III. First-Order Systems of Ordinary Differential Equations
7. Linear Planar Systems

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7. Linear Planar Systems

We now consider homogeneous linear systems of the form

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} x \\ y \end{pmatrix} = \mathbf{A}\begin{pmatrix} x \\ y \end{pmatrix}, \qquad \text{where} \quad \mathbf{A} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}, \tag{7.1}$$

where the coefficient matrix $\mathbf{A}$ is real and constant. Such a system is called *planar* because every solution of it can be thought of as tracing out a curve $(x(t), y(t))$ in the xy-plane as t varies.

7.1. **Phase-Plane Portraits.**

Of course, we have seen that solutions to system (7.1) can be expressed analytically as

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = e^{t\mathbf{A}}\begin{pmatrix} x_I \\ y_I \end{pmatrix}, \qquad \text{where} \quad \begin{pmatrix} x(0) \\ y(0) \end{pmatrix} = \begin{pmatrix} x_I \\ y_I \end{pmatrix}, \tag{7.2}$$

and $e^{t\mathbf{A}}$ is given by one of the following three formulas that depend upon the roots of the characteristic polynomial $p(z) = \det(z\mathbf{I} - \mathbf{A}) = z^2 - \mathrm{tr}(\mathbf{A})z + \det(\mathbf{A})$.

- If $p(z)$ has simple real roots $\mu \pm \nu$ with $\nu \neq 0$ then

$$e^{t\mathbf{A}} = e^{\mu t}\left[\cosh(\nu t)\mathbf{I} + \frac{\sinh(\nu t)}{\nu}(\mathbf{A} - \mu\mathbf{I})\right]. \tag{7.3a}$$

- If $p(z)$ has conjugate roots $\mu \pm i\nu$ with $\nu \neq 0$ then

$$e^{t\mathbf{A}} = e^{\mu t}\left[\cos(\nu t)\mathbf{I} + \frac{\sin(\nu t)}{\nu}(\mathbf{A} - \mu\mathbf{I})\right]. \tag{7.3b}$$

- If $p(z)$ has a double real root μ then

$$e^{t\mathbf{A}} = e^{\mu t}\left[\mathbf{I} + t\,(\mathbf{A} - \mu\mathbf{I})\right]. \tag{7.3c}$$

While such analytic formulas for these solutions are useful, often insight into their behavior can be quickly gained through a graph called a *phase-plane portrait* (or simply a *phase portrait*) of system (7.1).

As we observed above, any solution of (7.1) can be thought of as tracing out a curve $(x(t), y(t))$ in the xy-plane as t is varied through all real values. Each such curve is called an *orbit* or *trajectory* of the system. Any orbit traced out by a stationary solution consists of a single point — a so-called *stationary point*. For any orbit traced out by a nonstationary solution, the direction in which the solution moves as time increases is called the *orientation* or *direction of motion* of the orbit. More specifically, if $(x(t), y(t))$ is a nonstationary solution then the orientation at any point $(x(t), y(t))$ is the direction of the nonzero vector $(x'(t), y'(t))$, which is tangent to the orbit at $(x(t), y(t))$.

Fact 1. Every point in the xy-plane has exactly one orbit passing through it.

Reason. This is a consequence of the existence and uniqueness theorem. □

Fact 1 implies that orbits fill the xy-plane, and that orbits cannot cross or otherwise share a point. In particular, no orbit can cross itself. Any picture that indicates how these orbits fill the xy-plane and how they are oriented is called a phase-plane portrait of system (7.1).

A phase-plane portrait of system (7.1) can be either rendered by computer or sketched by hand. In either case there are three common approaches to creating one.

1. Plot orbits of the real eigensolutions of system (7.1).
2. Plot orbits of representative solutions either by analyzing or plotting formula (7.2) with $e^{t\mathbf{A}}$ given by (7.3) or by solving system (7.1) numerically.
3. Plot a direction field.

Because each of these approaches offer different advantages, it is common to employ more than one of them. Often all three approaches are employed when using a computer. The first approach should always be used because the eigensolutions typically are not picked up by the other two approaches.

Remark. Of course, the origin will be an orbit of system (7.1) for every $\mathbf{A}$. The solution that starts at the origin will stay at the origin. Points that give rise to solutions that do not move are called *stationary points*. In that case the entire orbit is a single point. Stationary points are sometimes called *equilibrium points*, *critical points*, or *fixed points*. This wealth of names is one measure of how important they are.

7.1.1. *Eigensolution Orbits.* Whether a phase-plane portrait is rendered by computer or sketched by hand, the first step is to identify and plot the orbits of all real eigensolutions of the linear system (7.1). If the matrix $\mathbf{A}$ has a real eigenpair $(\lambda, \mathbf{v})$ then system (7.1) has real eigensolutions of the form

$$\mathbf{x}(t) = ce^{\lambda t}\mathbf{v}\,, \quad \text{where } c \text{ is any nonzero real constant.} \tag{7.4}$$

The orbits of these solutions are called *eigensolution orbits*. They lie on the line $\mathbf{x} = c\mathbf{v}$ parametrized by c. This line is easy to plot; it is simply the line that passes through the origin and the point $\mathbf{v}$. There are three possibilities.

- If $\lambda > 0$ then the line $\mathbf{x} = c\mathbf{v}$ consists of three orbits: the origin, corresponding to $c = 0$, plus the two remaining half-lines, corresponding to $c > 0$ and $c < 0$. Because $\lambda > 0$ all solutions on the half-lines will move away from the origin as t increases. We indicate this orientation by placing arrowheads that point away from the origin on each half-line pointing away. If $\mathbf{v}$ lies on the x-axis then it should look something like the following picture.

  If $\mathbf{v}$ does not lie on the x-axis then this picture should be rotated accordingly.

- If $\lambda < 0$ then the line $\mathbf{x} = c\mathbf{v}$ consists of three orbits: the origin, corresponding to $c = 0$, plus the two remaining half-lines, corresponding to $c > 0$ and $c < 0$. Because $\lambda < 0$ all solutions on the half-lines will move towards the origin as t increases. We indicate this orientation by placing arrowheads that point towards the origin on each half-line. If $\mathbf{v}$ lies on the x-axis then it should look something like the following picture.

  If $\mathbf{v}$ does not lie on the x-axis then this picture should be rotated accordingly.

- If $\lambda = 0$ then every point on the line $\mathbf{x} = c\mathbf{v}$ is a stationary point, and thereby is an orbit. We indicate this case by placing circles on each half-line. If $\mathbf{v}$ lies on the x-axis then it should look something like the following picture.

If $\mathbf{v}$ does not lie on the x-axis then this picture should be rotated accordingly.

For every real eigenpair of $\mathbf{A}$ we should indicate these real eigensolution orbits in our phase-plane portrait before adding anything else. If $\mathbf{A}$ has no real eigenpairs then we are spared this step. We will illustrate this approach in the next two sections when we classify and sketch the phase-plane portraits that can arise from a system in the form (7.1).

Remark. Because real eigensolutions lie on the line $\mathbf{x} = c\mathbf{v}$ parametrized by c, they are sometimes called *line solutions*. This name can be misleading because none of the solutions moves along the entire line.

Example. Sketch the eigensolution orbits in the phase-plane for the planar system

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} x \\ y \end{pmatrix} = \mathbf{A}\begin{pmatrix} x \\ y \end{pmatrix}, \qquad \text{where} \quad \mathbf{A} = \begin{pmatrix} 0 & 2 \\ 1 & 1 \end{pmatrix}.$$

Solution. The characteristic polynomial of $\mathbf{A}$ is

$$p(z) = z^2 - \mathrm{tr}(\mathbf{A})z + \det(\mathbf{A}) = z^2 - z - 2 = (z+1)(z-2).$$

Therefore the eigenvalues of $\mathbf{A}$ are -1 and 2. Because

$$\mathbf{A} + \mathbf{I} = \begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix}, \qquad \mathbf{A} - 2\mathbf{I} = \begin{pmatrix} -2 & 2 \\ 1 & -1 \end{pmatrix},$$

we can read off that $\mathbf{A}$ has the eigenpairs

$$\left(-1, \begin{pmatrix} 2 \\ -1 \end{pmatrix}\right), \qquad \left(2, \begin{pmatrix} 1 \\ 1 \end{pmatrix}\right).$$

The corresponding real eigensolutions have the form

$$\mathbf{x}_1(t) = c_1 e^{-t}\begin{pmatrix} 2 \\ -1 \end{pmatrix}, \qquad \mathbf{x}_2(t) = c_2 e^{2t}\begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

The orbits associated with $\mathbf{x}_1(t)$ lie on the line through the point $(2, -1)$ in the xy-plane (the line $y = -\frac{1}{2}x$) and move towards the origin. The orbits associated with $\mathbf{x}_2(t)$ lie on the line through the point $(1, 1)$ in the xy-plane (the line $y = x$) and move away from the origin. These eigensolution orbits are shown below in Figure 7.1.

