III. First-Order Systems of Ordinary Differential Equations
9. Autonomous Planar Systems: Nonintegral Methods

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August 2, 2022

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For most autonomous planar systems the orbit equation (8.4) cannot be integrated. In this chapter we present techniques for understanding the phase portrait of such a system that do not require finding an integral. We consider first-order, autonomous, planar systems that have the form

$$x' = f(x,y)\,, \qquad y' = g(x,y)\,. \tag{9.1}$$

9.1. **Linearization about Stationary Solutions.** Recall from multivariable calculus that the Taylor approximations of $f(x,y)$ and $g(x,y)$ near a point (x_o,y_o) have the form

$$\begin{aligned} f(x,y) &= f(x_o,y_o) + (x-x_o)\,\partial_x f(x_o,y_o) + (y-y_o)\,\partial_y f(x_o,y_o) + \text{higher-order terms}\,,\\ g(x,y) &= g(x_o,y_o) + (x-x_o)\,\partial_x g(x_o,y_o) + (y-y_o)\,\partial_y g(x_o,y_o) + \text{higher-order terms}\,. \end{aligned}$$

If we drop the higher-order terms then these become linear approximations to $f(x,y)$ and $g(x,y)$ that are valid when (x,y) is near (x_o,y_o).

If (x_o,y_o) is a stationary solution of our system then $f(x_o,y_o) = g(x_o,y_o) = 0$ and these linear approximations become

$$\begin{aligned} f(x,y) &\approx (x-x_o)\,\partial_x f(x_o,y_o) + (y-y_o)\,\partial_y f(x_o,y_o)\,,\\ g(x,y) &\approx (x-x_o)\,\partial_x g(x_o,y_o) + (y-y_o)\,\partial_y g(x_o,y_o)\,. \end{aligned}$$

The idea of linearization is that when the solution $(x(t),y(t))$ of system (9.1) is near (x_o,y_o) then it can be approximated by the solution of the linear system obtained by replacing $f(x,y)$ and $g(x,y)$ with these linear approximations. More specifically, the idea is that $(x(t),y(t)) \approx (x_o+\tilde{x}(t), y_o+\tilde{y}(t))$ where $(\tilde{x}(t),\tilde{y}(t))$ satisfies the linear system

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix}\tilde{x}\\ \tilde{y}\end{pmatrix} = \begin{pmatrix}\partial_x f(x_o,y_o) & \partial_y f(x_o,y_o)\\ \partial_x g(x_o,y_o) & \partial_y g(x_o,y_o)\end{pmatrix}\begin{pmatrix}\tilde{x}\\ \tilde{y}\end{pmatrix}. \tag{9.2}$$

This linear system is called the *linearization of system* (9.1) *about* (x_o,y_o). Its coefficient matrix is $\mathbf{A} = \boldsymbol{\partial}\mathbf{f}(x_o,y_o)$ where $\boldsymbol{\partial}\mathbf{f}(x,y)$ is the matrix of partial derivatives given by

$$\boldsymbol{\partial}\mathbf{f}(x,y) = \begin{pmatrix}\partial_x f(x,y) & \partial_y f(x,y)\\ \partial_x g(x,y) & \partial_y g(x,y)\end{pmatrix}. \tag{9.3}$$

This matrix is often called the *Jacobian* matrix. Notice that the partial derivatives of f are the entries of its top row while those of g are the entries of its bottom row. The structure of $\boldsymbol{\partial}\mathbf{f}(x,y)$ can be recalled by first setting up the vector-valued function

$$\mathbf{f}(x,y) = \begin{pmatrix}f(x,y)\\ g(x,y)\end{pmatrix}.$$

Example. Compute the coefficient matrix of the linearization about each stationary point of the system

$$x' = y\,, \qquad y' = 4x - x^3\,.$$

Solution. We have seen that the stationary points of this system are

$$(-2,0)\,, \qquad (0,0)\,, \qquad (2,0)\,.$$

Because

$$\mathbf{f}(x,y) = \begin{pmatrix} f(x,y) \\ g(x,y) \end{pmatrix} = \begin{pmatrix} y \\ 4x - x^3 \end{pmatrix} ,$$

the Jacobian matrix for the system is

$$\boldsymbol{\partial}\mathbf{f}(x,y) = \begin{pmatrix} \partial_x f(x,y) & \partial_y f(x,y) \\ \partial_x g(x,y) & \partial_y g(x,y) \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 4 - 3x^2 & 0 \end{pmatrix} .$$

