III. First-Order Systems of Ordinary Differential Equations
8. Autonomous Planar Systems: Integral Methods

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Contents

Return to Main Webpage

8. Autonomous Planar Systems: Integral Methods

For the remainder of the course we will study first-order, autonomous, planar systems in the normal form

$$\frac{\mathrm{d}x}{\mathrm{d}t} = f(x,y)\,, \qquad \frac{\mathrm{d}y}{\mathrm{d}t} = g(x,y)\,. \tag{8.1}$$

The word "autonomous" has Greek roots and means "self governing". Such systems are called *autonomous* because they only depend on the state (x, y) and not on time t. What makes them *planar* is the fact that they have two dependent variables, x and y.

In the general discussions throughout this chapter we will assume that f and g are continuously differentiable over some domain D in the xy-plane. Therefore our basic existence and uniqueness theorem for initial-value problems applies whenever the initial data lies in D. When $f(x, y)$ and $g(x, y)$ are defined over the entire xy-plane and D is not specified explicitly then we will assume that D is the entire xy-plane.

As with linear planar systems that we studied in the last chapter, every solution of (8.1) can be thought of as tracing out a curve $(x(t), y(t))$ in the xy-plane — the so-called *phase-plane*. Each such curve is called an *orbit* or *trajectory* of the system.

Fact 1. Every point in the domain D has exactly one orbit passing through it.

Reason. This is a consequence of the existence and uniqueness theorem. □

Fact 1 implies that orbits fill the domain D, and that orbits cannot cross or otherwise share a point. We can gain insight into all solutions of system (8.1) by sketching a so-called *phase-plane portrait* that indicates how their orbits fill the domain D in the phase-plane.

This chapter presents techniques for understanding the phase-plane of system (8.1) that require either finding explicit solutions or finding an implicit relation that solutions satisfy. We call these techniques *integral methods* because they usually require finding a primitive. Sometimes integral methods can give a complete picture of the phase-plane of the system. At other times they have to be supplemented with *nonintegral methods* in order to get a complete picture of the phase-plane of the system. Nonintegral methods will be treated in the next chapter.

8.1. **Stationary Solutions and Points.**

A solution of system (8.1) called *stationary* if it does not depend on time. Because the time derivatives are zero for such a solution, it must satisfy the algebraic system

$$0 = f(x,y)\,, \qquad 0 = g(x,y)\,. \tag{8.2}$$

Conversely, if (x_o, y_o) satisfies this algebraic system then it is easily checked that $(x(t), y(t)) = (x_o, y_o)$ is a stationary solution of system (8.1). Therefore we can find all stationary solutions of the first-order system (8.1) by finding all solutions of the algebraic system (8.2).

System (8.2) consists of two algebraic equations for the two unknowns x and y. In general it can have no solutions, one solution, or many solutions.

Example. Find all stationary solutions of the system

$$\frac{\mathrm{d}x}{\mathrm{d}t} = y\,, \qquad \frac{\mathrm{d}y}{\mathrm{d}t} = 4x - x^3\,.$$

Solution. The stationary solutions satisfy

$$0 = y\,, \qquad 0 = 4x - x^3 = x(4 - x^2) = x(2 - x)(2 + x)\,.$$

The first of these equations implies $y = 0$, while the second implies $x = 0$, $x = 2$, or $x = -2$. Therefore the stationary solutions are

$$(0,0)\,, \qquad (2,0)\,, \qquad (-2,0)\,. \qquad \square$$

Example. Find all stationary solutions of the system

$$\frac{\mathrm{d}x}{\mathrm{d}t} = y + 9 - x^2\,, \qquad \frac{\mathrm{d}y}{\mathrm{d}t} = 2xy\,.$$

Solution. The stationary solutions satisfy

$$0 = y + 9 - x^2\,, \qquad 0 = 2xy\,.$$

The second of these equations implies that $x = 0$ or $y = 0$. Each of these possibilities spawns a logical thread that must be followed.

First, if $x = 0$ then the first equation becomes $0 = y + 9$, which implies $y = -9$. Hence, the thread from $x = 0$ leads to the stationary solution $(0, -9)$.

On the other hand, if $y = 0$ then the first equation becomes $0 = 9 - x^2$, which implies that $x = \pm 3$. Hence, the thread from $y = 0$ leads to the stationary solutions $(-3, 0)$ and $(3, 0)$.

By collecting the results from each thread, we have found the three stationary solutions

$$(0,-9)\,, \qquad (-3,0)\,, \qquad (3,0)\,. \qquad \square$$

Example. Find all stationary solutions of the system

$$\frac{\mathrm{d}x}{\mathrm{d}t} = (y - x)(x - 1)\,, \qquad \frac{\mathrm{d}y}{\mathrm{d}t} = (3 + 2x - x^2)y\,.$$

Solution. The stationary solutions satisfy

$$0 = (y - x)(x - 1)\,, \qquad 0 = (3 + 2x - x^2)y = (3 - x)(1 + x)y\,.$$

The first of these equations implies $y = x$ or $x = 1$. Each of these possibilities spawns a logical thread that must be followed.

First, if $y = x$ then the second equation becomes $0 = (3 - x)(1 + x)x$, which implies $x = 3$, $x = -1$, or $x = 0$. Hence, the thread from $y = x$ leads to the stationary solutions $(3, 3)$, $(-1, -1)$, and $(0, 0)$.

On the other hand, if $x = 1$ then the second equation becomes $0 = 4y$, which implies $y = 0$. Hence, the thread from $x = 1$ leads to the stationary solution $(1, 0)$.

By collecting the results from each thread, we have found the four stationary solutions

$$(3,3)\,, \quad (-1,-1)\,, \quad (0,0)\,, \quad (1,0)\,. \qquad \square$$

Remark. We could have found the stationary solutions in the last example by starting with the second equation. That equation implies that $x = 3$, $x = -1$, or $y = 0$. Each of these possibilities spawns a logical thread that must be followed.

Remark. In each of the above examples there were a finite number of stationary solutions. As the following example shows, this need not be the case.

Example. Find all stationary solutions of the system

$$\frac{\mathrm{d}x}{\mathrm{d}t} = y\,, \qquad \frac{\mathrm{d}y}{\mathrm{d}t} = -\sin(x) - y\,.$$

Solution. The stationary solutions satisfy

$$0 = y\,, \qquad 0 = -\sin(x) - y\,.$$

The first of these equations implies that $y = 0$, while the second implies that $\sin(x) = 0$, which implies that $x = n\pi$ for some integer n. Therefore the stationary solutions are

$$\cdots \quad (-2\pi, 0)\,, \quad (-\pi, 0)\,, \quad (0, 0)\,, \quad (\pi, 0)\,, \quad (2\pi, 0)\,, \quad \cdots\,. \qquad \square$$

8.2. **Semistationary Solutions.** A solution of system (8.1) called *semistationary* if one of the unknowns x or y does not vary with time. These special solutions of system (8.1) are easy to identify. They can be analyzed by the methods for first-order equations we studied at the beginning of the course.

There will be a semistationary solution in the form $(a, Y(t))$ if the right-hand side of the $\mathrm{d}x/\mathrm{d}t$ equation vanishes when $x = a$ for every y — i.e. if $f(a, y) = 0$ for every y. In that case $y = Y(t)$ must satisfy the first-order autonomous equation

$$\frac{\mathrm{d}y}{\mathrm{d}t} = g(a, y)\,. \tag{8.3a}$$

Sometimes we can find explicit or implicit solutions of this first-order equation.

There will be a semistationary solution in the form $(X(t), b)$ if the right-hand side of the $\mathrm{d}y/\mathrm{d}t$ equation vanishes when $y = b$ for every x — i.e. if $g(x, b) = 0$ for every x. In that case $x = X(t)$ must satisfy the first-order autonomous equation

$$\frac{\mathrm{d}x}{\mathrm{d}t} = f(x, b)\,. \tag{8.3b}$$

Sometimes we can find explicit or implicit solutions of this first-order equation.

The orbits of semistationary solutions are easy to plot in a phase-plane portrait without finding an analytic solution.

- Semistationary solutions in the form $(a, Y(t))$ lie on the line $x = a$ in the xy-plane. The direction of their orbits is found by transfering the phase-line portrait for the first-order autonomous equation (8.3a).
- Semistationary solutions in the form $(X(t), b)$ lie on the line $y = b$ in the xy-plane. The direction of their orbits is found by transfering the phase-line portrait for the first-order autonomous equation (8.3b).