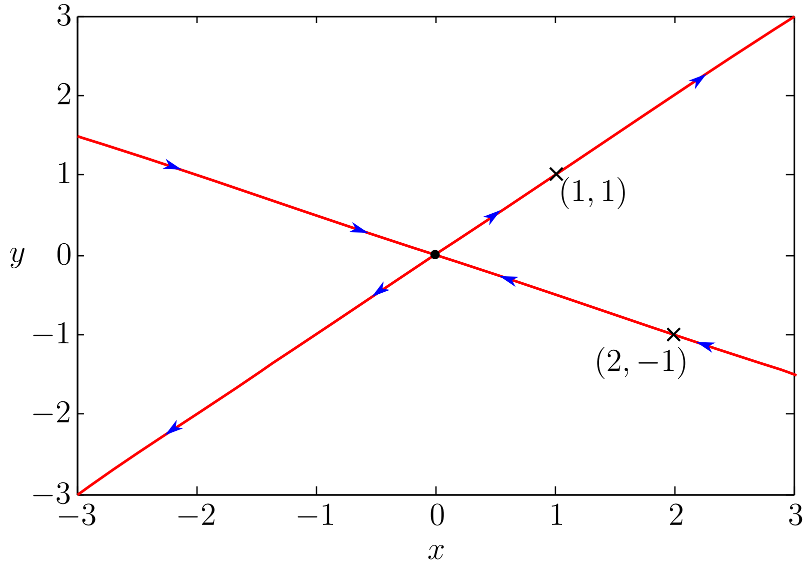


FIGURE 7.1. Eigensolution orbits (shown in red) for the example.

7.1.2. *Orbits of Representative Solutions.* The eigensolution orbits all lie on lines that pass through the origin. These lines divide the xy-plane into regions. To complete the phase portrait we must indicate what the other orbits look like in each of these regions. One way to do this is to draw the orbits of one or more representative solutions in each region. This can be done either by analyzing an analytic formula such as (7.2) or by solving system (7.1) numerically.

When rendering a phase-plane portrait by computer we can simply pick one or two initial-values (x_I, y_I) from each region of the xy-plane and have the computer plot $(x(t), y(t))$ given by formula (7.2) with $e^{t\mathbf{A}}$ given by (7.3) for an appropriate range of times t. Arrowheads are added to indicate the orientation of each orbit. Alternatively, we can approximate $(x(t), y(t))$ for an appropriate range of times t by having the computer solve the initial-value problem associated with system (7.1) and initial data (x_I, y_I). Because this alternative does not depend on having an explicit solution, this is what we will do for most nonlinear systems. However, here we are studying linear systems, so we can take advantage of having the explicit solution (7.2).

When sketching a phase-plane portrait by hand we analyze the behavior of a general solution. For example, we can analyze either the explicit formula (7.2) with $e^{t\mathbf{A}}$ given by (7.3) or a general solution constructed from eigenpairs, whichever is easier.

Example. Sketch the eigensolution orbits and some representive solutions in the phase-plane for the planar system

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} x \\ y \end{pmatrix} = \mathbf{A}\begin{pmatrix} x \\ y \end{pmatrix}, \qquad \text{where} \quad \mathbf{A} = \begin{pmatrix} 0 & 2 \\ 1 & 1 \end{pmatrix}.$$

Solution. We sketched the eigensolution orbits in the previous example. Here we are asked to add to that picture sketches of representative solutions. From the eigenpairs computed previously, we see that a general solution for this system is

$$\mathbf{x}(t) = c_1 e^{-t} \begin{pmatrix} 2 \\ -1 \end{pmatrix} + c_2 e^{2t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} .$$

When either $c_1 = 0$ or $c_2 = 0$ this reduces to an eigensolution whose orbit we have already sketched, so suppose that $c_1 \neq 0$ and $c_2 \neq 0$. Then when t is a large negative number we see that

$$\mathbf{x}(t) \approx c_1 e^{-t} \begin{pmatrix} 2 \\ -1 \end{pmatrix} ,$$

while when t is a large positive number we see that

$$\mathbf{x}(t) \approx c_2 e^{2t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} .$$

Therefore every such orbit will sweep from the line $y = -\frac{1}{2}x$ towards the line $y = x$ as time increases. Four such orbits are shown below in Figure 7.2, one in each region of the xy-plane separated by the eigensolution orbits.

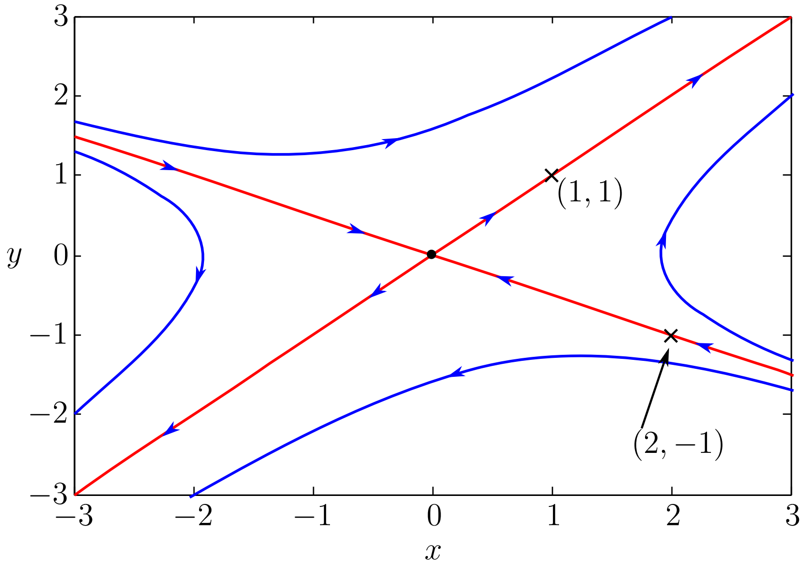


FIGURE 7.2. Representative orbits (shown in blue) added to Figure 7.1.

Remark. The foregoing phase-plane portrait is called a *saddle*. Such a portrait arises whenever **A** has both a negative and a positive eigenvalue.

Remark. We will illustrate this approach further in the next two sections when we classify and sketch the phase-plane portraits that can arise from a system in the form (7.1).

7.1.3. *Direction Fields.* Direction fields are another tool that can be used to indicate what orbits look like in the regions in the xy-plane that are separated by the eigensolution orbits. It is the crudest tool in our toolbox. Its virtue is that it can be applied to almost any first-order autonomous planar system in the form

$$\frac{\mathrm{d}x}{\mathrm{d}t} = f(x,y)\,, \qquad \frac{\mathrm{d}y}{\mathrm{d}t} = g(x,y)\,. \tag{7.5}$$

We assume that the functions $f(x,y)$ and $g(x,y)$ are defined over a set S in the xy-plane such that

- f and g are continuous over S,
- f and g are differentiable with respect to x and y over S,
- $\partial_x f$, $\partial_y f$, $\partial_x g$, and $\partial_y g$ are continuous over S.

Moreover, we assume that a rectangle $[x_L, x_R] \times [y_L, y_R]$ lies within the interior of S.

Remark. For the linear planar systems (7.1) that we are considering in this chapter, all of the above assumptions hold with S taken to be the entire xy-plane. In that case $[x_L, x_R] \times [y_L, y_R]$ can be any rectangle.

By Theorem 1.1 every point (x_I, y_I) in $[x_L, x_R] \times [y_L, y_R]$ has a unique curve $(X(t), Y(t))$ passing through it such that $(x, y) = (X(t), Y(t))$ is a solution of (7.5). This curve can be extended to the largest time interval $[a, b]$ such that $(X(t), Y(t))$ remains within the rectangle $[x_L, x_R] \times [y_L, y_R]$. It has a tangent vector given by

$$\frac{\mathrm{d}}{\mathrm{d}t}\big(X(t)\,,\,Y(t)\big) = \big(X'(t)\,,\,Y'(t)\big) = \big(f(X(t),Y(t))\,,\,g(X(t),Y(t))\big)\,.$$

In other words, the unique solution that goes through any point (x, y) in the rectangle $[x_L, x_R] \times [y_L, y_R]$ has the tangent vector $(f(x,y), g(x,y))$. A *direction field* for equation (7.5) over the rectangle $[x_L, x_R] \times [y_L, y_R]$ is a plot that shows the direction of this tangent vector with an arrow at each point of a grid in the rectangle $[x_L, x_R] \times [y_L, y_R]$. The idea is that these arrows might give us a correct picture of how the orbits move inside the rectangle.

We can produce such a direction field by using the MATLAB commands **meshgrid** and **quiver** as follows.

```
>> [X, Y] = meshgrid(x_L:h:x_R,y_L:k:y_R);
>> F = f(X, Y); G = g(X, Y);
>> L = sqrt(F.^2 + G.^2);
>> quiver(X, Y, F./L, G./L, ℓ)
>> axis tight, xlabel ’x’, ylabel ’y’
>> title ’Direction Field for dx/dt = f(x, y), dy/dt = g(x, y)’
```

Here h and k are the grid spacings for the intervals $[x_L, x_R]$ and $[y_L, y_R]$ respectively, which should have values of the form

$$h = \frac{x_R - x_L}{m}\,, \qquad k = \frac{y_R - y_L}{n}\,, \qquad \text{where } m \text{ and } n \text{ are positive integers.}$$

The meshgrid command creates an array of *grid points* in the rectangle $[x_L, x_R] \times [y_L, y_R]$ given by (x_i, y_j) where

$$x_i = x_L + ih \text{ for } i = 0,\, 1,\, \cdots\, m\,, \qquad y_j = y_L + jk \text{ for } j = 0,\, 1,\, \cdots\, n\,.$$

More precisely, meshgrid creates two arrays; the array X contains x_i in its ij^{th}-entry while the array Y contains y_j in its ij^{th}-entry. Then arrays F and G are computed that contain $f(x_i, y_j)$ and $g(x_i, y_j)$ respectively in their ij^{th}-entry. Next, an array L is computed that contains the length of the vector $(f(x_i, y_j), g(x_i, y_j))$ in its ij^{th}-entry. Finally, the quiver command plots an array of arrows of length ℓ so that the ij^{th}-arrow is centered at the grid point (x_i, y_j) and is pointing in the direction of the unit tangent vector

$$\left(\frac{f(x_i, y_j)}{\sqrt{f(x_i, y_j)^2 + g(x_i, y_j)^2}}, \frac{g(x_i, y_j)}{\sqrt{f(x_i, y_j)^2 + g(x_i, y_j)^2}} \right) .$$

The length ℓ should be smaller than h or k so that the plotted arrows will not overlap. Typically m and n will be about 20 to insure there will be enough arrows to give a complete picture of the direction field, but not so many that the plot becomes cluttered.