At $(0,0)$ the coefficient matrix of the linearized system is

$$\mathbf{A} = \boldsymbol{\partial}\mathbf{f}(0,0) = \begin{pmatrix} 0 & 1 \\ 4 & 0 \end{pmatrix} .$$

At $(-2,0)$ and $(2,0)$ the coefficient matrix of the linearized system is

$$\mathbf{A} = \boldsymbol{\partial}\mathbf{f}(\pm 2,0) = \begin{pmatrix} 0 & 1 \\ 4 - 12 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -8 & 0 \end{pmatrix} .$$

Example. Compute the coefficient matrix of the linearization about each stationary point of the system

$$x' = (y - x)(x - 1) , \qquad y' = (3 + 2x - x^2)y .$$

Solution. We have seen that the stationary points of this system are

$$(-1,-1) , \qquad (0,0) , \qquad (1,0) , \qquad (3,3) .$$

Because

$$\mathbf{f}(x,y) = \begin{pmatrix} f(x,y) \\ g(x,y) \end{pmatrix} = \begin{pmatrix} (y-x)(x-1) \\ (3+2x-x^2)y \end{pmatrix} = \begin{pmatrix} xy - x^2 - y + x \\ 3y + 2xy - x^2 y \end{pmatrix} ,$$

the Jacobian matrix for the system is

$$\boldsymbol{\partial}\mathbf{f}(x,y) = \begin{pmatrix} \partial_x f(x,y) & \partial_y f(x,y) \\ \partial_x g(x,y) & \partial_y g(x,y) \end{pmatrix} = \begin{pmatrix} y - 2x + 1 & x - 1 \\ 2y - 2xy & 3 + 2x - x^2 \end{pmatrix} .$$

At $(-1,-1)$ the coefficient matrix of the linearized system is

$$\mathbf{A} = \boldsymbol{\partial}\mathbf{f}(-1,-1) = \begin{pmatrix} -1+2+1 & -1-1 \\ -2-2 & 3-2-1 \end{pmatrix} = \begin{pmatrix} 2 & -2 \\ -4 & 0 \end{pmatrix} .$$

At $(0,0)$ the coefficient matrix of the linearized system is

$$\mathbf{A} = \boldsymbol{\partial}\mathbf{f}(0,0) = \begin{pmatrix} 1 & -1 \\ 0 & 3 \end{pmatrix} .$$

At $(1,0)$ the coefficient matrix of the linearized system is

$$\mathbf{A} = \boldsymbol{\partial}\mathbf{f}(1,0) = \begin{pmatrix} -2+1 & 1-1 \\ 0 & 3+2-1 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & 4 \end{pmatrix} .$$

At $(3,3)$ the coefficient matrix of the linearized system is

$$\mathbf{A} = \boldsymbol{\partial}\mathbf{f}(3,3) = \begin{pmatrix} 3-6+1 & 3-1 \\ 6-18 & 3+6-9 \end{pmatrix} = \begin{pmatrix} -2 & 2 \\ -12 & 0 \end{pmatrix} .$$

9.2. **Phase-Plane Portraits Near Stationary Points.** It is natural to ask when the phase-plane portrait of system (9.1) near the stationary point (x_o, y_o) looks like the phase-plane portrait of the linearized system with coefficient matrix $\mathbf{A} = \boldsymbol{\partial}\mathbf{f}(x_o, y_o)$. This is the case in some instances, but not in all. The following facts hold whenever $\boldsymbol{\partial}\mathbf{f}(x, y)$ is differentiable at (x_o, y_o).

