}e^{t\mathbf{D}}\mathbf{V}^{-1} &= \frac{e^{3t}}{2}\begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix}\begin{pmatrix} e^{i2t} & 0 \\ 0 & e^{-i2t} \end{pmatrix}\begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix} \\ &= \frac{e^{3t}}{2}\begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix}\begin{pmatrix} e^{i2t} & -ie^{i2t} \\ e^{-i2t} & ie^{-i2t} \end{pmatrix} = \frac{e^{3t}}{2}\begin{pmatrix} e^{i2t} + e^{-i2t} & -ie^{i2t} + ie^{-i2t} \\ ie^{i2t} - ie^{-i2t} & e^{i2t} + e^{-i2t} \end{pmatrix} \\ &= \frac{e^{3t}}{2}\begin{pmatrix} 2\cos(2t) & 2\sin(2t) \\ -2\sin(2t) & 2\cos(2t) \end{pmatrix} = e^{3t}\begin{pmatrix} \cos(2t) & \sin(2t) \\ -\sin(2t) & \cos(2t) \end{pmatrix}. \end{aligned}$$

Remark. Because $\mathbf{A}$ is real, $e^{t\mathbf{A}}$ must be real. As the previous example illustrates, the matrices $\mathbf{V}$ and $\mathbf{D}$ may not be real, but will always combine in formula (5.19) to yield the real result. In particular, formula (5.19) allows us to directly compute $e^{t\mathbf{A}}$ without computing a fundamental set of real solutions as an intermediate step.

Remark. Formula (5.19) is a more efficient way to compute $e^{t\mathbf{A}}$ than by formula (4.14) of the Natural Fundamental Set Method whenever $\mathbf{A}$ is diagonalizable and is not annihilated by a polynomial of small degree, which is usually the case when n is large. This is because computing the powers of $\mathbf{A}$ needed in formula (4.14) becomes time consuming for larger matrices. For example, it can require n^3 multiplications for each power of $\mathbf{A}$ needed beyond the first. This means that if $\mathbf{A}$ is annihilated by a polynomial of degree m then formula (4.14) can require $(m-2)n^3$ multiplications. In contrast, formula (5.19) typically requires $(n+1)n^2$ multiplications. This is because if $\mathbf{A}$ is diagonalizable with $\mathbf{A} = \mathbf{V}\mathbf{D}\mathbf{V}^{-1}$ then

$$\mathbf{A}^k = \mathbf{V}\mathbf{D}^k\mathbf{V}^{-1} \quad \text{for every positive integer } k\,.$$

Once $\mathbf{A}$ has been diagonalized, we can compute $\mathbf{A}^k$ by first computing $\mathbf{D}^k$, which requires $(k-1)n$ multiplications, and then computing $\mathbf{V}\mathbf{D}^k\mathbf{V}^{-1}$, which requires another $(n+1)n^2$ multiplications. This contrasts with the $(k-1)n^3$ multiplications it takes to compute $\mathbf{A}^k$ directly.

5.6. **Generalized Eigenpairs.** So far the only method we have studied for computing a fundamental matrix of system (5.5) for a general $n\times n$ matrix $\mathbf{A}$ is the fundamental set method for computing $e^{t\mathbf{A}}$ given by (4.14) in Chapter 4. The eigen methods presented so far in this chapter only yield n linearly independent solutions when $\mathbf{A}$ is either diagonalizable or 2×2. We now show how to extend eigen methods so that they yield n linearly independent solutions of system (5.5) for any $n\times n$ matrix $\mathbf{A}$.

Definition. If λ is an eigenvalue of a matrix $\mathbf{A}$ then a nonzero vector $\mathbf{v}$ that satisfies

$$(\mathbf{A}-\lambda\mathbf{I})^k\mathbf{v} = \mathbf{0} \quad \text{for some positive integer } k \tag{5.20}$$

is called a *generalized eigenvector* of $\mathbf{A}$ associated with λ, and $(\lambda, \mathbf{v})$ is called a *generalized eigenpair* of $\mathbf{A}$. The smallest k such that (5.20) holds is called its *degree*.