Remark. Often it is hard to figure out how the orbits move solely from the arrows of a direction field. Directions fields are most effective when used in conjunction with plots of eigensolution orbits and some representative solutions.

Example. Sketch a phase-plane portrait for the planar system

$$\frac{\mathrm{d}}{\mathrm{d}t} \begin{pmatrix} x \\ y \end{pmatrix} = \mathbf{A} \begin{pmatrix} x \\ y \end{pmatrix} , \qquad \text{where} \quad \mathbf{A} = \begin{pmatrix} 0 & 2 \\ 1 & 1 \end{pmatrix} .$$

Show the eigensolution orbits, some representive solutions, and a direction field.

Solution. We sketched the eigensolution orbits and some representative solutions in the previous examples. Here we are asked to add to that picture a direction field. We implement the Matlab code given above with $f(x, y) = 2y$ and $g(x, y) = x + y$. The result is shown below in Figure 7.3.

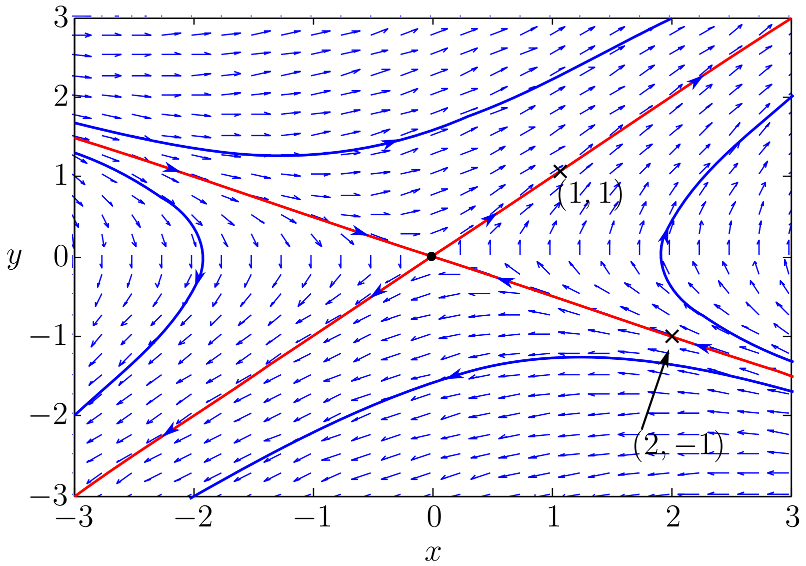


Figure 7.3. A direction field added to Figure 7.2.

7.2. **Classification of Phase-Plane Portraits.** We now classify the twenty types of phase-plane portrait that can arise from a system of the form (7.1). The most important of these are the fifteen types for which $\det(\mathbf{A}) \neq 0$. The first step in this classification is determined by eigenvalues of the coefficient matrix $\mathbf{A}$. There are three cases: $\mathbf{A}$ has two real eigenvalues, $\mathbf{A}$ has a conjugate pair of eigenvalues, $\mathbf{A}$ has one real eigenvalue.

7.2.1. *Two Real Eigenvalues.* Suppose $\mathbf{A}$ has two real eigenvalues $\lambda_1 < \lambda_2$. Let $(\lambda_1, \mathbf{v}_1)$ and $(\lambda_2, \mathbf{v}_2)$ be real eigenpairs of $\mathbf{A}$. We first plot the orbits that lie on the lines $\mathbf{x} = c_1\mathbf{v}_1$ and $\mathbf{x} = c_2\mathbf{v}_2$ as described above. Every other solution of system (7.1) has the form

$$\mathbf{x}(t) = c_1 e^{\lambda_1 t}\mathbf{v}_1 + c_2 e^{\lambda_2 t}\mathbf{v}_2 \,, \tag{7.6}$$

where both c_1 and c_2 are arbitrary nonzero real numbers. There are five possible types of portrait.

- If $\lambda_1 < 0 < \lambda_2$ then the nonzero solutions on the line $\mathbf{x} = c_1\mathbf{v}_1$ will approach the origin as $t \to \infty$ while the nonzero orbits on the line $\mathbf{x} = c_2\mathbf{v}_2$ will move away from the origin as t increases. It is clear that as $t \to \infty$ the solution (7.6) will approach the line $\mathbf{x} = c_2\mathbf{v}_2$ while as $t \to -\infty$ it will approach the line $\mathbf{x} = c_1\mathbf{v}_1$. This type is called a *saddle*.
- If $\lambda_1 < \lambda_2 < 0$ then every solution will approach the origin as $t \to \infty$. Because $e^{\lambda_1 t}$ decays to zero faster than $e^{\lambda_2 t}$ it is clear that the solution (7.6) behaves like $c_2 e^{\lambda_2 t}\mathbf{v}_2$ as $t \to \infty$. This means that all solutions not on the line $\mathbf{x} = c_1\mathbf{v}_1$ will approach the origin tangent to the line $\mathbf{x} = c_2\mathbf{v}_2$. This type is called a *nodal sink*.
- If $0 < \lambda_1 < \lambda_2$ then every solution will move away from the origin t increases. Because $e^{\lambda_2 t}$ decays to zero faster than $e^{\lambda_1 t}$ as $t \to -\infty$, it is clear that the solution (7.6) behaves like $c_1 e^{\lambda_1 t}\mathbf{v}_1$ as $t \to -\infty$. This means that all solutions not on the line $\mathbf{x} = c_2\mathbf{v}_2$ will emerge from the origin tangent to the line $\mathbf{x} = c_1\mathbf{v}_1$. This type is called a *nodal source*.
- If $\lambda_1 < \lambda_2 = 0$ then the line $\mathbf{x} = c_2\mathbf{v}_2$ is a line of stationary points. It is clear that as $t \to \infty$ every solution (7.6) will approach one of these stationary points as along a line that is parallel to the line $\mathbf{x} = c_1\mathbf{v}_1$. This means that all solutions not on the line of stationary points $\mathbf{x} = c_2\mathbf{v}_2$ will approach that "spine" along a "rib" that is parallel to the line $\mathbf{x} = c_1\mathbf{v}_1$. This type is called a *spinal sink*.
- If $0 = \lambda_1 < \lambda_2$ then the line $\mathbf{x} = c_1\mathbf{v}_1$ is a line of stationary points. It is clear that as t increases the solution (7.6) will move away from one of these stationary points along a line that is parallel to the line $\mathbf{x} = c_2\mathbf{v}_2$. This means that all solutions not on the line of stationary points $\mathbf{x} = c_1\mathbf{v}_1$ will move away from that "spine" along a "rib" that is parallel to the line $\mathbf{x} = c_2\mathbf{v}_2$. This type is called a *spinal source*.

Examples. If $\mathbf{A}$ has two real eigenvalues $\lambda_1 < \lambda_2$ then these five phase-plane portrait types might look like the following. Eigensolution orbits are shown in red.

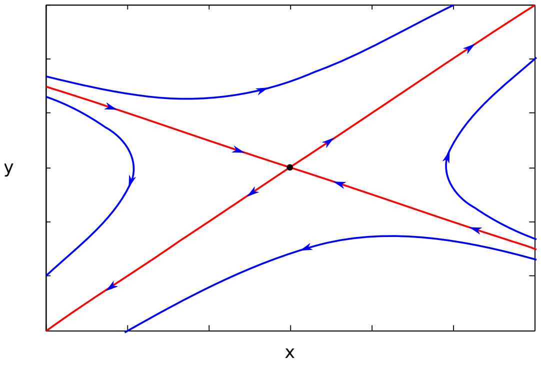
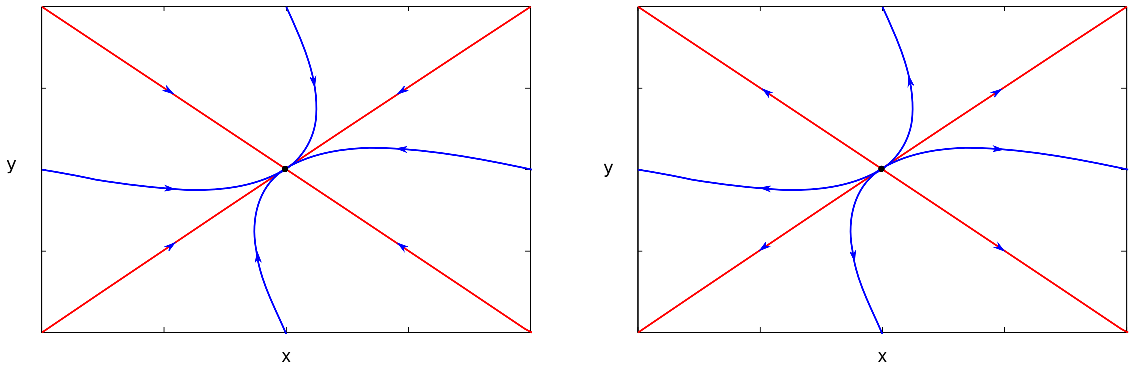