- If the origin is a *nodal sink*, *nodal source*, or *saddle* for the linearized system then the phase-plane portrait of the nonlinear system (9.1) will look similar near the stationary point (x_o, y_o). In particular, there will be orbits tangent to the lines passing through (x_o, y_o) in the directions of the eigenvectors of the linearized system. When an eigenvector of the linearized system has the form
$$\mathbf{v} = \begin{pmatrix} u_o \\ v_o \end{pmatrix} \quad \text{with } u_o \neq 0\,,$$
then the line passing through (x_o, y_o) in this direction is $y = y_o + \dfrac{v_o}{u_o}(x - x_o)$.
- If the origin is a *radial sink* or *radial source* for the linearized system then the phase-plane portrait of the nonlinear system (9.1) will look similar near the stationary point (x_o, y_o). In particular, every half line radiating from (x_o, y_o) will have exactly one orbit tangent to it.
- If the origin is a *twist sink* or *twist source* for the linearized system then the phase-plane portrait of the nonlinear system (9.1) will look similar near the stationary point (x_o, y_o). In particular, there will be orbits tangent to the one line passing through (x_o, y_o) in the direction of an eigenvector of the linearized system. They will twist in the same *clockwise* or *counterclockwise* sense.
- If the origin is a *spiral sink* or *spiral source* for the linearized system then the phase-plane portrait of the nonlinear system (9.1) will look similar near the stationary point (x_o, y_o). In particular, the orbits will spiral in the same *clockwise* or *counterclockwise* sense.
- If system (9.1) is *conservative* and the origin is a *center* for the linearized system then the phase-plane portrait of the nonlinear system (9.1) will look like a center near the stationary point (x_o, y_o). In particular, the orbits will loop around it with the same *clockwise* or *counterclockwise* sense.

Remark. The last fact above requires us to know that system (9.1) is conservative, but does not require us to compute and integral $H(x, y)$. In general, it is hard to check that a system is conservative. However, there are two cases that are easy to check. First, if system (9.1) is *separable* then it is conservative. Second, if system (9.1) satisfies $\partial_x f + \partial_y g = 0$ then it is *Hamiltonian* and thereby is conservative. Fortunately, these cases often arise in applications.

Remark. The only cases that do not appear on the above list are those for which 0 is an eigenvalue of $\mathbf{A}$ — i.e. those for which $\det(\mathbf{A}) = 0$. These are the cases where the origin is not the sole stationary point for the linearized system.

Remark. The foregoing list is longer than similar lists found in many textbooks, which often omit the radial, twist, or center cases. Some textbooks assert that we cannot conclude anything about the phase-plane portrait of the nonlinear system (9.1) near the stationary point (x_o, y_o) when the phase-plane portrait for the associated linearized system is a radial or twist case, *but this assertion is wrong.* The facts given in our list reflect a long line of work initiated by Henri Poincare in 1880 and completed by Philip Hartman in 1960. Their proofs lie beyond the scope of this course.

Remark. Analogous statements in higher dimensions were proved by Misha Guysinsky, Boris Hasselblatt, and Victoria Rayskin in 2003. Their results are an extension of Hartman-Grobman theory, which was developed around 1960.

Remark. In the last chapter we saw that if a nonlinear system has an integral $H(x, y)$ whose critical points are nondegenerate over a region of the xy-plane then all stationary points of the system in that region must be either saddles or centers. There are two ways in which this fact can be used in this chapter.

- If linearization shows that a stationary point is not a saddle or center then we can conclude that the system does not have an integral $H(x, y)$ over a region containing that point. In this case the nonlinear system is not conservative.
- If the linearization at every stationary point is either a saddle or a center then it makes sense to check if the system is conservative. Specifically, it makes sense to check if it is Hamiltonian, has a separable orbit equation, or has an integral.

Remark. Because this technique is easy to apply, it is often the first approach we will take. However, it only gives information about the phase-plane portrait *near stationary points.* In order to build a better understanding of the complete phase-plane portrait, it needs to be augmented by other techniques. We will illustrate how to do this after introducing some related notions of dynamical stability.

9.3. **Dynamical Stablity of Stationary Points.**

In Section 6.3 we introduced notions of dynamical stability of the origin for linear systems. Here we extend those notions to stationary points of the nonlinear system (9.1). Determining the *dynamical stability of a stationary point* addresses the question of whether as time increaces solutions near the stationary point either approach it, move away from it, or stay near it without approaching it.

Two basic notions of dynamical stability are the following.

> **Definition 9.1.** A stationary point is called *stable* if every solution that starts sufficiently near it will stay arbitrarily close to it. It is called *unstable* if it is not stable.

The language "every solution that starts sufficiently near it will stay arbitrarily close to it" is not very precise. Rather than formulate a more precise mathematical definition, we will build our understanding of these notions through examples. To begin with, it should be clear that for every system any stationary point is either stable or unstable. Roughly speaking, a stationary point will be unstable if at least one solution that starts near it will move away from it.