When

Example. Find all semistationary solutions of the system

$$\frac{\mathrm{d}x}{\mathrm{d}t} = y\,, \qquad \frac{\mathrm{d}y}{\mathrm{d}t} = 4x - x^3\,.$$

Solution. There is no value of x that makes the right-hand side of the $\mathrm{d}x/\mathrm{d}t$ equation zero. There is no value of y that makes the right-hand side of the $\mathrm{d}y/\mathrm{d}t$ equation zero. Therefore this system has no semistationary solutions. □

Example. Find all semistationary solutions of the system

$$\frac{\mathrm{d}x}{\mathrm{d}t} = y + 9 - x^2\,, \qquad \frac{\mathrm{d}y}{\mathrm{d}t} = 2xy\,.$$

Solution. There is no value of x that makes the right-hand side of the $\mathrm{d}x/\mathrm{d}t$ equation zero for every y. When $y = 0$ the right-hand side of the $\mathrm{d}y/\mathrm{d}t$ equation is zero for every x. When $y = 0$ the $\mathrm{d}x/\mathrm{d}t$ equation reduces to the first-order autonomous equation

$$\frac{\mathrm{d}x}{\mathrm{d}t} = 0 + 9 - x^2 = (3 - x)(3 + x)\,.$$

The solution of this equation with initial-value x_o at $t = 0$ is

$$x = 3\frac{(3 + x_o)e^{-6t} - (3 - x_o)}{(3 + x_o)e^{-6t} + (3 - x_o)}\,.$$

Therefore this system has the semistationary solutions

$$\left(3\frac{(3 + x_o)e^{-6t} - (3 - x_o)}{(3 + x_o)e^{-6t} + (3 - x_o)}\,,\, 0\right)\,. \qquad \square$$

Figure [??] shows these semistationary solutions with the stationary solutions.

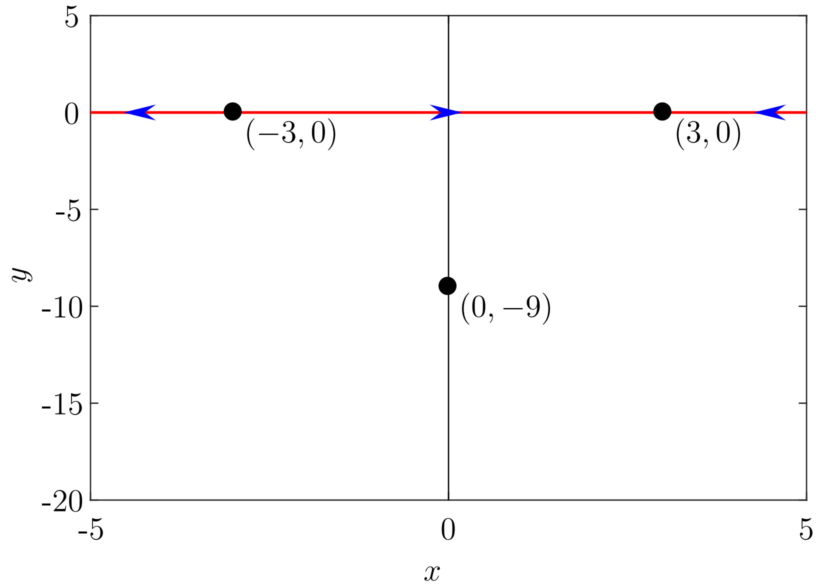

FIGURE 8.1. Stationary solutions (black circles) and semistationary solutions (red lines) for the second example system.

Remark. For $x_o = -3$, and $x_o = 3$ these recover the stationary solutions $(-3, 0)$ and $(3, 0)$ respectively. We thereby see that the stationary points $(-3, 0)$ and $(3, 0)$ are included in this semistationary solution, while the stationary point $(0, -9)$ is not.

Example. Find all semistationary solutions of the system

$$\frac{\mathrm{d}x}{\mathrm{d}t} = (y - x)(x - 1)\,, \qquad \frac{\mathrm{d}y}{\mathrm{d}t} = (3 + 2x - x^2)y\,.$$

Solution. When $x = 1$ the right-hand side of the $\mathrm{d}x/\mathrm{d}t$ equation is zero. When $x = 1$ the $\mathrm{d}y/\mathrm{d}t$ equation reduces to the first-order autonomous equation

$$\frac{\mathrm{d}y}{\mathrm{d}t} = (3 + 2 \cdot 1 - 1^2)y = 4y\,.$$

The solution of this equation with initial-value y_o at $t = 0$ is $y = y_o e^{4t}$.

When $y = 0$ the right-hand side of the $\mathrm{d}y/\mathrm{d}t$ equation is zero. When $y = 0$ the $\mathrm{d}x/\mathrm{d}t$ equation reduces to the first-order autonomous equation

$$\frac{\mathrm{d}x}{\mathrm{d}t} = (0 - x)(x - 1) = x(1 - x)\,.$$

The solution of this equation with initial-value x_o at $t = 0$ is $x = x_o e^t/(1 + x_o(e^t - 1))$.

Therefore this system has the semistationary solutions

$$\left(1\,,\, y_o e^{4t}\right)\,, \qquad \left(\frac{x_o e^t}{1 + x_o(e^t - 1)}\,,\, 0\right)\,. \qquad \square$$

Figure 8.2 shows these semistationary solutions with the stationary solutions.

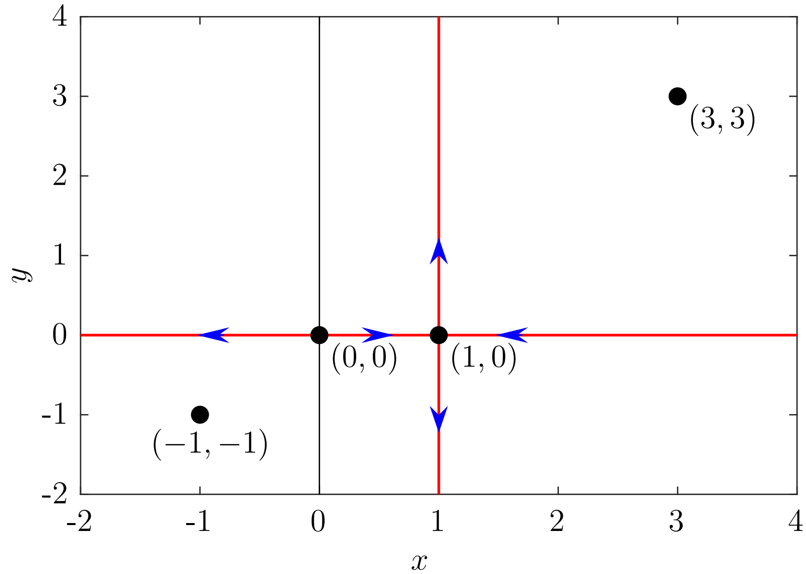

FIGURE 8.2. Stationary solutions (black circles) and semistationary solutions (red lines) for the first example system.

Remark. For $y_o = 0$, $x_o = 0$, and $x_o = 1$ these recover the stationary solutions $(1, 0)$, $(0, 0)$, and $(1, 0)$ respectively. We thereby see that the stationary point $(1, 0)$ is included in both semistationary solutions, the stationary point $(0, 0)$ is included in one semistationary solution, while the stationary points $(3, 3)$ and $(-1, -1)$ are not included in any semistationary solution.

8.3. **Orbit Equations and Integrals.** It is clear that every nonstationary solution $(x(t), y(t))$ of system (8.1) satisfies

$$-g(x, y)\,\frac{\mathrm{d}x}{\mathrm{d}t} + f(x, y)\,\frac{\mathrm{d}y}{\mathrm{d}t} = -g(x, y)\,f(x, y) + f(x, y)\,g(x, y) = 0\,.$$

Therefore the orbits these solutions trace out in the xy-plane lie on curves that satisfy the differential form equation

$$-\,g(x, y)\,\mathrm{d}x + f(x, y)\,\mathrm{d}y = 0\,. \tag{8.4}$$

Here we have adopted the convention of putting a minus in front of the $g(x, y)\,\mathrm{d}x$ term. This equation says that

$$\text{if } \begin{pmatrix} f(x, y) \\ g(x, y) \end{pmatrix} \neq \begin{pmatrix} 0 \\ 0 \end{pmatrix} \text{ then the orbit through } (x, y) \text{ is perpendicular to } \begin{pmatrix} -g(x, y) \\ f(x, y) \end{pmatrix}.$$

It says nothing at stationary points of system (8.1). Because (8.4) governs the relationship between x and y along orbits in the xy-plane, it is called an *orbit equation.*

Equation (8.4) can be recast as a first-order differential equation in two ways.

- In regions of the xy-plane where $f(x,y) \neq 0$ we can recast it as

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{g(x,y)}{f(x,y)}\,. \tag{8.5a}$$

  Here we are seeking y as a function of x.
- In regions of the xy-plane where $g(x,y) \neq 0$ we can recast it as

$$\frac{\mathrm{d}x}{\mathrm{d}y} = \frac{f(x,y)}{g(x,y)}\,. \tag{8.5b}$$

  Here we are seeking x as a function of y.

Equations (8.5a) and (8.5b) are equivalent to equation (8.4) except on the sets where $f(x,y) = 0$ and $g(x,y) = 0$ respectively. Like equation (8.4), they do not involve t. Rather, they govern the relationship between x and y along orbits in the xy-plane. For this reason they are also called *orbit equations.* If they are linear, separable, or have an exact differential form then their solution (possibly implicit) is reduced to finding some primitives.