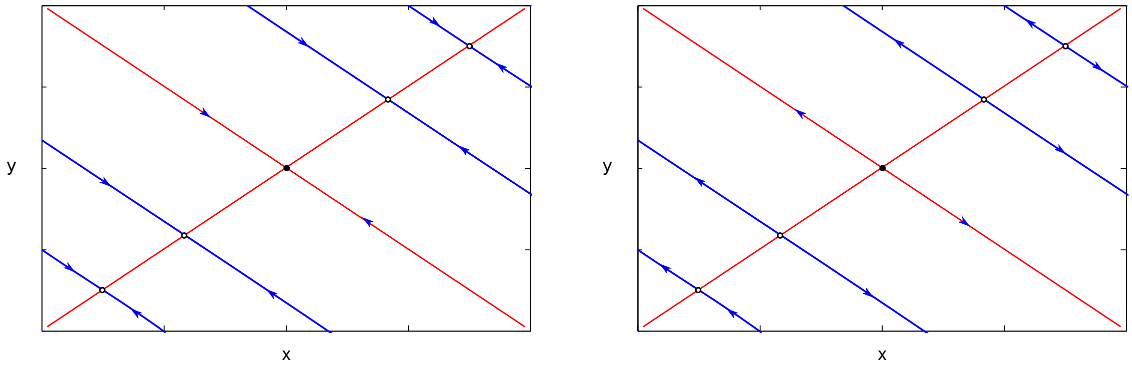
FIGURE 7.4. Saddle ($\lambda_1 < 0 < \lambda_2$)



(A) Nodal Sink ($\lambda_1 < \lambda_2 < 0$)

(B) Nodal Source ($0 < \lambda_1 < \lambda_2$)

FIGURE 7.5. Nodal Sink and Source



(A) Spinal Sink ($\lambda_1 < \lambda_2 = 0$)

(B) Spinal Source ($0 = \lambda_1 < \lambda_2$)

FIGURE 7.6. Spinal Sink and Source

Remark. Saddles, nodal sinks, and nodal sources occur more often than spinal sinks and spinal sources because having two nonzero eigenvalues is more likely than having an eigenvalue that is zero.

7.2.2. *A Conjugate Pair of Eigenvalues.* Suppose $\mathbf{A}$ has a conjugate pair of eigenvalues $\mu \pm i\nu$ with $\nu \neq 0$. There are no eigensolution orbits. The analytic solution (7.2) is

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = e^{t\mathbf{A}} \begin{pmatrix} x_I \\ y_I \end{pmatrix} = e^{\mu t} \left[\mathbf{I}\cos(\nu t) + (\mathbf{A} - \mu\mathbf{I}) \frac{\sin(\nu t)}{\nu} \right] \begin{pmatrix} x_I \\ y_I \end{pmatrix} . \tag{7.7}$$

The matrix inside the square brackets is a periodic function of t with period $2\pi/\nu$.

When $\mu = 0$ this solution will trace out an ellipse. But will it do so with a *clockwise* or *counterclockwise* rotation? We can determine the direction of rotation by considering what happens at special values of $\mathbf{x}$. For example,

$$\text{at } \mathbf{x} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \text{ we have } \frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} = \mathbf{A}\mathbf{x} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} a_{11} \\ a_{21} \end{pmatrix} .$$

This vector points up if $a_{21} > 0$, which indicates counterclockwise rotation, and points down if $a_{21} < 0$, which indicates clockwise rotation. Alternatively,

$$\text{at } \mathbf{x} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \text{ we have } \frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} = \mathbf{A}\mathbf{x} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} a_{12} \\ a_{22} \end{pmatrix} .$$

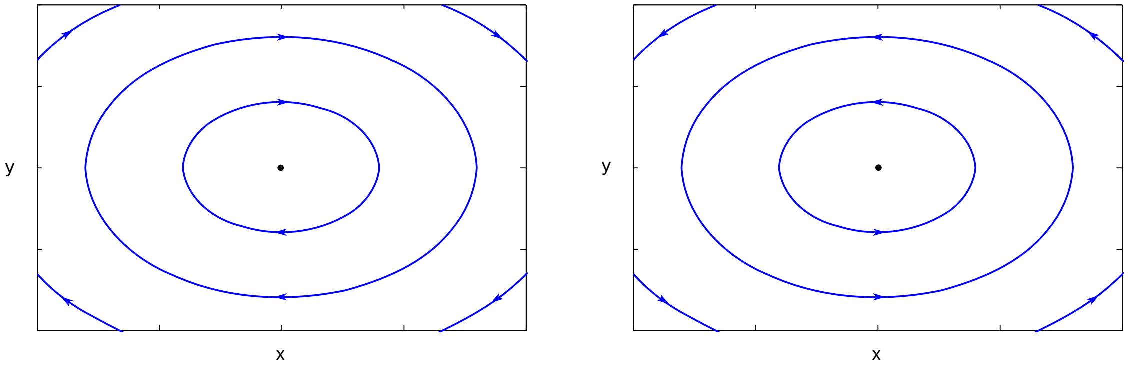
This vector points right if $a_{12} > 0$, which indicates clockwise rotation, and points left if $a_{12} < 0$, which indicates counterclockwise rotation. These arguements show that we can read off the direction of rotation from the sign of either a_{21} or a_{12}.

We will use the sign of a_{21} to determine the direction of rotation because it is consistent with the sign convention of the *right-hand rule* from vector calculus. If $a_{21} > 0$ then point the thumb of your right hand up and the other fingers will indicated a counterclockwise rotation. If $a_{21} < 0$ then point the thumb of your right hand down and the other fingers will indicated a clockwise rotation.

It is clear from (7.7) that when $\mu < 0$ all solutions will approach the origin as $t \to \infty$, while when $\mu > 0$ all solutions will move away from the origin as t increases. If we put this together with the information above, we see there are six possible types of portrait.

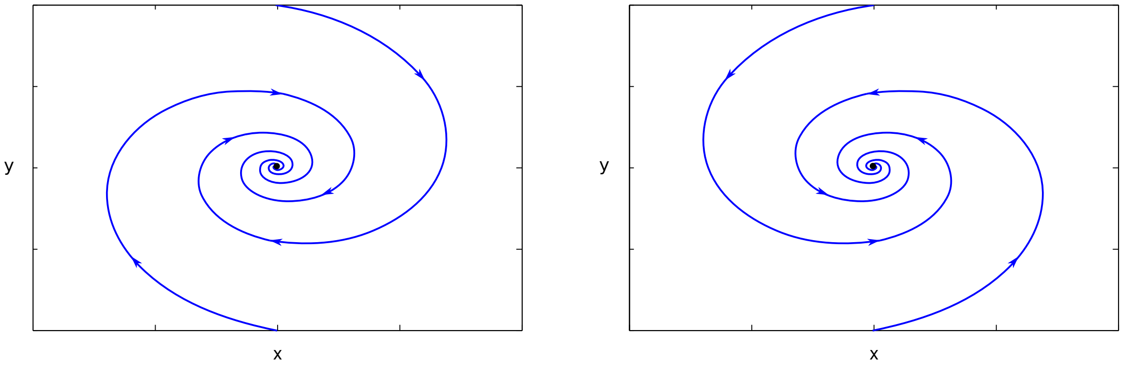
- If $\mu = 0$ then all orbits are ellipses around the origin. This type is called a *center*. The center is *clockwise* if $a_{21} < 0$, and is *counterclockwise* if $a_{21} > 0$.
- If $\mu < 0$ then all solutions spiral into the origin as $t \to \infty$. This type is called a *spiral sink*. The spiral is *clockwise* if $a_{21} < 0$, and is *counterclockwise* if $a_{21} > 0$.
- If $\mu > 0$ then all solutions spiral away from the origin as $t \to \infty$. This type is called a *spiral source*. The spiral is *clockwise* if $a_{21} < 0$, and is *counterclockwise* if $a_{21} > 0$.

Examples. If $\mathbf{A}$ has a conjugate pair of eigenvalues $\mu \pm i\nu$ with $\nu \neq 0$ then these six phase-plane portrait types might look like the following.