Two additional notions of dynamical stability are the following.

Definition 9.2. A stationary point is called *attracting* if every solution that starts near it will approach it as $t \to \infty$. It is called *repelling* if every solution that starts near it but not at it will move away from it.

It should be clear that if a stationary point is attracting then it is stable, and that if it is repelling then it is unstable. As we know from our study of linear systems in Section 6.3, these implications do not go the other way.

Remark. Definitions 9.1 and 9.2 extend to any stationary point of a nonlinear system the notions of dynamical stability that had been introduced by Definitions 7.1 and 7.2 respectively for the origin and linear systems.

Remark. The term *asymptotically stable* is often used for *attracting*, but there is no analogous term for *repelling*.

It is natural to ask when the stability of a stationary point (x_o, y_o) of system (9.1) can be determined from the stability of the origin for the linearized system with coefficient matrix $\mathbf{A} = \boldsymbol{\partial}\mathbf{f}(x_o, y_o)$. This is the case in some instances, but not in all.

- If the origin is *attracting* or *repelling* for the linearized system then the stationary solution (x_o, y_o) of the nonlinear system (9.1) has the same property.
- If system (9.1) is *conservative* and the origin is a *center* for the linearized system then the stationary solution (x_o, y_o) of the nonlinear system (9.1) is stable.
- If the coefficient matrix $\mathbf{A}$ of the linearized system has an eigenvalue with a positive real part then the stationary solution (x_o, y_o) of the nonlinear system (9.1) is unstable.

Remark. In the first two cases we know that the phase-plane portrait of system (9.1) near the stationary point (x_o, y_o) looks like the phase-plane portrait of the linearized system, from which we can read off the stability. However, in the third case we can determine that the stationary point is unstable even when the phase-plane portrait of the linearized system is a linear source, which does not tell us what the phase-plane portrait of system (9.1) looks like near the stationary point (x_o, y_o).

9.4. **Phase-Plane Portraits.** The starting point for understanding the phase-plane portrait of any system (9.1) is to find its stationary points and to analyze the linearized system associated with each of them. By combining this with techniques from Chapter [??] and from later in this chapter we now have the following general strategy for understanding phase-plane portraits.

(1) **Stationary Points.** Find all the stationary points. These are all points (a, b) such that

$$f(a, b) = 0\,, \qquad g(a, b) = 0\,.$$

At each stationary point (a, b) compute the Jacobian matrix $\mathbf{A} = \boldsymbol{\partial}\mathbf{f}(a, b)$.

- If $\det(\mathbf{A}) = 0$ then there is no further information.
- If $\det(\mathbf{A}) \neq 0$ and the phase-plane portarit of the associated linearized system is either a *node, saddle, radial, twist,* or *spiral* then the phase-plane portrait of system (9.1) looks similar near the stationary point (x_o, y_o).

- If $\det(\mathbf{A}) \neq 0$ and the phase-plane portarit of the associated linearized system is a *center* then the phase-plane portrait of system (9.1) looks similar near the stationary point (x_o, y_o) provided *system (9.1) is conservative.*

(2) **Semistationary Orbits.** Find all the semistationary orbits. These lie on any vertical line $x = a$ such that

$$f(a, y) = 0 \quad \text{for every } y\,,$$

or on any horizontal line $y = b$ such that

$$g(x, b) = 0 \quad \text{for every } x\,.$$

The direction of the arrows on any such vertical line $x = a$ is determined by the phase-line portrait of the equation

$$y' = g(a, y)\,.$$

The direction of the arrows on and such horizontal line $y = b$ is determined by the phase-line portrait of the equation

$$x' = f(x, b)\,.$$

Nearby orbits will have arrows in the same direction.

(3) **Level Sets of an Integral.** If possible, find an integral of system (9.1). These are functions $H(x, y)$ such that $H(x, y) = c$ is an implicit general solution of the *orbit differential form*

$$g(x, y)\,\mathrm{d}x - f(x, y)\,\mathrm{d}y = 0\,.$$

This can be done by the methods of Part I when this differential form is separable, exact, or has an integrating factor. (See Chapter [??] for more details.) This can be done only over regions of the phase-plane within which every stationary point is either a *saddle* or a *center*.