Remark. A generalized eigenvector has degree 1 if and only if it is an eigenvector.

The following fact generalizes recipe (5.10b), which we used to construct a second solution for 2×2 matrices that are not diagonalizable.

Fact 11. If $(\lambda, \mathbf{v})$ is a generalized eigenpair of $\mathbf{A}$ of degree k then a solution of system (5.5) is

$$\mathbf{x}(t) = e^{\lambda t}\left(\mathbf{v} + t\,(\mathbf{A}-\lambda\mathbf{I})\mathbf{v} + \cdots + \frac{t^{k-1}}{(k-1)!}\,(\mathbf{A}-\lambda\mathbf{I})^{k-1}\mathbf{v}\right). \tag{5.21}$$

Reason. By (5.20) and (5.21) we see that

$$\begin{aligned}(\mathbf{A}-\lambda\mathbf{I})\mathbf{x}(t) &= e^{\lambda t}\left((\mathbf{A}-\lambda\mathbf{I})\mathbf{v} + t\,(\mathbf{A}-\lambda\mathbf{I})^2\mathbf{v} + \cdots + \frac{t^{k-1}}{(k-1)!}\,(\mathbf{A}-\lambda\mathbf{I})^{k}\mathbf{v}\right)\\ &= e^{\lambda t}\left((\mathbf{A}-\lambda\mathbf{I})\mathbf{v} + t\,(\mathbf{A}-\lambda\mathbf{I})^2\mathbf{v} + \cdots + \frac{t^{k-2}}{(k-2)!}\,(\mathbf{A}-\lambda\mathbf{I})^{k-1}\mathbf{v}\right).\end{aligned}$$

Then differentiating (5.21) and using the above relation yields

$$\begin{aligned}\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t}(t) &= \lambda e^{\lambda t}\left(\mathbf{v} + t\,(\mathbf{A}-\lambda\mathbf{I})\mathbf{v} + \cdots + \frac{t^{k-1}}{(k-1)!}\,(\mathbf{A}-\lambda\mathbf{I})^{k-1}\mathbf{v}\right)\\ &\quad + e^{\lambda t}\left((\mathbf{A}-\lambda\mathbf{I})\mathbf{v} + \cdots + \frac{t^{k-2}}{(k-2)!}\,(\mathbf{A}-\lambda\mathbf{I})^{k-1}\mathbf{v}\right)\\ &= \lambda\mathbf{x}(t) + (\mathbf{A}-\lambda\mathbf{I})\mathbf{x}(t)\\ &= \mathbf{A}\mathbf{x}(t)\,.\end{aligned}$$

Therefore $\mathbf{x}(t)$ given by (5.21) is a solution of system (5.5). $\square$

The reason the above recipe yields n linearly independent solutions of system (5.5) is the following fact from linear algebra.

Fact 12. If $\mathbf{A}$ is an $n\times n$ matrix and λ is an eigenvalue of $\mathbf{A}$ that is a root of $p_{\mathbf{A}}(z)$ with multiplicity m then there exists m generalized eigenpairs, $(\lambda, \mathbf{v}_1)$, $(\lambda, \mathbf{v}_2)$, $\cdots$, $(\lambda, \mathbf{v}_m)$, such that the generalized vectors $\mathbf{v}_1$, $\mathbf{v}_2$, $\cdots$, $\mathbf{v}_m$ are linearly independent.

Remark. The proof of this fact is far beyond the scope of this course. However, you do not need to know the proof of this fact to use it.

Fact 12 does not give us a simple way to compute the m generalized eigenvectors $\mathbf{v}_1$, $\mathbf{v}_2$, $\cdots$, $\mathbf{v}_m$. If n is not too large then we can compute m linearly independent solutions of the algebraic system

$$(\mathbf{A} - \lambda\mathbf{I})^m\mathbf{v} = \mathbf{0}\,.$$

However if m and n are both not small then computing the powers $(\mathbf{A} - \lambda\mathbf{I})^m$ might take some time. In general we can start by looking for all solutions of

$$(\mathbf{A} - \lambda\mathbf{I})\mathbf{v} = \mathbf{0}\,.$$

If there are m linearly independent solutions of this system then we are done. (This will be the case when $\mathbf{A}$ is diagonalizable.) Otherwise we can look for all solutions of

$$(\mathbf{A} - \lambda\mathbf{I})^2\mathbf{v} = \mathbf{0}\,.$$

There will be solutions of this system that are not solutions of the previous system. If there are m linearly independent solutions of this system then we are done. Otherwise we continue to the next power. We will always find solutions of the system associated with each successive power that are not solutions of the previous system until we have found m linearly independent generalized eigenvectors. This can happen for a power much less than m.