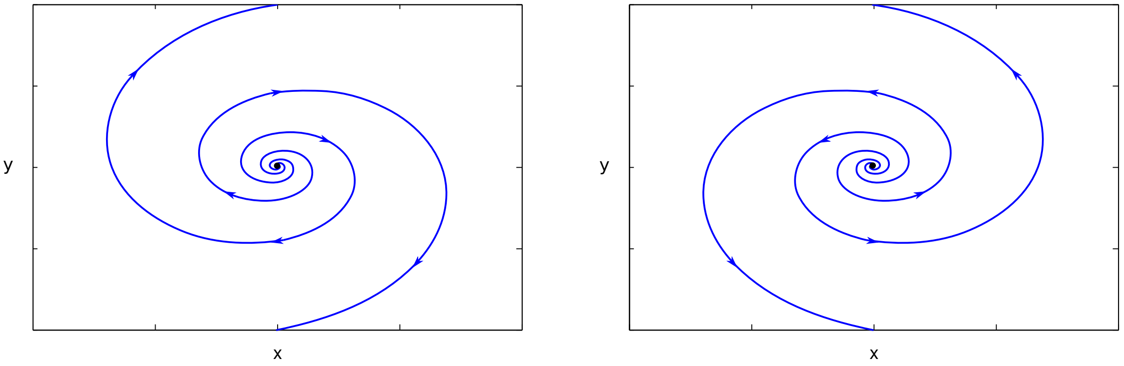
(A) Clockwise Center ($a_{21} < 0$) (B) Counterclockwise Center ($a_{21} > 0$)

Figure 7.7. Centers ($\mu = 0$)



(A) Clockwise Spiral Sink ($a_{21} < 0$) (B) Counterclockwise Spiral Sink ($a_{21} > 0$)

Figure 7.8. Spiral Sinks ($\mu < 0$)



(A) Clockwise Spiral Source ($a_{21} < 0$) (B) Counterclockwise Spiral Source ($a_{21} > 0$)

Figure 7.9. Spiral Sources ($\mu > 0$)

Remark. In general settings spirals occur more often than centers because having eigenvalues with a nonzero real part is more likely than having imaginary eigenvalues. However, there are special settings in which spirals cannot occur, but centers can.

7.2.3. *One Real Eigenvalue.* Suppose $\mathbf{A}$ has one real eigenvalue μ. By (7.2) and (7.3c) the analytic solution is

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = e^{t\mathbf{A}} \begin{pmatrix} x_I \\ y_I \end{pmatrix} = e^{\mu t} \left[\mathbf{I} + t \left(\mathbf{A} - \mu \mathbf{I} \right) \right] \begin{pmatrix} x_I \\ y_I \end{pmatrix} . \tag{7.8}$$

There are two subcases.

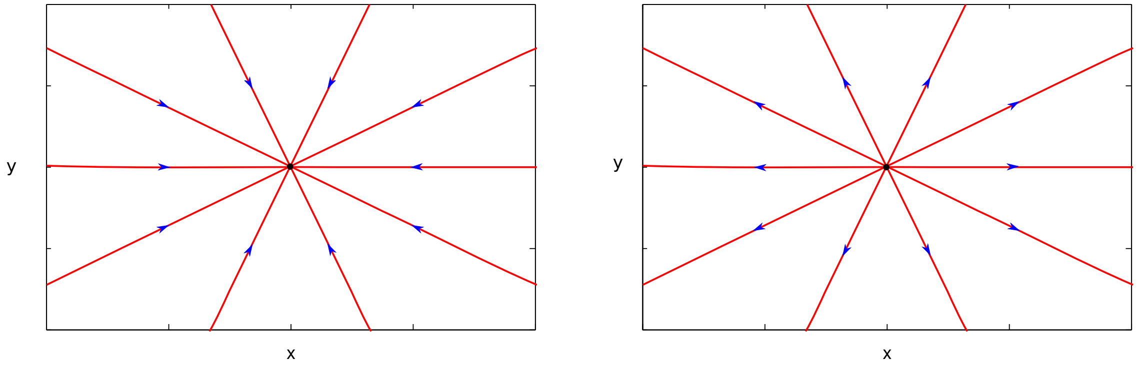
The simplest subcase is when $\mathbf{A} = \mu\mathbf{I}$. This subcase is easy to spot because the matrix $\mathbf{A}$ is simply a multiple of the identity matrix $\mathbf{I}$. It follows that every nonzero vector is an eigenvector of $\mathbf{A}$. In this subcase the analytic solution (7.8) reduces to

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = e^{t\mathbf{A}} \begin{pmatrix} x_I \\ y_I \end{pmatrix} = e^{\mu t} \begin{pmatrix} x_I \\ y_I \end{pmatrix} .$$

There are three possible types of portrait.

- If $\mu < 0$ all solutions radially approach the origin as $t \to \infty$. This type is called a *radial sink*.
- If $\mu > 0$ all solutions radially move away from the origin as $t \to \infty$. This type is called a *radial source*.
- If $\mu = 0$ then all solutions are stationary. This type is called *null*.

Examples. If $\mathbf{A} = \mu\mathbf{I}$ with $\mu \neq 0$ then the phase-plane portraits will look like the following. Because all nonzero solutions are eigensolutions, they are shown in red.



(A) Radial Sink ($\mu < 0$)

(B) Radial Source ($\mu > 0$)

FIGURE 7.10. Radial Sink and Source

Remark. The case $\mu = 0$ yields a phase-plane portrait in which every point is a stationary point, so it is not shown.

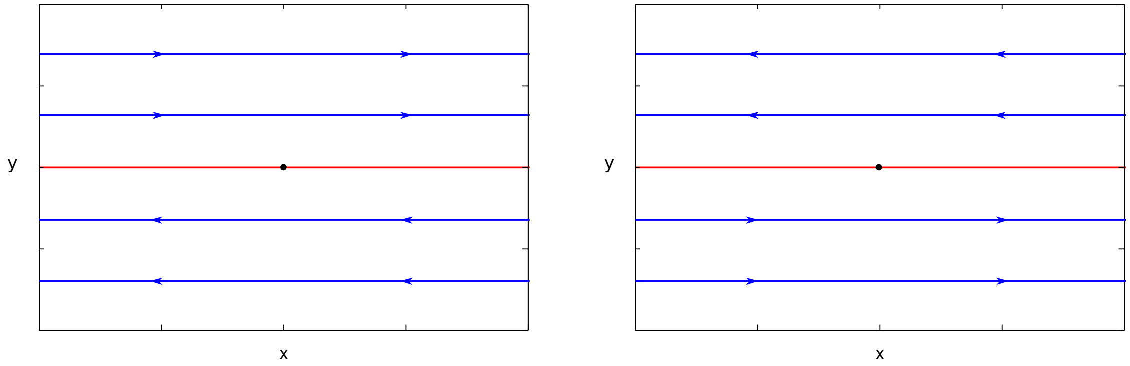
Remark. Because the subcase $\mathbf{A} = \mu\mathbf{I}$ is so special, these portraits do not arise as often as those for the next subcase.

The other subcase is when $\mathbf{A} \neq \mu\mathbf{I}$. In this subcase $\mathbf{A}$ will have the eigenpair $(\mu, \mathbf{v})$ where $\mathbf{v}$ is proportional to any nonzero column of $\mathbf{A} - \mu\mathbf{I}$. We first plot the orbits that lie on the line $\mathbf{x} = c\mathbf{v}$ as described above. There are six possible types of portrait.

- If $\mu = 0$ then all solutions on the line $\mathbf{x} = c\mathbf{v}$ are stationary. All other solutions will move parallel to this line. This type is called a *shear*. The shear is *clockwise* if either $a_{21} < 0$ or $a_{12} > 0$, and is *counterclockwise* if either $a_{21} > 0$ or $a_{12} < 0$.
- If $\mu < 0$ then all solutions on the line $\mathbf{x} = c\mathbf{v}$ move towards the origin as t increases. Because $e^{\mu t}$ will decay to zero faster than $e^{\mu t}t$, and because the columns of $\mathbf{A} - \mu\mathbf{I}$ are proportional to $\mathbf{v}$, it is clear from (7.8) that every solution approaches the origin tangent to the line $\mathbf{x} = c\mathbf{v}$. This type is called a *twist sink*. The twist is *clockwise* if either $a_{21} < 0$ or $a_{12} > 0$, and is *counterclockwise* if either $a_{21} > 0$ or $a_{12} < 0$.
- If $\mu > 0$ then all solutions on the line $\mathbf{x} = c\mathbf{v}$ move away from the origin as t increases. Because $e^{\mu t}$ will decay to zero faster than $e^{\mu t}t$ as $t \to -\infty$, and because the columns of $\mathbf{A} - \mu\mathbf{I}$ are proportional to $\mathbf{v}$, it is clear from (7.8) that every solution emerges from the origin tangent to the line $\mathbf{x} = c\mathbf{v}$. This type is called a *twist source*. The twist is *clockwise* if either $a_{21} < 0$ or $a_{12} > 0$, and is *counterclockwise* if either $a_{21} > 0$ or $a_{12} < 0$.

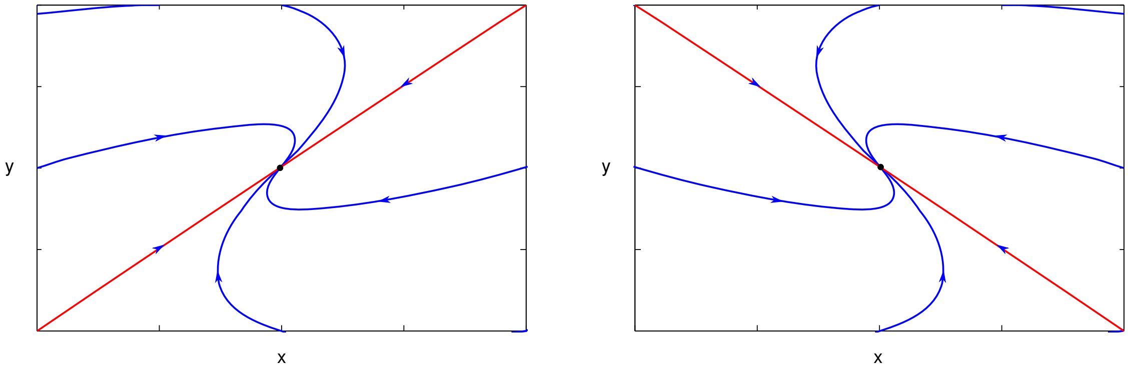
Remark. For a twist or a shear we can have $a_{21} = 0$ or $a_{12} = 0$ but we cannot have $a_{21} = a_{12} = 0$. Therefore we can always determine the direction of rotation from either a_{21} or a_{12}. Alternatively, we can combine the a_{21} test and the a_{12} test into a single test on $a_{21} - a_{12}$ that works for every spiral, center, twist, or shear. Namely, if $a_{21} - a_{12} < 0$ the rotation is clockwise, while if $a_{21} - a_{12} > 0$ the rotation is counterclockwise. This can be recalled with that aid of the "right-hand" rule.