The critical points of the integral $H(x, y)$ will be the stationary points of system (9.1) that lie within the region over which $H(x, y)$ is defined. Each of these stationary points will be either a *saddle* or a *center*. The critical values of $H(x, y)$ are its values at these critical points.

- Sketch level sets $H(x, y) = c$ every c that is a critical value corresponding to a saddle. These will contain all the *separatrices*.
- Sketch level sets $H(x, y) = c$ for at least one value of c that lies between adjacent critical values of $H(x, y)$.

When $H(x, y) = c$ is a simple enough then the associated level set can be sketched by hand. Otherwise it can be plotted by computer, — for example, by using the Matlab command “contour”. The direction of the arrows along these orbits can often be inferred from nearby semistationary orbits or the orientation of orbits near stationary points.

(4) **Numerically Approximate Orbits.** When an integral of the orbit differential form cannot be found then we can plot numerical solutions of system (9.1) — for example, generated using the Runge-Kutta method or the Matlab command “ode45”.

Now for some examples.

Example. Determine the stability and classify each stationary point of the system

$$x' = y\,, \qquad y' = 4x - x^3\,.$$

Sketch the phase-plane portrait of the system near each stationary point.

Solution. We have seen that its stationary points are

$$(-2,0)\,, \qquad (0,0)\,, \qquad (2,0)\,.$$

At $(0,0)$ the coefficient matrix of the linearized system was shown to be

$$\mathbf{A} = \begin{pmatrix} 0 & 1 \\ 4 & 0 \end{pmatrix}.$$

Its characteristic polynomial is $p(z) = z^2 - 4 = (z+2)(z-2)$. Its eigenvalues thereby are -2 and 2. Hence, the stationary point $(0,0)$ is a *saddle* and is *unstable.* Because

$$\mathbf{A} + 2\mathbf{I} = \begin{pmatrix} 2 & 1 \\ 4 & 2 \end{pmatrix}, \qquad \mathbf{A} - 2\mathbf{I} = \begin{pmatrix} -2 & 1 \\ 4 & -2 \end{pmatrix},$$

we see that $\mathbf{A}$ has the eigenpairs

$$\left(-2\,, \begin{pmatrix} 1 \\ -2 \end{pmatrix}\right), \qquad \left(2\,, \begin{pmatrix} 1 \\ 2 \end{pmatrix}\right).$$

There is one orbit that approaches $(0,0)$ tangent to each half of the line $y = -2x$. There is also one orbit that emerges from $(0,0)$ tangent to each half of the line $y = 2x$.

At $(-2,0)$ and $(2,0)$ the coefficient matrix of the linearized system was shown to be

$$\mathbf{A} = \begin{pmatrix} 0 & 1 \\ -8 & 0 \end{pmatrix}.$$

Its characteristic polynomial is $p(z) = z^2 + 8$. Its eigenvalues thereby are the conjugate pair $\pm i\sqrt{8}$ while $a_{21} = -8 < 0$. Therefore the origin is a clockwise center for the linearized system. However, we cannot conclude that the stationary points $(-2,0)$ and $(2,0)$ are clockwise centers in the phase-plane portrait of the nonlinear system without additional information.

There are at least two ways to see that the nonlinear system is conservative without computing an integral $H(x,y)$. For example, because $\partial_x f(x,y) + \partial_y g(x,y) = 0$, the system is Hamiltonian, and thereby is conservative. Alternatively, the orbit equation is

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{4x - x^3}{y}\,,$$

which is separable, whereby the system is conservative. Because the nonlinear system is conservative, we can now conclude that the stationary points $(-2,0)$ and $(2,0)$ are *clockwise centers* and are *stable.*