Example. Try to solve the orbit equation for the system

$$\frac{\mathrm{d}x}{\mathrm{d}t} = y\,, \qquad \frac{\mathrm{d}y}{\mathrm{d}t} = 4x - x^3\,.$$

Solution. For this system orbit equation (8.4) is

$$-(4x - x^3)\,\mathrm{d}x + y\,\mathrm{d}y = 0\,.$$

This has the separated form

$$y\,\mathrm{d}y = (4x - x^3)\,\mathrm{d}x\,,$$

which can be integrated to obtain the implicit general solution

$$\tfrac{1}{2}y^2 = 2x^2 - \tfrac{1}{4}x^4 + c\,.$$

Upon solving for y we find the explicit solutions

$$y = \pm\sqrt{4x^2 - \tfrac{1}{2}x^4 + 2c}\,.$$

Example. Try to solve the orbit equation for the system

$$\frac{\mathrm{d}x}{\mathrm{d}t} = y + 9 - x^2\,, \qquad \frac{\mathrm{d}y}{\mathrm{d}t} = 2xy\,.$$

Solution. For this system orbit equation (8.4) is

$$-2xy\,\mathrm{d}x + (y + 9 - x^2)\,\mathrm{d}y = 0\,.$$

This equation does not have a separated form. However, because

$$\partial_y(-2xy) = -2x = \partial_x(y + 9 - x^2)\,,$$

we see that this differential form is exact. We thereby know that there is an $H(x,y)$ such that

$$\partial_x H(x,y) = -2xy\,, \qquad \partial_y H(x,y) = y + 9 - x^2\,.$$

These equations can be integrated to obtain $H(x,y) = \frac{1}{2}y^2 + 9y - x^2y$. Therefore the orbit equation has an implicit general solution

$$\tfrac{1}{2}y^2 + 9y - x^2y = c\,.$$

Example. Try to solve the orbit equation for the system

$$\frac{\mathrm{d}x}{\mathrm{d}t} = (y-x)(x-1)\,, \qquad \frac{\mathrm{d}y}{\mathrm{d}t} = (3+2x-x^2)y\,.$$

Solution. For this system orbit equation (8.5a) is

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{(3+2x-x^2)y}{(y-x)(x-1)}\,.$$

This equation is not linear or separable. Later we will show that it does not have an integral. Therefore we cannot find a general solution of this orbit equation. However, it has the stationary solution $y = 0$.

Similarly, for this system orbit equation (8.5b) is

$$\frac{\mathrm{d}x}{\mathrm{d}y} = \frac{(y-x)(x-1)}{(3+2x-x^2)y}\,,$$

which has the stationary solution $x = 1$. □

8.3.1. *Hamiltonian Systems.* The orbit equation (8.4) is

$$-g(x,y)\,\mathrm{d}x + f(x,y)\,\mathrm{d}y = 0\,.$$

This differential form is exact when

$$\partial_x f(x,y) + \partial_y g(x,y) = 0\,. \tag{8.6}$$

In that case there exists a function $H(x,y)$ such that

$$\partial_x H(x,y) = -g(x,y)\,, \qquad \partial_y H(x,y) = f(x,y)\,.$$

System (8.1) thereby is seen to have the form

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \partial_y H(x,y)\,, \qquad \frac{\mathrm{d}y}{\mathrm{d}t} = -\partial_x H(x,y)\,. \tag{8.7}$$

Such a system is said to be a *Hamiltonian* system while H is called its *Hamiltonian*. The Hamiltonian H is determined uniquely up to an additive constant. Because we have assumed that f and g are continuously differentiable, H will be twice continuously differentiable.

Example. Consider the system

$$\frac{\mathrm{d}x}{\mathrm{d}t} = y\,, \qquad \frac{\mathrm{d}y}{\mathrm{d}t} = 4x - x^3\,.$$

Show it is Hamiltonian and find its Hamiltonian $H(x,y)$.

Solution. The system is Hamiltonian because

$$\partial_x f(x,y) + \partial_y g(x,y) = \partial_x y + \partial_y (4x - x^3) = 0\,.$$

We can find $H(x, y)$ by solving

$$\partial_x H(x, y) = -4x + x^3 \,, \qquad \partial_y H(x, y) = y \,.$$

By integrating the second equation we find that $H(x, y) = \frac{1}{2}y^2 + h(x)$. By substituting this into the first equation we find that $h'(x) = -4x + x^3$. By setting $h(x) = -2x^2 + \frac{1}{4}x^4$ we obtain the Hamiltonian

$$H(x, y) = \tfrac{1}{2}y^2 - 2x^2 + \tfrac{1}{4}x^4 \,.$$

□

Many systems can be put into Hamiltonian form. For example, every second-order equation of the form

$$\frac{\mathrm{d}^2 x}{\mathrm{d}t^2} = g(x) \,,$$

can be recast as the first-order system

$$\frac{\mathrm{d}x}{\mathrm{d}t} = y \,, \qquad \frac{\mathrm{d}y}{\mathrm{d}t} = g(x) \,.$$

This system is Hamiltonian with $H(x, y) = \frac{1}{2}y^2 - G(x)$ where $G'(x) = g(x)$ because

$$\partial_x H(x, y) = -G'(x) = -g(x) \,, \qquad \partial_y H(x, y) = y \,.$$

More generally, consider any first-order system of the form

$$\frac{\mathrm{d}x}{\mathrm{d}t} = f(y) \,, \qquad \frac{\mathrm{d}y}{\mathrm{d}t} = g(x) \,.$$

This system is Hamiltonian with $H(x, y) = F(y) - G(x)$ where $F'(y) = f(y)$ and $G'(x) = g(x)$ because

$$\partial_x H(x, y) = -G'(x) = -g(x) \,, \qquad \partial_y H(x, y) = F'(y) = f(y) \,.$$

Remark. We can determine the phase-plane portrait of a Hamiltonian system by analyzing the Hamiltonian $H(x, y)$. We will postpone the discussion of how this is done until after we have introduced a larger class of systems to which the same techniques can be applied.

8.3.2. *Conservative Systems.* More generally, the orbit equation (8.3) has the equivalent differential form

$$-\rho(x, y)\, g(x, y)\, \mathrm{d}x + \rho(x, y)\, f(x, y)\, \mathrm{d}y = 0 \,,$$

where $\rho(x, y)$ is a nonzero factor that we will assume is also continuously differentiable. We hope to find a $\rho(x, y)$ that makes this differential form exact, in which case it is called an *integrating factor.* This will be the case when $\rho(x, y)$ satisfies

$$\partial_x[f(x, y)\, \rho] + \partial_y[g(x, y)\, \rho] = 0 \,. \tag{8.8}$$

When we can find such a ρ then there exists a function $H(x, y)$ such that

$$\partial_x H(x, y) = -\rho(x, y)\, g(x, y) \,, \qquad \partial_y H(x, y) = \rho(x, y)\, f(x, y) \,.$$

System (8.1) thereby is seen to have the form

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{1}{\rho(x, y)}\, \partial_y H(x, y) \,, \qquad \frac{\mathrm{d}y}{\mathrm{d}t} = -\frac{1}{\rho(x, y)}\, \partial_x H(x, y) \,. \tag{8.9}$$

Such a system is said to be *conservative* because if $x = X(t)$, $y = Y(t)$ is any solution of it then

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}H(X,Y) &= \partial_x H(X,Y)\frac{\mathrm{d}X}{\mathrm{d}t} + \partial_y H(X,Y)\frac{\mathrm{d}Y}{\mathrm{d}t} \\ &= \partial_x H(X,Y)\,\frac{1}{\rho(X,Y)}\,\partial_y H(X,Y) - \partial_y H(X,Y)\,\frac{1}{\rho(X,Y)}\,\partial_x H(X,Y) = 0\,.\end{aligned}$$

In other words, if $x = X(t)$, $y = Y(t)$ is any solution of (8.9) then $H(X(t), Y(t))$ is constant. Hamiltonian systems correspond to the cases when $\rho(x,y) = 1$. In the more general setting, H is called an *integral* of the system. Given a choice of ρ, the integral H is determined uniquely up to an additive constant. Because we have assumed that f, g, and ρ are continuously differentiable, H will be twice continuously differentiable.

Example. The drag force $\mathbf{f}_{\text{drag}}$ acting on an object moving with velocity $\mathbf{u}$ through air can be modeled as

$$\mathbf{f}_{\text{drag}} = -\eta_{\text{air}} A |\mathbf{u}|\mathbf{u}\,,$$

where η_{air} is the density of air, A is the aerodynamic cross-section of the object, and $|\mathbf{u}|$ is the magnitude of $\mathbf{u}$. This force has magnitude proportional to $|\mathbf{u}|^2$ and acts in the direction opposite to $\mathbf{u}$. If the mass of the object is m and the gravitational field is uniform then the velocity $\mathbf{u}$ will evolve as

$$\frac{\mathrm{d}u}{\mathrm{d}t} = -ku\sqrt{u^2+v^2}\,, \qquad \frac{\mathrm{d}v}{\mathrm{d}t} = -kv\sqrt{u^2+v^2} - a\,,$$

where u and v are the horizontal and vertical components of $\mathbf{u}$, $a > 0$ is the magnitude of the gravitational acceleration, and k is given by $k = \eta_{\text{air}} A/m$. Find an integral for this system and use it to give an implicit general solution.