Examples. When $\mathbf{A}$ has one real eigenvalue μ and $\mathbf{A} \neq \mu\mathbf{I}$ then these six phase-plane portrait types might look like the following. Eigensolution orbits are shown in red.



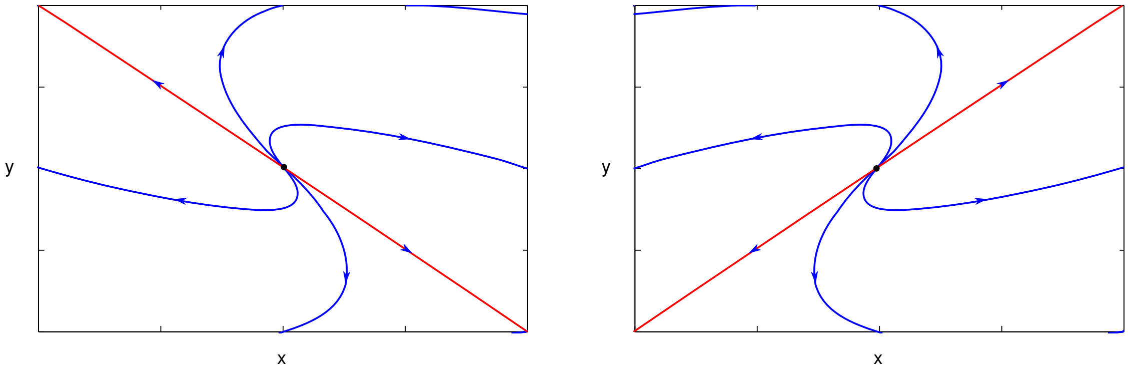
(A) Clockwise ($a_{21} < 0$ or $a_{12} > 0$) (B) Counterclockwise ($a_{21} > 0$ or $a_{12} < 0$)

FIGURE 7.11. Shears ($\mu = 0$)



(A) Clockwise ($a_{21} < 0$ or $a_{12} > 0$) (B) Counterclockwise ($a_{21} > 0$ or $a_{12} < 0$)

Figure 7.12. Twist Sinks ($\mu < 0$)



(A) Clockwise ($a_{21} < 0$ or $a_{12} > 0$) (B) Counterclockwise ($a_{21} > 0$ or $a_{12} < 0$)

Figure 7.13. Twist Sources ($\mu > 0$)

Remark. Twist sinks and twist sources occur more often than shears because having a nonzero eigenvalue is more likely than an eigenvalue that is zero.

Remark. Many textbooks call radial sinks and sources *proper nodes* and twist sinks and sources *improper nodes*. While traditional, this terminology is both more cumbersome and much less desciptive than that used here. Moreover, it often leaves students with the impression that improper nodes are rare. However, the opposite is true; improper nodes are much more common than proper nodes. Other books refer to radial sinks and sources as *star* sinks and sources.

Remark. Many textbooks do not include spinal sinks and sources, shears, or null in their classification of phase portraits. These are the five types for which $\det(\mathbf{A}) = 0$. The fifteen types for which $\det(\mathbf{A}) \neq 0$ play a big role in the analysis of nonlinear systems, so they are the most important to know.

7.2.4. *Summary of the Types.* The phase portrait of a system $\mathbf{x}' = \mathbf{A}\mathbf{x}$ is classified into one of twenty types as follows.

☐ If $\mathbf{A}$ has two real eigenvalues then the type is either a *saddle, node,* or *spine.*
- If they have opposite signs then the type is a *saddle.*
- If both are negative then the type is a *nodal sink.*
- If both are positive then the type is a *nodal source.*
- If one is zero and the other is negative then the type is a *spinal sink.*
- If one is zero and the other is positive then the type is a *spinal source.*

Examples of these five types are shown in Figures 7.4 – 7.6. Two real eigenpairs are needed to sketch these portraits. The first three are most important.

☐ If $\mathbf{A}$ has one real eigenvalue and $\mathbf{A} = \mu\mathbf{I}$ then the type is either a *radial* or *null.*
- If the eigenvalue is negative then the type is *radial sink.*
- If the eigenvalue is positive then the type is *radial source.*
- If the eigenvalue is zero then the type is *null.*

The first two types are shown in Figure 7.10. There are many real eigenpairs, but none are need to sketch these portraits. The first two are most important.

☐ If $\mathbf{A}$ has one real eigenvalue and $\mathbf{A} \neq \mu\mathbf{I}$ then the type is either a *shear* or *twist.*
- If the eigenvalue is negative then the type is a *twist sink.*
  - ○ If $a_{21} < 0$ or $a_{12} > 0$ then it is a *clockwise twist sink.*
  - ○ If $a_{21} > 0$ or $a_{12} < 0$ then it is a *counterclockwise twist sink.*
- If the eigenvalue is positive then the type is a *twist source.*
  - ○ If $a_{21} < 0$ or $a_{12} > 0$ then it is a *clockwise twist source.*
  - ○ If $a_{21} > 0$ or $a_{12} < 0$ then it is a *counterclockwise twist source.*
- If the eigenvalue is zero then the type is a *shear.*
  - ○ If $a_{21} < 0$ or $a_{12} > 0$ then it is a *clockwise shear.*
  - ○ If $a_{21} > 0$ or $a_{12} < 0$ then it is a *counterclockwise shear.*

Examples of these six types are shown in Figures 7.11 – 7.13. One real eigenpair is needed to sketch these portraits. The first four are most important.

☐ If $\mathbf{A}$ has a conjugate pair of eigenvalues then the type is either a *center* or *spiral.*
- If their real parts are negative then the type is a *spiral sink.*
  - ○ If $a_{21} < 0$ (or $a_{12} > 0$) then it is a *clockwise spiral sink.*
  - ○ If $a_{21} > 0$ (or $a_{12} < 0$) then it is a *counterclockwise spiral sink.*
- If their real parts are positive then the type is a *spiral source.*
  - ○ If $a_{21} < 0$ (or $a_{12} > 0$) then it is a *clockwise spiral source.*
  - ○ If $a_{21} > 0$ (or $a_{12} < 0$) then it is a *counterclockwise spiral source.*
- If their real parts are zero then the type is a *center.*
  - ○ If $a_{21} < 0$ (or $a_{12} > 0$) then it is a *clockwise center.*
  - ○ If $a_{21} > 0$ (or $a_{12} < 0$) then it is a *counterclockwise center.*

Examples of these six types are shown in Figures 7.7 – 7.9. There are no real eigenpairs. All six are important.

Rather than seeing this as a list of twenty types to be memorized, this classification is easier to master if it is broken down into steps.

First, each phase portrait type belongs to one of four catagories determined by the ☐ step in the foregoing complete list, which we isolate below.

☐ If **A** has two real eigenvalues then the type is either a *saddle, node,* or *spine.* The portrait is one of five types. Two real eigenpairs are needed for a sketch.

☐ If **A** has one real eigenvalue and $\mathbf{A} = \mu\mathbf{I}$ then the type is either a *radial* or *null.* The portrait is one of three types. No real eigenpair is needed for a sketch.

☐ If **A** has one real eigenvalue and $\mathbf{A} \neq \mu\mathbf{I}$ then the type is either a *twist* or *shear.* The portrait is one of six types. One real eigenpair is needed for a sketch.

☐ If **A** has a conjugate pair of eigenvalues then the type is either a *spiral* or *center.* The portrait is one of six types. There are no real eigenpairs.

Remark. Whether **A** has two real eigenvalues, one real eigenvalue, or a conjugate pair of eigenvalues can be read off from the discriminant of its characteristic polynomial.

Second, these four catagories split into five, three, three, and three subcatagories respectively, determined by the • step in the complete list, which we isolate below.

☐ If **A** has two real eigenvalues then the type is either a *saddle, node,* or *spinal.*
- If they have opposite signs then the type is a *saddle.*
- If both are negative then the type is a *nodal sink.*
- If both are positive then the type is a *nodal source.*
- If one is zero and the other is negative then the type is a *spinal sink.*
- If one is zero and the other is positive then the type is a *spinal source.*

☐ If **A** has one real eigenvalue and $\mathbf{A} = \mu\mathbf{I}$ then the type is either a *radial* or *null.*
- If the eigenvalue is negative then the type is *radial sink.*
- If the eigenvalue is positive then the type is *radial source.*
- If the eigenvalue is zero then the type is *null.*

☐ If **A** has one real eigenvalue and $\mathbf{A} \neq \mu\mathbf{I}$ then the type is either a *twist* or *shear.*
- If the eigenvalue is negative then the type is a *twist sink.*
- If the eigenvalue is positive then the type is a *twist source.*
- If the eigenvalue is zero then the type is a *shear.*

☐ If **A** has a conjugate pair of eigenvalues then the type is either a *spiral* or *center.*
- If their real parts are negative then the type is a *spiral sink.*
- If their real parts are positive then the type is a *spiral source.*
- If their real parts are zero then the type is a *center.*

Remark. This information about the eigenvalues of **A** can be read off from $\mathrm{tr}(\mathbf{A})$ and $\det(\mathbf{A})$, or equivalently from the mean and discriminant of its characteristic polynomial.