The sketch resulting from the foregoing analysis is shown in Figure 9.1 below. □

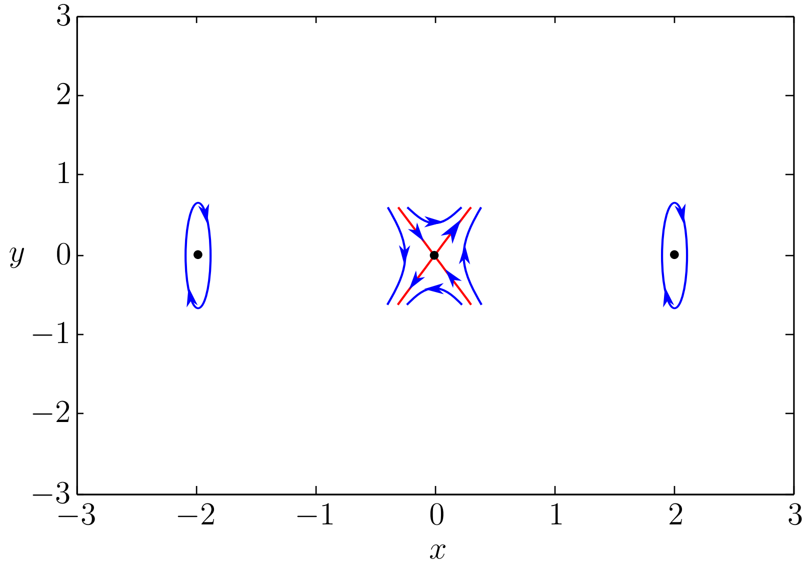

FIGURE 9.1. Sketch of the phase-plane portrait for Example 1 near each stationary point as determined by linearization.

Remark. Figure 9.1 should be compared with the fairly complete phase-plane portrait for Example 1 shown in Figure 9.2 below, which was obtained in Chapter 8 by plotting several level sets of the integral $H(x,y) = \frac{1}{2}y^2 - 2x^2 + \frac{1}{4}x^4$.

Remark. All that needs to be added to Figure 9.1 to suggest the details in Figure 9.2 are the separatrices shown in red in Figure 9.2. These lie on the level set associated with $(0,0)$, which is given implicitly by

$$\tfrac{1}{2}y^2 - 2x^2 + \tfrac{1}{4}x^4 = H(x,y) = H(0,0) = 0\,.$$

Upon solving this for y we find that

$$y = \pm\sqrt{4x^2 - \tfrac{1}{2}x^4} \qquad \text{for } -\sqrt{8} \leq x \leq \sqrt{8}\,.$$

By using the methods of first-semester calculus to add a sketch of this one level set to Figure 9.1, we can gain a better understanding of the complete phase-plane portrait.

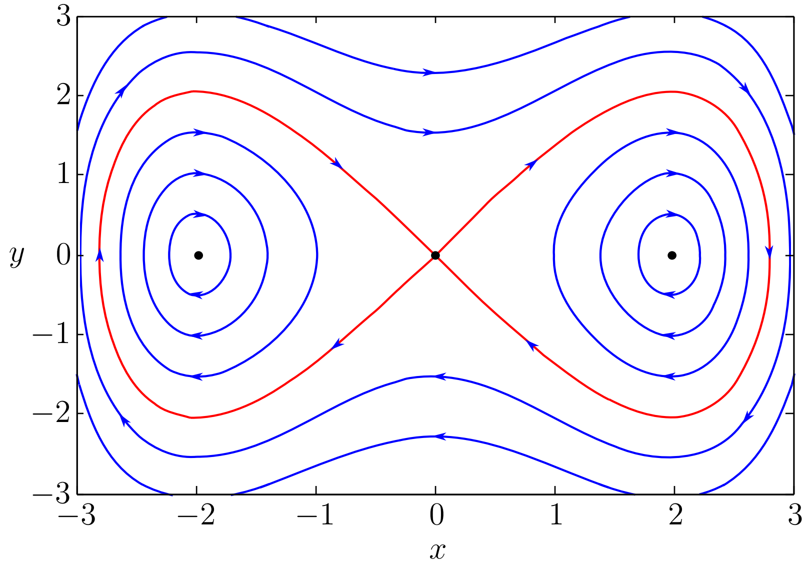

FIGURE 9.2. Sketch of a phase-plane portrait for Example 1 determined by level sets. Notice that this gives a more complete picture of the phase-plane portrait.

Example. Determine the stability and classify each stationary point of the system

$$x' = (y - x)(x - 1)\,, \qquad y' = (3 + 2x - x^2)y\,.$$

Sketch the phase-plane portrait of the system near each stationary point.