One case for which there is a relatively direct algorithm for computing m linearly independent generalized eigenvectors is when there is only one linearly independent eigenvector associated with λ. In that case we can generate m linearly independent generalized eigenvectors $\mathbf{v}_1$, $\mathbf{v}_2$, $\cdots$, $\mathbf{v}_m$ by finding any nonzero solution of the system of m systems

$$(\mathbf{A} - \lambda\mathbf{I})\mathbf{v}_1 = \mathbf{0}\,, \quad (\mathbf{A} - \lambda\mathbf{I})\mathbf{v}_2 = \mathbf{v}_1\,, \quad \cdots \quad (\mathbf{A} - \lambda\mathbf{I})\mathbf{v}_m = \mathbf{v}_{m-1}\,. \tag{5.22}$$

This system does not have a unique solution because we can add any multiple of $\mathbf{v}_1$ to each $\mathbf{v}_k$ and still have a solution. However, we do not need the general solution of this system — any solution will do. A set of generalized eigenvectors generated in this way is called an *eigenchain* of *length* m. It can be checked from (5.22) that for every $k = 1$, $\cdots$, m these vectors satisfy

$$(\mathbf{A} - \lambda\mathbf{I})^j\mathbf{v}_k = \mathbf{v}_{k-j} \quad \text{when } j < k\,, \quad \text{and} \quad (\mathbf{A} - \lambda\mathbf{I})^k\mathbf{v}_k = \mathbf{0}\,. \tag{5.23}$$

In particular, each $\mathbf{v}_k$ is a generalized eigenvector of degree k.

Given an eigenchain of length m associated with λ, we can generate m linearly independent solutions of the system (5.5) by

$$\begin{aligned} \mathbf{x}_1(t) &= e^{\lambda t}\mathbf{v}_1\,, \\ \mathbf{x}_2(t) &= e^{\lambda t}\big(\mathbf{v}_2 + t\,\mathbf{v}_1\big)\,, \\ \mathbf{x}_3(t) &= e^{\lambda t}\big(\mathbf{v}_3 + t\,\mathbf{v}_2 + \tfrac{1}{2}t^2\mathbf{v}_1\big)\,, \\ &\ \ \vdots \\ \mathbf{x}_m(t) &= e^{\lambda t}\big(\mathbf{v}_m + t\,\mathbf{v}_{m-1} + \cdots + \tfrac{1}{(m-1)!}t^{m-1}\mathbf{v}_1\big)\,, \end{aligned} \tag{5.24}$$

Here each $\mathbf{x}_k(t)$ is obtained by setting $\mathbf{v} = \mathbf{v}_k$ in recipe (5.21) and using relations (5.23).

EXERCISES ON EIGEN METHODS

For problems 1-6 find eigenpairs for the following matrices.

(1) $\mathbf{A} = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$

Short Answer
Solution

(2) $\mathbf{A} = \begin{pmatrix} 3 & -2 \\ 3 & -1 \end{pmatrix}$

Short Answer
Solution

(3) $\mathbf{A} = \begin{pmatrix} 3 & -2 \\ 2 & -2 \end{pmatrix}$

Short Answer
Solution

(4) $\mathbf{A} = \begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix}$

Short Answer
Solution

(5) $\mathbf{A} = \begin{pmatrix} 3 & 2 & 2 \\ 1 & 4 & 1 \\ -2 & -4 & -1 \end{pmatrix}$

Short Answer
Solution

(6) $\mathbf{A} = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$

Short Answer
Solution

For problems 7 - 9 find all the eigenvectors associated with the given eigenvalue.

(7) $\mathbf{A} = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{pmatrix}$ with eigenvalue $\lambda = 1$.