Remark. This is a famous problem from the earliest days of the subject. It was considered by members of the Bernoulli family, but none could solve it. However, as we will see below, it can be solved by the integrating factor method that Euler introduced in 1734.

Solution. The only stationary solution of this system is

$$u = 0\,, \qquad v = -\sqrt{a/k}\,.$$

The only semistationary solutions of this system have the form $(0, V(t))$ where $v = V(t)$ satisfies the first-order equation

$$\frac{\mathrm{d}v}{\mathrm{d}t} = -k|v|v - a\,.$$

We solved this equation earlier in the course.

We now seek solutions with $u \neq 0$. The system is not separable. Setting

$$f(u,v) = -ku\sqrt{u^2+v^2}\,, \qquad g(u,v) = -kv\sqrt{u^2+v^2} - a\,,$$

we see that

$$\partial_u f = -k\sqrt{u^2+v^2} - \frac{ku^2}{\sqrt{u^2+v^2}}\,, \qquad \partial_v g = -k\sqrt{u^2+v^2} - \frac{kv^2}{\sqrt{u^2+v^2}}\,.$$

The system is not Hamiltonian because

$$\partial_u f + \partial_v g = -3k\sqrt{u^2+v^2} \neq 0\,.$$

Therefore we must seek an integrating factor.

An integrating factor $\rho(u,v)$ must satisfy the linear partial differential equation

$$\begin{aligned} 0 &= \partial_u[f\rho] + \partial_v[g\rho] = f\partial_u\rho + g\partial_v\rho + (\partial_u f + \partial_v g)\rho \\ &= -ku\sqrt{u^2+v^2}\partial_u\rho - \left(kv\sqrt{u^2+v^2} + a\right)\partial_v\rho - 3k\sqrt{u^2+v^2}\rho\,. \end{aligned}$$

By setting $\partial_v\rho = 0$ this reduces to the linear ordinary differential equation

$$0 = u\partial_u\rho + 3\rho\,.$$

This has the nonzero solution $\rho = 1/u^3$ for $u \neq 0$.

Because $\rho = 1/u^3$ is an integrating factor wherever $u \neq 0$, we can find an integral $H(u,v)$ such that

$$\partial_u H(u,v) = \frac{1}{u^3}\left(kv\sqrt{u^2+v^2} + a\right)\,, \qquad \partial_v H(u,v) = -\frac{1}{u^3}ku\sqrt{u^2+v^2}\,.$$

We can integrate the second equation as

$$H(u,v) = -\int \frac{k}{u^2}\sqrt{u^2+v^2}\,\mathrm{d}v = -k\int\sqrt{1+r^2}\,\mathrm{d}r\,, \qquad \text{where } v = ur\,.$$

By using the calculus fact

$$\int\sqrt{1+r^2}\,\mathrm{d}r = \tfrac{1}{2}\left(r\sqrt{1+r^2} + \log\left(r + \sqrt{1+r^2}\right)\right) + c\,,$$

we obtain

$$H(u,v) = -\frac{k}{2}\left(\frac{v}{u}\sqrt{1+\frac{v^2}{u^2}} + \log\left(\frac{v}{u} + \sqrt{1+\frac{v^2}{u^2}}\right)\right) + h(u)\,.$$

But, by using the same calculus fact, this implies that

$$\partial_u H(u,v) = -k\sqrt{1+\frac{v^2}{u^2}}\;\partial_u\left(\frac{v}{u}\right) + h'(u) = k\sqrt{u^2+v^2}\;\frac{v}{u^3} + h'(u)\,,$$

which when placed into the first equation yields

$$h'(u) = \frac{a}{u^3}\,.$$

By taking $h(u) = -\frac{1}{2}a/u^2$ we obtain the integral

$$H(u,v) = -\frac{k}{2}\left(\frac{v}{u}\sqrt{1+\frac{v^2}{u^2}} + \log\left(\frac{v}{u} + \sqrt{1+\frac{v^2}{u^2}}\right)\right) - \frac{a}{2u^2}\,.$$

Therefore when $u \neq 0$ the orbit equation has an implicit general solution given by

$$k\left(\frac{v}{u}\sqrt{1+\frac{v^2}{u^2}} + \log\left(\frac{v}{u} + \sqrt{1+\frac{v^2}{u^2}}\right)\right) + \frac{a}{u^2} = c\,.$$

This cannot be solved analytically either for u as an explicit function of v or for v as an explicit function of u. $\square$

8.4. **Phase-Plane Portraits from Integrals.** Here we consider the class of conservative systems in the form

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{1}{\rho(x,y)}\,\partial_y H(x,y)\,, \qquad \frac{\mathrm{d}y}{\mathrm{d}t} = -\frac{1}{\rho(x,y)}\,\partial_x H(x,y)\,, \quad \text{with } \rho(x,y) \neq 0\,. \tag{8.10}$$

In particular, the class of Hamiltonian systems is included by taking $\rho(x,y) = 1$. Here we show how the global phase-plane portrait of such a system can be determined by analyzing its integral $H(x,y)$.

8.4.1. *Critical Points and Level Sets.* We begin with two basic facts.

Fact 1. A point (x_o, y_o) is a stationary solution of the conservative system (8.10) if and only if it is a critical point of $H(x,y)$.

Reason. Recall from multivariable calculus that a point is a *critical point* of $H(x,y)$ if and only if it satifies

$$\partial_x H(x,y) = 0\,, \qquad \partial_y H(x,y) = 0\,.$$

But these are exactly the equations that characterize stationary solutions of (8.10). $\square$

Fact 2. If $(x(t), y(t))$ is any solution to the conservative system (8.10) then $H(x(t), y(t))$ is a constant.

Reason. The multivariable chain rule and system (8.10) yield

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}H(x,y) &= \partial_x H(x,y)\,\frac{\mathrm{d}x}{\mathrm{d}t} + \partial_y H(x,y)\,\frac{\mathrm{d}y}{\mathrm{d}t} \\ &= \partial_x H(x,y)\,\frac{1}{\rho(x,y)}\,\partial_y H(x,y) + \partial_y H(x,y)\left(-\frac{1}{\rho(x,y)}\,\partial_x H(x,y)\right) \\ &= 0\,.\end{aligned}$$

Therefore $H(x(t), y(t))$ is a constant. $\square$

This last fact states that orbits of system (8.10) lie in so-called *level sets* of $H(x,y)$ — namely, on sets in the xy-plane of the form

$$\big\{(x,y)\,:\,H(x,y) = c\big\} \qquad \text{for some constant } c\,. \tag{8.11}$$

This set will be empty unless c is in the range of H. If c is in the range of H then c is called a *critical value* of H if $c = H(x_o, y_o)$ for some point (x_o, y_o) that is a critical point of the function H, and is called a *noncritical value* of H otherwise.

- If c is a noncritical value of H then the associated level set will consist of one or more disjoint curves, each of which is a single orbit. These curves will be either a closed loop, corresponding to a periodic orbit, or an unbounded curve, corresponding to an orbit that becomes unbounded as $t \to -\infty$ or as $t \to \infty$.

- If c is a critical value of H then the associated level set will consist of one or more critical points and possibly other curves that can either loop from a critical point to itself, connect two critical points, connect a critical point to infinity, run from infinity to infinity, or be a closed loop. We will illustrate most of these possibilities with examples below.

8.4.2. *Nondegenerate Critical Points.* For simplicity we will consider only integrals H with critical points that are *nondegenerate*. Recall from multivariable calculus that a critical point (x_o, y_o) of H is said to be nondegenerate if

$$\det\big(\boldsymbol{\partial}^2 H(x_o, y_o)\big) \neq 0\,, \tag{8.12}$$

where $\boldsymbol{\partial}^2 H(x, y)$ denotes the Hessian matrix of second derivatives, which is given by

$$\boldsymbol{\partial}^2 H(x, y) = \begin{pmatrix} \partial_{xx} H(x, y) & \partial_{xy} H(x, y) \\ \partial_{yx} H(x, y) & \partial_{yy} H(x, y) \end{pmatrix}. \tag{8.13}$$

Nondegenerate critical points are *isolated* in the sense that they can be surrounded by a box in which there are no other critical points. At a nondegenerate critical point (x_o, y_o) of H there are just three possibilities.

- It can be a saddle point of H. In this case the level sets of H near (x_o, y_o) extend away from (x_o, y_o). Hence, (x_o, y_o) is a *saddle* in the phase-plane portrait of system (8.10).
- It can be a local minimizer of H. In this case the level sets of H near (x_o, y_o) are closed loops, representing periodic orbits. Hence, (x_o, y_o) is a *center* in the phase-plane portrait of system (8.10). Moreover, it can be shown that (x_o, y_o) will be a *clockwise center* when $\rho(x_o, y_o) > 0$, and a *counterclockwise center* when $\rho(x_o, y_o) < 0$.
- If can be a local maximizer of H. In this case the level sets of H near (x_o, y_o) are closed loops, representing periodic orbits. Hence, (x_o, y_o) is a *center* in the phase-plane portrait of system (8.10). Moreover, it can be shown that (x_o, y_o) will be a *counterclockwise center* when $\rho(x_o, y_o) > 0$, and a *clockwise center* when $\rho(x_o, y_o) < 0$.