Solution. We have seen that the stationary points of this system are

$$(-1, -1)\,, \qquad (0, 0)\,, \qquad (1, 0)\,, \qquad (3, 3)\,.$$

At $(-1, -1)$ the coefficient matrix of the linearized system was shown to be

$$\mathbf{A} = \begin{pmatrix} 2 & -2 \\ -4 & 0 \end{pmatrix}\,.$$

Its characteristic polynomial is $p(z) = z^2 - 2z - 8 = (z + 2)(z - 4)$. Its eigenvalues thereby are -2 and 4. Hence, the stationary point $(-1, -1)$ is a *saddle* and is *unstable*. Because

$$\mathbf{A} + 2\mathbf{I} = \begin{pmatrix} 4 & -2 \\ -4 & 2 \end{pmatrix}\,, \qquad \mathbf{A} - 4\mathbf{I} = \begin{pmatrix} -2 & -2 \\ -4 & -4 \end{pmatrix}\,,$$

we see that $\mathbf{A}$ has the eigenpairs

$$\left(-2\,, \begin{pmatrix} 1 \\ 2 \end{pmatrix}\right)\,, \qquad \left(4\,, \begin{pmatrix} 1 \\ -1 \end{pmatrix}\right)\,.$$

There is one orbit that approaches $(-1, -1)$ tangent to each half of the line $y = 2x + 1$. There is also one orbit that emerges from $(-1, -1)$ tangent to each half of the line $y = -x - 2$.

At $(0,0)$ the coefficient matrix of the linearized system was shown to be

$$\mathbf{A} = \begin{pmatrix} 1 & -1 \\ 0 & 3 \end{pmatrix} .$$

Because $\mathbf{A}$ is upper triangular, we can read off that its eigenvalues are 1 and 3. Hence, the stationary point $(0,0)$ is a *nodal source* and is *repelling*. Because

$$\mathbf{A} - \mathbf{I} = \begin{pmatrix} 0 & -1 \\ 0 & 2 \end{pmatrix} , \qquad \mathbf{A} - 3\mathbf{I} = \begin{pmatrix} -2 & -1 \\ 0 & 0 \end{pmatrix} ,$$

we see that $\mathbf{A}$ has the eigenpairs

$$\left(1 , \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right) , \qquad \left(3 , \begin{pmatrix} 1 \\ -2 \end{pmatrix} \right) .$$

There is one orbit that emerges from $(0,0)$ tangent to each half of the line $y = -2x$. All other orbits emerge from $(0,0)$ tangent to the x-axis.

At $(1,0)$ the coefficient matrix of the linearized system was shown to be

$$\mathbf{A} = \begin{pmatrix} -1 & 0 \\ 0 & 4 \end{pmatrix} .$$

Because $\mathbf{A}$ is diagonal, we can read off that its eigenpairs are

$$\left(-1 , \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right) , \qquad \left(4 , \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right) .$$

Hence, the stationary point $(1,0)$ is a *saddle* and is *unstable*. There is one orbit that approaches $(1,0)$ tangent to each half of the x-axis. There is also one orbit that emerges from $(1,0)$ tangent to each half of the line $x = 1$.

At $(3,3)$ the coefficient matrix of the linearized system was shown to be

$$\mathbf{A} = \begin{pmatrix} -2 & 2 \\ -12 & 0 \end{pmatrix} .$$

Its characteristic polynomial is $p(z) = z^2 + 2z + 24 = (z+1)^2 + 23$. Its eigenvalues thereby are the conjugate pair $-1 \pm i\sqrt{23}$ while $a_{21} = -12 < 0$. Hence, the stationary point $(3,3)$ is a *clockwise spiral sink* and is *attracting*.

The sketch resulting from the foregoing analysis is shown in Figure 9.3 below. $\square$