Finally, the three subcatagories in the last two catagories split into clockwise and counterclockwise subsubcatagories determined by the ∘ step in the complete list.

Remark. This information can usually be read off from the sign of the a_{21} entry of **A**. If $a_{21} = 0$ then it can always be read off from the sign of the a_{12} entry of **A**.

7.2.5. *Mean-Discriminant Plane.* We can visualize the relationships between various types of phase-plane portrait for linear systems through the mean-discriminant plane. By completing the square of the characteristic polynomial we bring it into the form

$$p(z) = z^2 - \mathrm{tr}(\mathbf{A})z + \det(\mathbf{A}) = (z-\mu)^2 - \delta\,,$$

where the *mean* μ and *discriminant* δ are given by

$$\mu = \tfrac{1}{2}\mathrm{tr}(\mathbf{A})\,, \qquad \delta = \mu^2 - \det(\mathbf{A})\,.$$

Recall that δ is called the discriminant because it determines the root structure of the characteristic polynomial:

- when $\delta > 0$ there are the two simple real roots $\mu \pm \sqrt{\delta}$;
- when $\delta = 0$ there is the one double real root μ;
- when $\delta < 0$ there is the conjugate pair of roots $\mu \pm i\sqrt{|\delta|}$.

In all cases μ is the average of the roots, which is why it is called the mean.

The types of phase-plane portraits that arise for different values of μ and δ in the $\mu\delta$-plane are shown in the figure below.

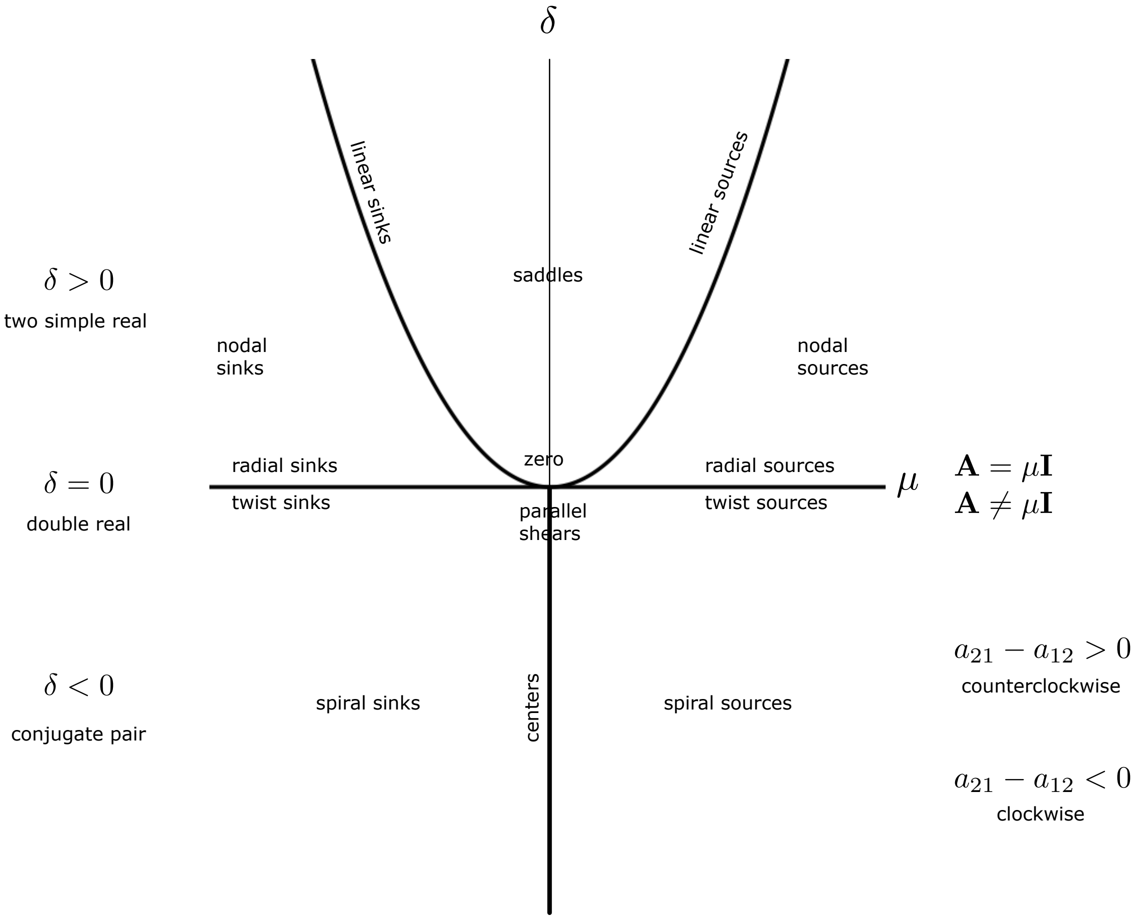


Figure 7.14. Mean-Discriminant Plane

In the upper half-plane ($\delta > 0$) the matrix $\mathbf{A}$ has the two simple real eigenvalues $\mu \pm \sqrt{\delta}$. These will both be negative whenever $\mu < -\sqrt{\delta}$, which is the region labeled *nodal sinks* in the figure. These will both be positive whenever $\sqrt{\delta} < \mu$, which is the region labeled *nodal sources* in the figure. These will have opposite signs whenever $-\sqrt{\delta} < \mu < \sqrt{\delta}$, which is the region labeled *saddles* in the figure. These three regions are seperated by the parabola $\delta = \mu^2$. There is one negative and one zero eigenvalue along the branch of this parabola where $\mu = -\sqrt{\delta}$, which is labeled *spinal sinks* in the figure. There is one zero and one positive eigenvalue along the branch of this parabola where $\mu = \sqrt{\delta}$, which is labeled *spinal sources* in the figure. There are five types of phase portrait in the upper half-plane.

On the μ-axis ($\delta = 0$) the matrix $\mathbf{A}$ has the single real eigenvalue μ. There are two cases to consider: the case when $\mathbf{A} = \mu\mathbf{I}$ which we label above the μ-axis, and the case when $\mathbf{A} \neq \mu\mathbf{I}$ which we label below the μ-axis.

- When $\mathbf{A} = \mu\mathbf{I}$ every nonzero vector is an eigenvector associated with the eigenvalue μ. This eigenvalue is negative whenever $\mu < 0$, which is the half-line labeled *radial sink* in the figure. This eigenvalue is positive whenever $\mu > 0$, which is the half-line labeled *radial source* in the figure. This eigenvalue is zero whenever $\mu = 0$, in which case $\mathbf{A} = \mu\mathbf{I} = \mathbf{0}$, which is why the origin is labeled *null* in the figure. There are three types of phase portrait represented here.
- When $\mathbf{A} \neq \mu\mathbf{I}$ the eigenvectors associated with the eigenvalue μ are proportional to any nonzero column of $\mathbf{A} - \mu\mathbf{I}$. This eigenvalue is negative whenever $\mu < 0$, which is the half-line labeled *twist sink* in the figure. This eigenvalue is positive whenever $\mu > 0$, which is the half-line labeled *twist source* in the figure. This eigenvalue is zero whenever $\mu = 0$, in which case the origin is labeled *shears* in the figure. In each of these subcases we can determine the rotation of the orbits (*clockwise* or *counterclockwise*) either by the a_{21} test or the a_{12} test or by the $a_{21} - a_{12}$ test. There are six types of phase portrait represented here.

In the lower half-plane ($\delta < 0$) the matrix $\mathbf{A}$ has the conjugate pair of eigenvalues $\mu \pm i\sqrt{|\delta|}$. These will have negative real parts whenever $\mu < 0$, which is the region labeled *spiral sinks* in the figure. These will have positive real parts whenever $\mu > 0$, which is the region labeled *spiral sources* in the figure. These will be purely imaginary whenever $\mu = 0$, which is the half-line labeled *centers* in the figure. In each case we can determine the rotation of the orbits (*clockwise* or *counterclockwise*) by the a_{21} test. There are six types of phase portrait in the lower half-plane.

Remark. Most textbooks present a similar picture, but use the trace-determinant plane rather than the mean-discriminant plane. Therefore the pictures are not the same! The advantage of the mean-discriminant plane picture is its clean separation of the cases of two real, one real, and conjugate pair eigenvalues into the upper half-plane, μ-axis, and lower half-plane respectively. This allows a cleaner presentation of the tests for the rotation of the orbits (clockwise or counterclockwise). It also allows a cleaner presentation of the nine types of phase portrait on the μ-axis.

7.3. **Dynamical Stability of the Origin.** The origin of the phase-plane plays a special role for linear systems. This is because any solution $(x(t), y(t))$ of system (7.1) that starts at the origin will stay at the origin. In other words, the origin is an orbit for every linear system (7.1). Determining the *dynamical stability of the origin* addresses the question of whether as time increases solutions near the origin either approach the origin, move away from the origin, or stay near the origin without approaching it.

Two basic notions of dynamical stability are the following.