Remark. To get the rotations of the centers correct we must carefully adhere to the sign conventions in system (8.10).

Now we recall two sets of criteria from multivariable calculus that determine when a nondegenerate critical point is a saddle point, a local minimizer, or a local maximizer.

The first set of criteria are based on the *eigenvalues* of the Hessian matrix $\boldsymbol{\partial}^2 H(x_o, y_o)$. The Hessian matrix is symmetric because $\partial_{xy} H(x, y) = \partial_{yx} H(x, y)$. Therefore it has real eigenvalues. If (x_o, y_o) is a nondegenerate critical point of H then the eigenvalues of $\boldsymbol{\partial}^2 H(x_o, y_o)$ are nonzero and there are three possibilites.

- (x_o, y_o) is a saddle point when $\boldsymbol{\partial}^2 H(x_o, y_o)$ has one eigenvalue of each sign.
- (x_o, y_o) is a local minimizer when $\boldsymbol{\partial}^2 H(x_o, y_o)$ has two positive eigenvalues.
- (x_o, y_o) is a local maximizer when $\boldsymbol{\partial}^2 H(x_o, y_o)$ has two negative eigenvalues.

If the eigenvalues of $\boldsymbol{\partial}^2 H(x_o, y_o)$ are easy to find then these criteria are easy to use. For example, when $\boldsymbol{\partial}^2 H(x_o, y_o)$ is diagonal then its eigenvalues are just its diagonal entries.

The second set of criteria are based on the *determinant* and *trace* of the Hessian matrix $\boldsymbol{\partial}^2 H(x_o, y_o)$, which often are easier to compute than its eigenvalues. Recall that the characteristic polynomial of $\boldsymbol{\partial}^2 H(x_o, y_o)$ is

$$p(z) = z^2 - \mathrm{tr}\big(\boldsymbol{\partial}^2 H(x_o, y_o)\big) z + \det\big(\boldsymbol{\partial}^2 H(x_o, y_o)\big) \,.$$

The eigenvalues of $\boldsymbol{\partial}^2 H(x_o, y_o)$ are the roots, λ_1 and λ_2, of this polynomial. Therefore

$$p(z) = (z - \lambda_1)(z - \lambda_2) = z^2 - (\lambda_1 + \lambda_2) z + \lambda_1 \lambda_2 \,.$$

Upon comparing these two expressions for $p(z)$ we see that

$$\mathrm{tr}\big(\boldsymbol{\partial}^2 H(x_o, y_o)\big) = \lambda_1 + \lambda_2 \,, \qquad \det\big(\boldsymbol{\partial}^2 H(x_o, y_o)\big) = \lambda_1 \lambda_2 \,.$$

This shows that λ_1 and λ_2 have opposite signs if $\det\big(\boldsymbol{\partial}^2 H(x_o, y_o)\big) < 0$. It also shows λ_1 and λ_2 have the same sign if $\det\big(\boldsymbol{\partial}^2 H(x_o, y_o)\big) > 0$, in which case that sign is given by $\mathrm{tr}\big(\boldsymbol{\partial}^2 H(x_o, y_o)\big)$. Therefore we have the following criteria.

- (x_o, y_o) is a saddle point when $\det\big(\boldsymbol{\partial}^2 H(x_o, y_o)\big) < 0$.
- (x_o, y_o) is a local minimizer when $\det\big(\boldsymbol{\partial}^2 H(x_o, y_o)\big) > 0$ and $\mathrm{tr}\big(\boldsymbol{\partial}^2 H(x_o, y_o)\big) > 0$.
- (x_o, y_o) is a local maximizer when $\det\big(\boldsymbol{\partial}^2 H(x_o, y_o)\big) > 0$ and $\mathrm{tr}\big(\boldsymbol{\partial}^2 H(x_o, y_o)\big) < 0$.

These criteria are easier to apply when the eigenvalues of $\boldsymbol{\partial}^2 H(x_o, y_o)$ are not obvious.

8.4.3. *Examples.* When system (8.10) has only nondegenerate critical points then its global phase-plane portrait can be sketched as follows.

1. Determine the critical points of H and plot them. They are the stationary points of system (8.10).
2. Check that the critical points are nondegenerate and identify them as saddle points, local minimizers, or local maximizers.
3. If $\rho(x_o, y_o) > 0$ then local minimizers will be clockwise centers and local maximizers will be counterclockwise centers. If $\rho(x_o, y_o) < 0$ then the rotation is reversed.
4. Sketch the level set associated with each saddle point. This will partition the phase-plane into regions.
5. Sketch enough level sets for noncritical values so there is at least one orbit inside each region.
6. Determine the direction of each orbit either by inferring it from the centers or by determining it directly from system (8.10).

Example. Sketch a phase portrait for the system

$$\frac{\mathrm{d}x}{\mathrm{d}t} = y \,, \qquad \frac{\mathrm{d}y}{\mathrm{d}t} = 4x - x^3 \,.$$

Solution. We have shown already that this is a Hamiltonian system with

$$H(x, y) = \tfrac{1}{2} y^2 - 2x^2 + \tfrac{1}{4} x^4 \,.$$

We also have shown already that the critical points of H are

$$(0,0)\,, \qquad (-2,0)\,, \qquad (2,0)\,.$$

The Hessian matrix is

$$\boldsymbol{\partial}^2 H(x,y) = \begin{pmatrix} -4+3x^2 & 0 \\ 0 & 1 \end{pmatrix}\,.$$

Because

$$\boldsymbol{\partial}^2 H(0,0) = \begin{pmatrix} -4 & 0 \\ 0 & 1 \end{pmatrix}\,, \qquad \boldsymbol{\partial}^2 H(\pm 2,0) = \begin{pmatrix} 8 & 0 \\ 0 & 1 \end{pmatrix}\,,$$

we see that $(0,0)$ is a *saddle point* while $(-2,0)$ and $(2,0)$ are *local minimizers*. Hence, the phase-plane portrait is a *saddle* near $(0,0)$ and is a *clockwise center* near $(-2,0)$ and $(2,0)$.

The saddle point $(0,0)$ has value $H(0,0) = 0$. The level set associated with the critical value 0 consists of all points (x,y) that satisfy $H(x,y) = 0$. This equation can be written as

$$y^2 = 4x^2 - \tfrac{1}{2}x^4\,,$$

which has solutions

$$y = \pm 2x\sqrt{1 - \tfrac{1}{8}x^2} \qquad \text{for every } x \text{ in } \left[-\sqrt{8}, \sqrt{8}\right].$$

A graph shows two orbits that emerge from the origin tangent to the lines $y = \pm 2x$, move out to either $(\pm\sqrt{8}, 0)$ while taking extreme values of $y = \pm 2\sqrt{2}$ when $x = \pm 2$, and return to the origin. These orbits are called *separatrices* because they separate the periodic orbits that go around one center from the periodic orbits that go around both centers. Figure 8.3 shows how these orbits are produced by the intersection of $H(x,y)$ with level planes.

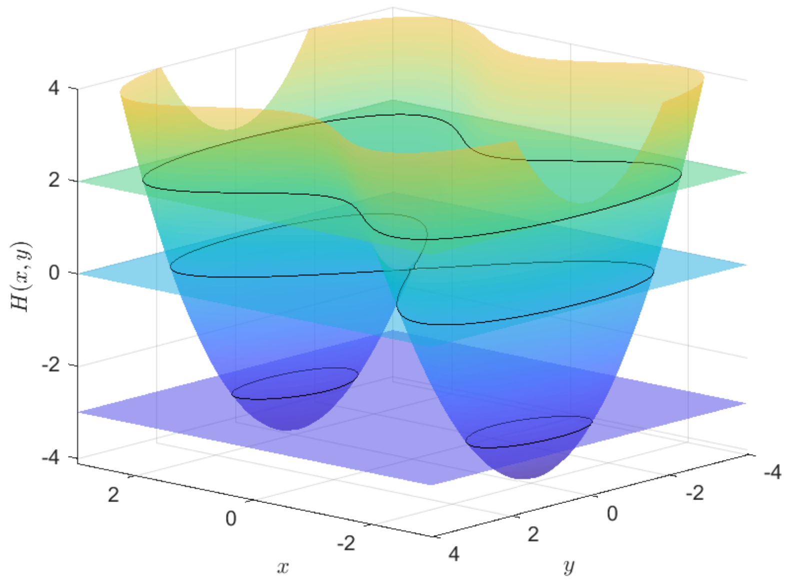

FIGURE 8.3. Level curves of $H(x,y)$ are orbits.