Solution

(8) $\mathbf{A} = \begin{pmatrix} 2 & -1 & 0 \\ 0 & 3 & 0 \\ -7 & -7 & -5 \end{pmatrix}$ with eigenvalue $\lambda = 2$.

Solution

(9) $\mathbf{A} = \begin{pmatrix} 3 & 7 & 5 & 5 \\ 0 & 1 & 0 & 0 \\ 0 & -5 & -2 & -5 \\ 0 & 2 & 0 & 3 \end{pmatrix}$ with eigenvalue $\lambda = 3$.

Solution

(10) Find the eigenpairs for a matrix of the form

$$\mathbf{A} = \begin{pmatrix} a & b \\ -b & a \end{pmatrix}$$

Solution

(11) Find the eigenpairs for a matrix of the form

$$\mathbf{A} = \begin{pmatrix} a & b \\ b & -a \end{pmatrix}$$

Solution

(12) Consider the following matrix

$$\mathbf{A} = \begin{pmatrix} 2\alpha & 1 \\ -1 & 0 \end{pmatrix}.$$

For what values of α are the eigenvalues real, conjugate pairs, or repeated?

Solution

For problems 13- 19 find a general (real-valued) solution to the system of differential equations. As $t \to \infty$, what happens to the solutions?

(13) $\mathbf{x}' = \begin{pmatrix} 1 & 1 \\ 4 & -2 \end{pmatrix} \mathbf{x}$

Short Answer
Solution

(14) $\mathbf{x}' = \begin{pmatrix} 1 & -1 \\ 1 & 3 \end{pmatrix} \mathbf{x}$

Short Answer
Solution

(15) $\mathbf{x}' = \begin{pmatrix} 5 & -1 \\ 3 & 1 \end{pmatrix} \mathbf{x}$

Short Answer
Solution

(16) $\mathbf{x}' = \begin{pmatrix} -1 & -4 \\ 1 & -1 \end{pmatrix} \mathbf{x}$

Short Answer
Solution

(17) $\mathbf{x}' = \begin{pmatrix} 1 & 2 \\ -5 & -1 \end{pmatrix} \mathbf{x}$

Short Answer
Solution

(18) $\mathbf{x}' = \begin{pmatrix} 4 & -2 \\ 8 & -4 \end{pmatrix} \mathbf{x}$

Short Answer
Solution

(19) $\mathbf{x}' = \begin{pmatrix} 1 & 1 & 2 \\ 1 & 2 & 1 \\ 2 & 1 & 1 \end{pmatrix} \mathbf{x}$

Short Answer
Solution

For problems 20- 23 find a general (real-valued) solution the system of differential equations and construct a fundamental matrix, $\Psi(t)$.

(20) $\mathbf{x}' = \begin{pmatrix} 3 & 6 \\ -1 & -2 \end{pmatrix} \mathbf{x}$

Short Answer
Solution

(21) $\mathbf{x}' = \begin{pmatrix} 1 & -1 \\ 5 & -3 \end{pmatrix} \mathbf{x}$

Short Answer
Solution

(22) $\mathbf{x}' = \begin{pmatrix} 4 & -1 \\ 1 & 2 \end{pmatrix} \mathbf{x}$

Short Answer
Solution

(23) $\mathbf{x}' = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} \mathbf{x}$

Short Answer
Solution

For problems 24- 26 solve the initial value problem.

(24) $\mathbf{x}' = \begin{pmatrix} -3 & \sqrt{2} \\ \sqrt{2} & -2 \end{pmatrix} \mathbf{x}, \qquad \mathbf{x}(0) = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$

Short Answer
Solution

(25) $\mathbf{x}' = \begin{pmatrix} -3 & 2 \\ -1 & -1 \end{pmatrix} \mathbf{x}, \qquad \mathbf{x}(0) = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$

Short Answer
Solution

(26) $\mathbf{x}' = \begin{pmatrix} 2 & \frac{3}{2} \\ -\frac{3}{2} & -1 \end{pmatrix} \mathbf{x}, \qquad \mathbf{x}(0) = \begin{pmatrix} 3 \\ -2 \end{pmatrix}$

Short Answer
Solution

For problems 27- 28 compute $e^{t\mathbf{A}}$ *using the fundamental matrix,* $\Psi(t)$.