> **Definition 7.1.** We say that the origin is *stable* if every solution that starts sufficiently near it will stay arbitrarily close to it. We say that the origin is *unstable* if it is not stable.

The language "every solution that starts sufficiently near it will stay arbitrarily close to it" is not very precise. Rather than formulate a more precise mathematical definition, we will build our understanding of these notions through examples. To begin with, it should be clear that for every system the origin is either stable or unstable. Roughly speaking, the origin will be unstable if at least one solution that starts near it will move away from it.

Two additional notions of dynamical stability are the following.

> **Definition 7.2.** We say that the origin is *attracting* if every solution that starts near it will approach it as $t \to \infty$. We say that the origin is *repelling* if every solution that starts near it but not at it will move away from it.

It should be clear that if the origin is attracting then it is stable, and that if it is repelling then it is unstable. These implications do not go the other way. Indeed, the following examples show that there are systems for which the origin is stable but not attracting, and systems which the origin is unstable but not repelling.

Examples. For linear planar systems we can determine the dynamical stability of the origin by examining their phase-plane portraits given in Figures 7.4 - 7.13. We see that the dynamical stability of the origin for each of them is given by the following table.

attracting (so also stable)	stable (but not attracting)	unstable (but not repelling)	repelling (so also unstable)
nodal sinks	centers	saddles	nodal sources
radial sinks	spinal sinks	spinal sources	radial sources
twist sinks	null	shears	twist sources
spiral sinks			spiral sources

Table 7.1. Dynamical Stability of the Origin for Linear Planar Systems

Remark. Many textbooks use the term *asymptotically stable* for attracting but have no term for repelling.

Remark. In Section 9.3 these notions of dynamical stability will be extended to any stationary point of a nonlinear system. Here we apply them to the origin in the setting of linear planar systems.

EXERCISES ON LINEAR PLANAR SYSTEMS

For problem 1- 8, given the eigenpairs, $(\lambda, \mathbf{v})$, sketch the corresponding phase portrait. (The matrix $\mathbf{A}$ is provided when necessary.)

(1) $\left(1, \begin{pmatrix} 2 \\ 1 \end{pmatrix}\right)$ $\left(2, \begin{pmatrix} -1 \\ 1 \end{pmatrix}\right)$

Solution

(2) $\left(-1, \begin{pmatrix} 2 \\ 1 \end{pmatrix}\right)$ $\left(-3, \begin{pmatrix} -1 \\ 4 \end{pmatrix}\right)$

Solution

(3) $\left(5, \begin{pmatrix} 1 \\ 2 \end{pmatrix}\right)$ $\left(3, \begin{pmatrix} 1 \\ -1 \end{pmatrix}\right)$

Solution

(4) $\left(0, \begin{pmatrix} 1 \\ 1 \end{pmatrix}\right)$ $\left(-2, \begin{pmatrix} -5 \\ 1 \end{pmatrix}\right)$

Solution

(5) $\left(2, \begin{pmatrix} 3 \\ 1 \end{pmatrix}\right)$ $\left(-1, \begin{pmatrix} 1 \\ -1 \end{pmatrix}\right)$

Solution

(6) $\left(-2, \begin{pmatrix} -1 \\ 1 \end{pmatrix}\right)$ $\left(-4, \begin{pmatrix} 1 \\ 1 \end{pmatrix}\right)$

Solution

(7) $\left(-1 + i2, \begin{pmatrix} 2i \\ 1 \end{pmatrix}\right)$ $\left(-1 - i2, \begin{pmatrix} -2i \\ 1 \end{pmatrix}\right)$ $\mathbf{A} = \begin{pmatrix} -1 & -4 \\ 1 & -1 \end{pmatrix}$

Solution

(8) $\left(i, \begin{pmatrix} 1 \\ 2 - i \end{pmatrix}\right)$ $\left(-i, \begin{pmatrix} 1 \\ 2 + i \end{pmatrix}\right)$ $\mathbf{A} = \begin{pmatrix} 2 & -1 \\ 5 & -2 \end{pmatrix}$

Solution

For problems 9- 20 sketch the phase portraits of the following system of differential equations. Determine the stability of each system (whether the system is stable or unstable and whether or not it is attracting or repelling)

(9) $\mathbf{x}' = \begin{pmatrix} -2 & 0 \\ 0 & -2 \end{pmatrix} \mathbf{x}$

Solution

(10) $\frac{\mathrm{d}}{\mathrm{d}t} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$

Solution

(11) $\mathbf{x}' = \begin{pmatrix} -1 & -1 \\ 1 & -3 \end{pmatrix} \mathbf{x}$

Solution

(12) $\mathbf{x}' = \begin{pmatrix} 2 & \frac{3}{2} \\ -\frac{3}{2} & -1 \end{pmatrix} \mathbf{x}$

Solution

(13) $\dfrac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}1 & \frac{1}{2}\\4 & 2\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}$

Solution

(14) $\mathbf{x}' = \begin{pmatrix}2 & 2\\-1 & 4\end{pmatrix}\mathbf{x}$

Solution

(15) $\dfrac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}3 & 0\\0 & 3\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}$

Solution

(16) $\mathbf{x}' = \begin{pmatrix}4 & -1\\1 & 2\end{pmatrix}\mathbf{x}$

Solution

(17) $\mathbf{x}' = \begin{pmatrix}-3 & 2\\-2 & 1\end{pmatrix}\mathbf{x}$

Solution

(18) $\mathbf{x}' = \begin{pmatrix}1 & 2\\-5 & -1\end{pmatrix}\mathbf{x}$

Solution

(19) $\dfrac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}-2 & 1\\-1 & -2\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}$

Solution

(20) $\dfrac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}2 & -1\\0 & 3\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}$

Solution

For problems 21- 26 sketch the phase portrait as well as the trajectory associated with the initial condition. Determine the stability of each system. (Make sure to clearly label the trajectory)

(21) $\dfrac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}0 & -2\\2 & 0\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}, \qquad \begin{pmatrix}x(0)\\y(0)\end{pmatrix} = \begin{pmatrix}2\\-1\end{pmatrix}$

Solution

(22) $\mathbf{x}' = \begin{pmatrix}-3 & 2\\-1 & -1\end{pmatrix}\mathbf{x}, \qquad \mathbf{x}(0) = \begin{pmatrix}1\\-2\end{pmatrix}$

Solution

(23) $\mathbf{x}' = \begin{pmatrix}3 & -2\\3 & -1\end{pmatrix}\mathbf{x}, \qquad \mathbf{x}(0) = \begin{pmatrix}3\\1\end{pmatrix}$

Solution

(24) $\dfrac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}2 & 4\\-1 & -2\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}, \qquad \begin{pmatrix}x(0)\\y(0)\end{pmatrix} = \begin{pmatrix}1\\1\end{pmatrix}$

Solution

(25) $\dfrac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}1 & 1\\4 & -2\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}, \qquad \begin{pmatrix}x(0)\\y(0)\end{pmatrix} = \begin{pmatrix}0\\2\end{pmatrix}$

Solution

(26) $\mathbf{x}' = \begin{pmatrix}5 & -1\\3 & 1\end{pmatrix}\mathbf{x}, \qquad \mathbf{x}(0) = \begin{pmatrix}1\\\frac{1}{2}\end{pmatrix}$

Solution

(27) Consider the linear system given by

$$\mathbf{x}' = \begin{pmatrix} 1 & \alpha \\ 0 & 1 \end{pmatrix} \mathbf{x}.$$

Sketch the phase portrait, and classify the behavior and stability near the origin when $\alpha = 1$, $\alpha = -1$ and $\alpha = 0$. Can you see how the phase portrait might transform as α moves from 1 to 0 to -1?

Solution

(28) Consider the linear system given by

$$\mathbf{x}' = \begin{pmatrix} 1 & \alpha \\ -\alpha & 1 \end{pmatrix} \mathbf{x}.$$

Sketch the phase portrait and classify the behavior and stability near the origin when $\alpha = 1$, $\alpha = -1$ and $\alpha = 0$. Can you see how the phase portrait might transform as α moves from 1 to 0 to -1?

Solution

(29) Consider the linear system given by

$$\mathbf{x}' = \begin{pmatrix} 2\alpha & 1 \\ -1 & 0 \end{pmatrix} \mathbf{x}.$$

Sketch the phase portrait and classify the stability near the origin when $\alpha = 2$, $\alpha = 1$, $\alpha = .5$, $\alpha = 0$, $\alpha = -.5$, $\alpha = -1$, $\alpha = -2$. There is a critical value α_0 such that $\alpha > \alpha_0$ and $\alpha < \alpha_0$ have different stability (i.e. the system changes from unstable to stable). What is α_0?

Solution

Problems 30- 33 apply to the planar system $\mathbf{x}' = \mathbf{A}\mathbf{x}$.

(30) Show that the origin is attracting if and only if $\operatorname{tr}(\mathbf{A}) < 0$ and $\det(\mathbf{A}) > 0$.

Solution

(31) Show that the origin is repelling if and only if $\operatorname{tr}(\mathbf{A}) > 0$ and $\det(\mathbf{A}) > 0$.

Solution

(32) Is it true that the origin is stable if and only if $\operatorname{tr}(\mathbf{A}) \leq 0$ and $\det(\mathbf{A}) \geq 0$? State why or why not.

Solution

(33) Is it true that the origin is unstable if and only if $\det(\mathbf{A}) \leq 0$? State why or why not.

Solution

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