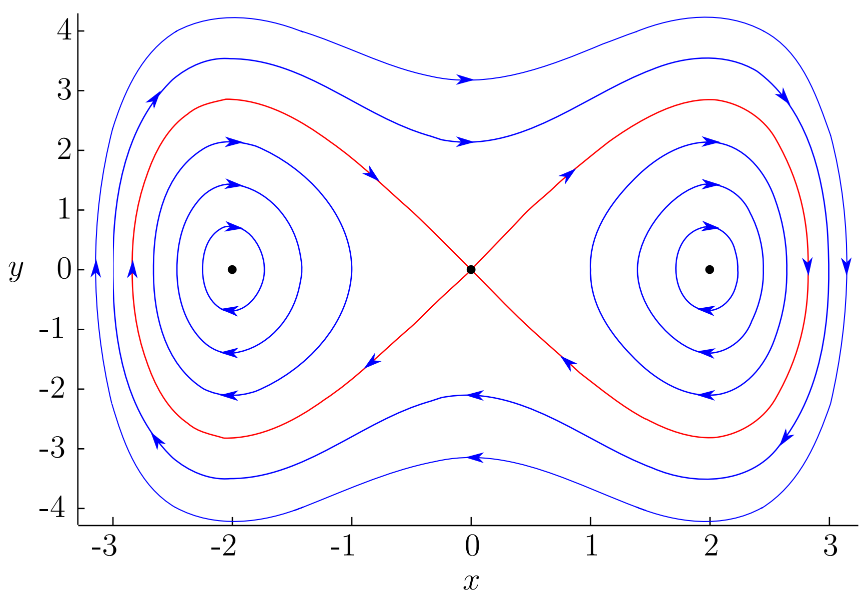

FIGURE 8.4. Phase portrait for the system $x' = y$ and $y' = 4x - x^3$. The separatrices are shown in red.

The direction of the orbits on the separatrices can be inferred from the clockwise rotation of the orbits around each center. Then the clockwise rotation of the orbits that go around both centers can be inferred from the direction of the orbits on the separatrices. Alternatively, because $x' = y$ we see that orbits move to the right when

$y > 0$ and move to the left when $y < 0$. The resulting phase-plane portrait is shown below in Figure 8.1.

Remark. The above example illustrates why level sets associated with saddle points are more important than those associated with local extremizers. Because $(-2, 0)$ and $(2, 0)$ are the only local minimizers and they have the same value $H(\pm 2, 0) = -4$, and because $H(x, y) \to \infty$ as $x^2 + y^2 \to \infty$, these local minimizers are global minimizers. Hence, the level set associated with the critical value -4 consists of just the local minimizers $(-2, 0)$ and $(2, 0)$. This can also be seen from the fact that the equation $H(x, y) = -4$ can be written as

$$\tfrac{1}{2}y^2 + \tfrac{1}{4}(x^2 - 4)^2 = 0\,,$$

which is satisfied only when $(x, y) = (\pm 2, 0)$.

Example. An undamped pendulum of mass m and length ℓ that moves in a plane is governed by the equation

$$m\ell\,\theta'' = -ma\sin(\theta)\,,$$

where a is the acceleration of gravity and θ is the angle (in radians) of the deviation of the pendulum from its downward rest position. This can be recast as the first-order system

$$x' = y\,, \qquad y' = -\omega^2 \sin(x)\,,$$

where $x = \theta$, $y = \theta'$, and $\omega > 0$ is the natural frequency of the pendulum, which is determined by $\omega^2 = a/\ell$. Sketch a phase portrait for this system.

Solution. The stationary points of this system satisfy

$$0 = y\,, \qquad 0 = -\omega^2 \sin(x)\,.$$

Therefore there are an infinite number of stationary points given by

$$(n\pi, 0) \quad \text{for every integer } n\,.$$

When n is even the pendulum is in its downward rest position. When n is odd the pendulum is in its upward rest position.

We have seen that systems of this form are Hamiltonian with

$$H(x, y) = \tfrac{1}{2}y^2 - \omega^2 \cos(x)\,.$$

Because the Hessian is

$$\boldsymbol{\partial}^2 H(x, y) = \begin{pmatrix} \partial_{xx}H(x, y) & \partial_{xy}H(x, y) \\ \partial_{yx}H(x, y) & \partial_{yy}H(x, y) \end{pmatrix} = \begin{pmatrix} \omega^2\cos(x) & 0 \\ 0 & 1 \end{pmatrix}\,.$$

we see that

$$\boldsymbol{\partial}^2 H(n\pi, 0) = \begin{pmatrix} (-1)^n\omega^2 & 0 \\ 0 & 1 \end{pmatrix}\,.$$

Because this is a diagonal matrix, its eigenvalues can be read off from its diagonal entries. We thereby see that its eigenvalues are $(-1)^n\omega^2$ and 1.

- We see that when n is even the critical point $(n\pi, 0)$ is a local minimizer of H, whereby the stationary point $(n\pi, 0)$ is a *clockwise center*.
- We see that when n is odd the critical point $(n\pi, 0)$ is a saddle point of H, whereby the stationary point $(n\pi, 0)$ is a *saddle*.

We now sketch the level sets associated with the saddle points of H. Because $H(n\pi, 0) = \omega^2$ when n is odd, there is just one such level set, which consists of all points (x, y) that satisfy

$$H(x, y) = \tfrac{1}{2}y^2 - \omega^2 \cos(x) = \omega^2 \,.$$

Upon solving this equation for y we find that

$$y = \pm 2\omega \sqrt{\frac{1 + \cos(x)}{2}} = \pm 2\omega \cos(\tfrac{1}{2}x) \,.$$

In the last step we used the half-angle identity $\cos(\frac{1}{2}x) = \sqrt{(1 + \cos(x))/2}$. These curves run between adjacent saddle points. They are *separatrices* because they separate the periodic orbits that go around one center from the nonperiodic orbits.

The direction of the orbits on the separatrices can be inferred from the clockwise rotation of the orbits around each center. Then the direction of the nonperiodic orbits can be inferred from the direction of the orbits on the separatrices. Alternatively, because $x' = y$ we see that orbits move to the right when $y > 0$ and move to the left when $y < 0$. The resulting phase-plane portrait for $\omega = 1$ is shown below in Figure 8.2.

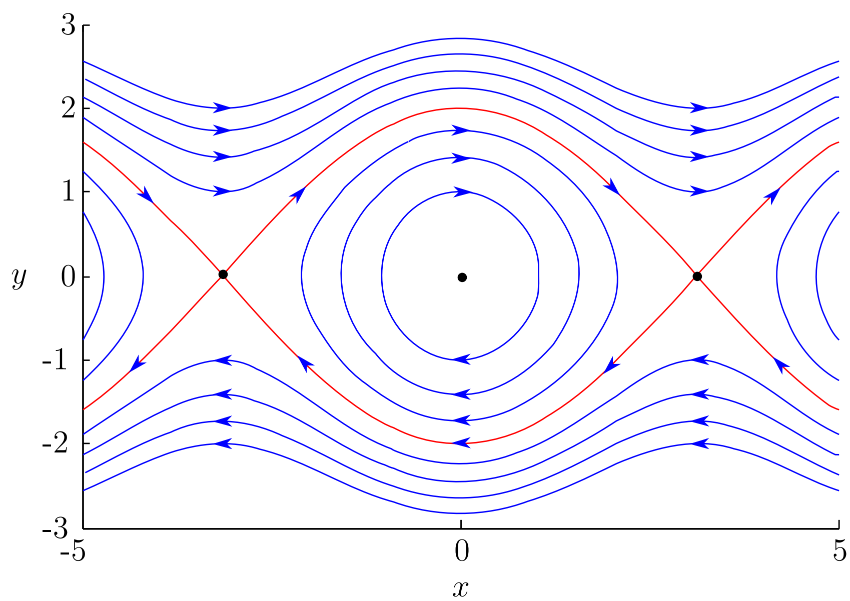

Figure 8.5. Phase portrait for the system $x' = y$ and $y' = -\sin(x)$. The separatrices are shown in red and are given by $y = \pm 2\cos(\frac{1}{2}x)$.

Remark. This example also illustrates why level sets associated with saddle points are more important than those associated with local extermizers. Because $H(n\pi, 0) = -\omega^2$ when n is even, all the local minimizers lie on the level set given by $H(x, y) = -\omega^2$. But for every (x, y) we have

$$H(x, y) = \tfrac{1}{2}y^2 - \omega^2 \cos(x) \geq -\omega^2 \cos(x) \geq -\omega^2 \,,$$

with equality only when $y = 0$ and $\cos(x) = 1$. Therefore this level set consists of just the local minimizers, which are seen to be global minimizers.

Remark. The downward rest positions are the centers. They are *stable* in the sense that every orbit that starts nearby a center will stay nearby that center. More precisely, if we displace the pendulum slightly from its downward rest position and release it with a small initial velocity then the pendulum will just make small periodic oscillations about its downward rest position. The upward rest positions are the saddles. They are *unstable* in the sense that orbits that start nearby a saddle will move far away from that saddle. More precisely, if we displace the pendulum slightly from its upward rest position and release it with a small initial velocity then the pendulum will generally either make large oscillations or spin around. These ideas will be developed further in the next chapter.