(27) $\mathbf{A} = \begin{pmatrix} 5 & 3 \\ 5 & 3 \end{pmatrix}$

Short Answer

Solution

(28) $\mathbf{A} = \begin{pmatrix} 1 & -4 \\ 4 & -7 \end{pmatrix}$

Short Answer
Solution

(29) Recall exercise 10 from the chapter on Linear Systems: General Methods and Theory. That exercise considered a coupled system of pendula governed by the second-order system of differential equations

$$\theta_1'' = -\frac{g}{\ell}\theta_1 - \frac{k}{m_1}(\theta_1 - \theta_2)\,,$$
$$\theta_2'' = -\frac{g}{\ell}\theta_2 + \frac{k}{m_2}(\theta_1 - \theta_2)\,.$$

This can be written as a first order linear system $\mathbf{x}' = \mathbf{A}\mathbf{x}$, where

$$\mathbf{A} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -\frac{g}{\ell} - \frac{k}{m_1} & 0 & \frac{k}{m_1} & 0 \\ 0 & 0 & 0 & 1 \\ \frac{k}{m_2} & 0 & -\frac{g}{\ell} - \frac{k}{m_1} & 0 \end{pmatrix}.$$

Find a general (real-valued) solution to this system.

Solution

(30) Show that any $n \times n$ Hermitian matrix ($\mathbf{A}^* = \mathbf{A}$) has only real eigen-values.

Solution

(31) Show that any $n \times n$ anti-Hermitian matrix ($\mathbf{A}^* = -\mathbf{A}$) has only imaginary eigen-values.

Solution

For problems 32- 33 show that the following matrices are diagonalizable first. Then, diagonalize them.

(32) $\mathbf{A} = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$

Short Answer
Solution

(33) $\mathbf{A} = \begin{pmatrix} 3 & -2 \\ 4 & -1 \end{pmatrix}$

Short Answer
Solution

(34) For which values of α is the following matrix diagonalizable?

$$\mathbf{A} = \begin{pmatrix} 1 & \alpha \\ 0 & 1 \end{pmatrix}$$

Solution

For problems 35- 36 calculate $e^{t\mathbf{A}}$ via diagonalization.

(35) $\mathbf{A} = \begin{pmatrix} 2 & -1 \\ 3 & -2 \end{pmatrix}$

Short Answer
Solution

(36) $\mathbf{A} = \begin{pmatrix} 1 & 5 \\ -1 & 3 \end{pmatrix}$

Short Answer
Solution

(37) Suppose that $\mathbf{V}$ is an invertible $n \times n$ matrix. Prove the following formula for a given $n \times n$ matrix $\mathbf{A}$.

$$e^{t\mathbf{VAV}^{-1}} = \mathbf{V}e^{t\mathbf{A}}\mathbf{V}^{-1}.$$

(Hint: Show that both sides satisfy the same initial value problem.)

Solution

(38) It is a well know result in linear algebra that any 2×2 non-diagonalizable matrix can be written as $\mathbf{A} = \mathbf{VJV}^{-1}$, where $\mathbf{V}$ is the matrix of generalized eigenvectors associated with the eigenvaue a and $\mathbf{J}$ is the matrix of the form

$$\mathbf{J} = \begin{pmatrix} a & 1 \\ 0 & a \end{pmatrix}.$$

Write an explicit formula for $e^{t\mathbf{J}}$ by computing the n^{th} power of $\mathbf{J}$ and using the series representation of $e^{t\mathbf{J}}$. Use this to arrive at a formula for $e^{t\mathbf{A}}$ in terms of $\mathbf{V}$? Check that this is consistent with the general formula for a 2×2 matrix with eigen-value a of multiplicity 2. (Hint: use the result from problem 37.)

Solution

Navigation to Other Chapters

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Ordinary Differential Equations

I. First-Order Ordinary Differential Equations

II. Higher-Order Linear Ordinary Differential Equations

III. First-Order Systems of Ordinary Differential Equations