8.5. **Reduced Equations and Analytic Solutions.** Sometimes solutions of system (8.1) can be found by solving two first-order equations. The first of these is the *orbit equation*. Its solutions describe sets in the xy-plane within which orbits lie. The second is a so-called *reduced equation* that restricts system (8.1) to a curve in the xy-plane given by a solution of the orbit equation.

Here are two ways in which to restrict system (8.1) to the solution of an orbit equation.

- If we can find an explicit solution $y = Y(x)$ of an orbit equation then we can try to find x as a function of t by solving

$$\frac{\mathrm{d}x}{\mathrm{d}t} = f(x, Y(x))\,. \tag{8.14a}$$

  This autonomous equation can be solved implicitly if we can find a primitive of $1/f(x, Y(x))$. If we are able to find a solution $x(t)$ of this equation then a solution of system (8.1) is given by

$$\big(x(t)\,,\, y(t)\big) = \big(x(t)\,,\, Y(x(t))\big)\,.$$

- If we can find an explicit solution $x = X(y)$ of an orbit equation then we can try to find y as a function of t by solving

$$\frac{\mathrm{d}y}{\mathrm{d}t} = g(X(y), y)\,. \tag{8.14b}$$

  This autonomous equation can be solved implicitly if we can find a primitive of $1/g(X(y), y)$. If we are able to find a solution $y(t)$ of this equation then a solution of system (8.1) is given by

$$\big(x(t)\,,\, y(t)\big) = \big(X(y(t))\,,\, y(t)\big)\,.$$

Equations (8.14a) and (8.14b) are called *reduced equations* because they restrict system (8.1) to a set in the xy-plane given by a solution of an orbit equation. We will give some examples of their use in the next subsection.

It is often difficult or impossible to solve the orbit equation (8.4). And even when we can find an explicit solution of an orbit equation, it is often difficult or impossible to find the primitive needed to solve the associated reduced equation (8.14).

Example. Try to solve the system

$$\frac{\mathrm{d}x}{\mathrm{d}t} = y\,, \qquad \frac{\mathrm{d}y}{\mathrm{d}t} = 4x - x^3\,.$$

Solution. For this system orbit equation (8.4) is

$$-(4x - x^3)\,\mathrm{d}x + y\,\mathrm{d}y = 0\,.$$

This has the separated form

$$y\,\mathrm{d}y = (4x - x^3)\,\mathrm{d}x\,,$$

which can be integrated to obtain

$$\tfrac{1}{2}y^2 = 2x^2 - \tfrac{1}{4}x^4 + \tfrac{1}{2}c\,.$$

Here we put the $\frac{1}{2}$ in front of the c because we see that the equation must be multiplied by 2 in order to solve for y. Upon solving for y we find the explicit solutions

$$y = \pm\sqrt{4x^2 - \tfrac{1}{2}x^4 + c}\,.$$

The reduced equation then becomes

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \pm\sqrt{4x^2 - \tfrac{1}{2}x^4 + c}\,,$$

which can be solved implicitly in terms of an integral as

$$t = \pm \int \frac{1}{\sqrt{4x^2 - \frac{1}{2}x^4 + c}}\,\mathrm{d}x\,.$$

This integral cannot be evaluated analytically for every c by methods from first-year calculus. That evaluation requires the use of *elliptic functions*, which lie beyond the scope of this course. It can be evaluated analytically for $c = 0$ by methods from first-year calculus, but we will not do that here. □

Remark. In the previous example we could integrate the orbit equation easily, but could not do the same for the reduced equation. This is often the case when the orbit equation can be integrated. In the next example we cannot even find a general solution of the orbit equation. This will be the case for most orbit equations.

Example. Try to solve the system

$$\frac{\mathrm{d}x}{\mathrm{d}t} = y + 9 - x^2\,, \qquad \frac{\mathrm{d}y}{\mathrm{d}t} = 2xy\,.$$

Solution. For this system orbit equation (8.4) is

$$-2xy\,\mathrm{d}x + (y + 9 - x^2)\,\mathrm{d}y = 0\,.$$

This equation does not have a separated form. However, because

$$\partial_y(-2xy) = -2x = \partial_x(y + 9 - x^2)\,,$$

we see that this differential form is exact. We thereby know that there is an $H(x, y)$ such that

$$\partial_x H(x, y) = -2xy\,, \qquad \partial_y H(x, y) = y + 9 - x^2\,.$$

These equations can be integrated to obtain $H(x,y) = \frac{1}{2}y^2 - 9y - x^2y$. Therefore the orbit equation has the implicit general solution

$$\tfrac{1}{2}y^2 + 9y - x^2y = c\,.$$

Example. Try to solve the system

$$\frac{\mathrm{d}x}{\mathrm{d}t} = (y-x)(x-1)\,, \qquad \frac{\mathrm{d}y}{\mathrm{d}t} = (3+2x-x^2)y\,.$$

Solution. For this system orbit equation (8.5a) is

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{(3+2x-x^2)y}{(y-x)(x-1)}\,.$$

This equation is not linear or separable. Because we cannot find an integral, we cannot find a general solution of this orbit equation. However, it has the stationary solution $y = 0$. The associated reduced equation is

$$\frac{\mathrm{d}x}{\mathrm{d}t} = (0-x)(x-1) = x - x^2\,.$$

If $x = X(t)$ is a solution of this equation then $(X(t), 0)$ is a solution of our system.

Similarly, for this system orbit equation (8.5b) is

$$\frac{\mathrm{d}x}{\mathrm{d}y} = \frac{(y-x)(x-1)}{(3+2x-x^2)y}\,,$$

which has the stationary solution $x = 1$. The associated reduced equation is

$$\frac{\mathrm{d}y}{\mathrm{d}t} = (3+2\cdot 1 - 1^2)y = 4y\,.$$

Because $y = ce^{4t}$ is a solution of this equation, $(1, ce^{4t})$ is a solution of our system. $\square$

Remark. The solutions found in the last example were the semistationary solutions that we found earlier. This reflects the following general facts.

- Semistationary solutions of system (8.1) in the form $(X(t), b)$ correspond to $y = b$ being a stationary solution of the orbit equation (8.5a).
- Semistationary solutions of system (8.1) in the form $(a, Y(t))$ correspond to $x = a$ being a stationary solution of the orbit equation (8.5b).

Therefore this approach does not yield new solutions unless we can find nonstationary solutions of an orbit equation.

Exercises on Integral Methods

Find all the stationary solutions to the following system of differential equations.

(1) $\frac{\mathrm{d}x}{\mathrm{d}t} = 1 + 2y, \qquad \frac{\mathrm{d}y}{\mathrm{d}t} = 1 - 3x^2$

Solution

(2) $\frac{\mathrm{d}x}{\mathrm{d}t} = x - y, \qquad \frac{\mathrm{d}y}{\mathrm{d}t} = y + 2x$

Solution

(3) $\frac{\mathrm{d}x}{\mathrm{d}t} = y, \qquad \frac{\mathrm{d}y}{\mathrm{d}t} = x(x-2)(y-2)$

Solution

(4) $\frac{\mathrm{d}x}{\mathrm{d}t} = -(2+y)y, \qquad \frac{\mathrm{d}y}{\mathrm{d}t} = x(1-y)$

Solution

(5) $\frac{\mathrm{d}x}{\mathrm{d}t} = x(1-x-y), \qquad \frac{\mathrm{d}y}{\mathrm{d}t} = (1-y)(2+x)$

Solution

(6) $\frac{\mathrm{d}x}{\mathrm{d}t} = (2+x)(y-x), \qquad \frac{\mathrm{d}y}{\mathrm{d}t} = y(2+x-x^2)$

Solution

For problems $7 - 10$*, find the semistationary solutions. If they do not exist, state as such.*

(7) $\frac{\mathrm{d}x}{\mathrm{d}t} = y, \qquad \frac{\mathrm{d}y}{\mathrm{d}t} = x^2$

Short Answer
Solution

(8) $\frac{\mathrm{d}x}{\mathrm{d}t} = y(x-1), \qquad \frac{\mathrm{d}y}{\mathrm{d}t} = (x-2)(y-2)$

Short Answer
Solution

(9) $\frac{\mathrm{d}x}{\mathrm{d}t} = (x^2-2x)(y+1), \qquad \frac{\mathrm{d}y}{\mathrm{d}t} = y(2+x^2)$

Short Answer
Solution

(10) $\frac{\mathrm{d}x}{\mathrm{d}t} = x(1+y), \qquad \frac{\mathrm{d}y}{\mathrm{d}t} = (y^2-y)(x+1)$

Short Answer
Solution

(11) Show that if $(x(t), y(t))$ is a solution to a Hamiltonian system, then $H(x(t), y(t))$ is some constant.

Solution

(12) Consider the Hamiltonian system

$$x' = \partial_y H(x,y), \quad y' = -\partial_x H(x,y)$$

and some function $g(x,y)$ which satisfies

$$\partial_x g(x,y)\partial_y H(x,y) = \partial_x H(x,y)\partial_y g(x,y).$$

Show that $g(x(t), y(t))$ is constant.

Solution

Show the system is Hamiltonian and find its Hamiltonian $H(x, y)$

(13) $\frac{\mathrm{d}x}{\mathrm{d}t} = y, \qquad \frac{\mathrm{d}y}{\mathrm{d}t} = -x + \frac{x^3}{6}$

Short Answer
Solution

(14) $\frac{\mathrm{d}x}{\mathrm{d}t} = (y - x), \qquad \frac{\mathrm{d}y}{\mathrm{d}t} = (y - x)(2 + x)$

Short Answer
Solution

(15) $\frac{\mathrm{d}x}{\mathrm{d}t} = 1 + 2y, \qquad \frac{\mathrm{d}y}{\mathrm{d}t} = 1 - 3x^2$

Short Answer
Solution

(16) $\frac{\mathrm{d}x}{\mathrm{d}t} = x - xy, \qquad \frac{\mathrm{d}y}{\mathrm{d}t} = -y + \frac{y^2}{2}$

Short Answer
Solution

(17) $\frac{\mathrm{d}x}{\mathrm{d}t} = -x - x^2, \qquad \frac{\mathrm{d}y}{\mathrm{d}t} = y + 2xy$

Short Answer
Solution

(18) $\frac{\mathrm{d}x}{\mathrm{d}t} = x(-2 + \frac{x}{2} + \frac{x^2}{3}), \qquad \frac{\mathrm{d}y}{\mathrm{d}t} = y(2 - x - x^2)$

Short Answer
Solution

Draw the phase portraits for the following system of equations

(19) $\frac{\mathrm{d}x}{\mathrm{d}t} = y, \qquad \frac{\mathrm{d}y}{\mathrm{d}t} = 2x - x^3$

Solution

(20) $\frac{\mathrm{d}x}{\mathrm{d}t} = 2y, \qquad \frac{\mathrm{d}y}{\mathrm{d}t} = -3x$

Solution

(21) $\frac{\mathrm{d}x}{\mathrm{d}t} = -x + y + x^2, \qquad \frac{\mathrm{d}y}{\mathrm{d}t} = y - 2xy$

Solution

(22) $\frac{\mathrm{d}x}{\mathrm{d}t} = 2 - y, \qquad \frac{\mathrm{d}y}{\mathrm{d}t} = 4 - x^2$

Solution

(23) $\frac{\mathrm{d}x}{\mathrm{d}t} = y, \qquad \frac{\mathrm{d}y}{\mathrm{d}t} = -x + \frac{x^2}{4}$

Solution

(24) $x' = x^2 + y - 1, \qquad y' = 2xy$

Solution

(25) $\frac{\mathrm{d}x}{\mathrm{d}t} = y - 2, \qquad \frac{\mathrm{d}y}{\mathrm{d}t} = \sin(x)$

Solution

(26) Consider a body falling through the air with drag as described in the text. Recall that the equations of motion are

$$\frac{\mathrm{d}}{\mathrm{d}t}u = -ku\sqrt{u^2 + v^2}, \qquad \frac{\mathrm{d}}{\mathrm{d}t}v = -kv\sqrt{u^2 + v^2} - a.$$

where u and v are the horizonal and vertical components of the velocity $\mathbf{u}$, $a > 0$ is the accelleration due to gravity and $k = \rho_{\mathrm{air}} A/m$, where ρ_{air} is the density of air, A is the crosssectional area of the falling body, and m is the mass of the body.

Often times a good candidate for a Hamiltonian for a physical system is the total mechanical energy. In this case it is given by

$$E(t) = \frac{1}{2}m|\mathbf{u}(t)|^2 + ma\int_0^t v(s)\,\mathrm{d}s.$$

Show that the system is always losing energy, i.e.

$$\frac{\mathrm{d}}{\mathrm{d}t}E(t) = -\rho_{\text{air}}A|\mathbf{u}|^3 \leq 0.$$

What is the rate of energy loss at the stationary solution (i.e. at terminal velocity?).

Solution

(27) Suppose a particle of mass m is the moving in the presence of an attractive central potential $V(r) = -\dfrac{1}{\alpha r^\alpha}$, here α is a positive integer. It is well known from Physics that the distance r of the particle from the origin is govered by the equation

$$\frac{\mathrm{d}^2 r}{\mathrm{d}t^2} - \frac{L^2}{mr^3} + \frac{1}{mr^{\alpha+1}} = 0, \quad r > 0,$$

where L is the angular momentum.

(a) Write this second order differential equation as a first order planar system and show that it is Hamiltonian. Give its Hamiltonian H.
(b) Solve the differential equation for r in the case $\alpha = 2$, $r(0) = r_0 > 0$, and $r'(0) = 0$ by using the Hamiltonian to reduce the equations of motion for r to a first order seperable differential equation.
(c) In your solution for part (b) suppose the angular momentum $L > 1$. What is the behavior of r as $t \to \infty$.
(d) In your solution for part (b), suppose $L < 1$. After what time t_0 does the solution become undefined? What does $r(t)$ approach as $t \to t_0$. What happens to $p(t)$ as $t \to t_0$?
(e) In your solution for part (b), what happens when $L = 1$.

Solution

(28) Consider Newton's equations for a one-dimensional particle of mass m in a potential $V(x)$,

$$m\frac{\mathrm{d}^2}{\mathrm{d}t^2}x = -V'(x).$$

(a) Write this equation as a planar system. Show that this system is Hamiltonian and find its' Hamiltonian H.
(b) Using the Hamiltonian to reduce the system, show that the evolution of x can be given implicitly in terms of the formula,

$$t = \pm\int \frac{1}{\sqrt{c - \frac{2}{m}V(x)}}\,\mathrm{d}x.$$

Solution

(29) Consider a Hamiltonian of the form $H(x,y) = \frac{1}{2}y^2 + V(x)$, and the corresponding Hamiltonian system

$$x' = \partial_y H(x,y), \quad y' = -\partial_x H(x,y).$$

We will assume that $V(x)$ is a smooth function. Show the following

(a) Equilibrium points of are of the form $(x_0, 0)$, where x_0 is a critical point of $V(x)$.

(b) If x_0 is a (strict) local maximum of $V(x)$, then $(x_0, 0)$ is a saddle point for the system.

(c) If x_0 is a (strict) local minimum of $V(x)$, then $(x_0, 0)$ is a locally a center

Solution

(30) **(Liouville's Theorem)** Consider a Hamiltonian system $x' = \partial_y H(x,y)$, $y' = -\partial_x H(x,y)$ and define

$$\mathbf{\Phi}_t(x_0, y_0) = \begin{pmatrix} \varphi_t(x_0, y_0) \\ \psi_t(x_0, y_0) \end{pmatrix} := \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$$

to be the unique solution $(x(t), y(t))$ to the Hamiltonian system with initial conditions $(x(0), y(0)) = (x_0, y_0)$. Let A be a set of initial points in $\mathbb{R}^2$, then

$$\mathbf{\Phi}_t(A) = \{\mathbf{\Phi}(x,y) : (x,y) \text{ is in } A\}$$

is the evolution of that set of initial points under the dynamics of the system (think of a blob of ink moving in a fluid). Answer the following:

(a) Show that

$$\boldsymbol{\partial}\mathbf{\Phi}_t(x,y) = \begin{pmatrix} \partial_x \varphi_t(x,y) & \partial_y \varphi_t(x,y) \\ \partial_x \psi_t(x,y) & \partial_y \psi_t(x,y) \end{pmatrix}$$

satisfies

$$\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{\partial}\mathbf{\Phi}_t(x,y) = \begin{pmatrix} \partial_{yx} H(\mathbf{\Phi}_t(x,y)) & \partial_{yy} H(\mathbf{\Phi}_t(x,y)) \\ -\partial_{xx} H(\mathbf{\Phi}_t(x,y)) & -\partial_{xy} H(\mathbf{\Phi}_t(x,y)) \end{pmatrix} \boldsymbol{\partial}\mathbf{\Phi}_t(x,y).$$

(b) Use Liouville's Wronskian Theorem to conclude that

$$\frac{\mathrm{d}}{\mathrm{d}t} \det\left(\boldsymbol{\partial}\mathbf{\Phi}_t(x,y)\right) = 0.$$

(c) Finally, use the following multivariable calculus formula for the volume of a set

$$\mathrm{Vol}\left(\mathbf{\Phi}_t(A)\right) = \iint_A \det\left(\boldsymbol{\partial}\mathbf{\Phi}_t(x,y)\right)\, \mathrm{d}x\mathrm{d}y$$

to show that $\mathrm{Vol}\left(\mathbf{\Phi}_t(A)\right) = \mathrm{Vol}\left(A\right)$ (i.e. Hamiltonian evolution preserves the volume of sets of initial points).

Solution

Navigation to Other Chapters

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