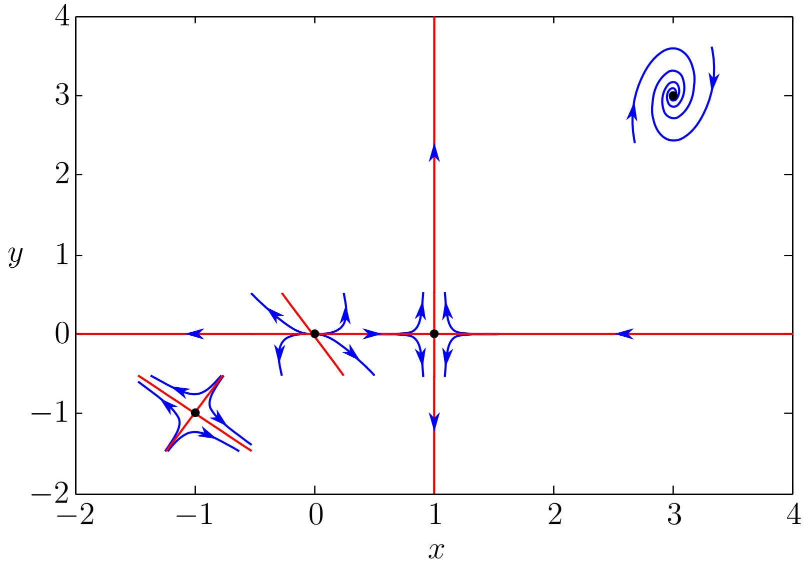

Figure 9.3. Sketch of the phase-plane portrait for Example 2 showing the semistationary solutions and the result of the linearized analysis near each stationary point.

Remark. Figure 9.3 should be compared with the phase-plane portrait for Example 2 shown below, which was obtained by numerical integration using the Runge-Kutta method.

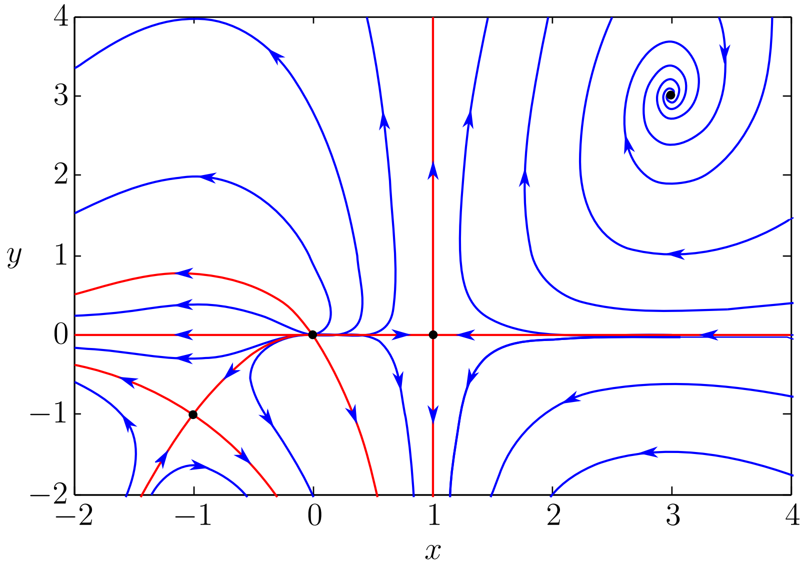

Figure 9.4. Sketch of a phase-plane portrait for Example 2 determined by numerical integration. Notice that this gives a more complete picture of the phase-plane portrait.

Remark. Because linearization shows that the stationary points $(0,0)$ and $(3,3)$ are not saddles or centers, we know the nonlinear system is not conservative over the entire phase-plane. The next example shows that care must be taken when drawing such conclusions.

Example. Determine the stability and classify each stationary point of the system

$$u' = -ku\sqrt{u^2+v^2}\,, \qquad v' = -kv\sqrt{u^2+v^2} - a\,.$$

Sketch the phase-plane portrait of the system near each stationary point.

Solution. We have seen that the only stationary point of this system is

$$\Big(0\,,\,-\sqrt{a/k}\Big)\,.$$

Because

$$\mathbf{f}(u,v) = \begin{pmatrix} -ku\sqrt{u^2+v^2} \\ -kv\sqrt{u^2+v^2} - a \end{pmatrix}\,,$$

the Jacobian matrix for this system is

$$\boldsymbol{\partial}\mathbf{f}(u,v) = \begin{pmatrix} -k\sqrt{u^2+v^2} - k\dfrac{u^2}{\sqrt{u^2+v^2}} & -k\dfrac{uv}{\sqrt{u^2+v^2}} \\ -k\dfrac{vu}{\sqrt{u^2+v^2}} & -k\sqrt{u^2+v^2} - k\dfrac{v^2}{\sqrt{u^2+v^2}} \end{pmatrix}\,.$$

The coefficient matrix of the linearization at $(0, -\sqrt{a/k})$ is

$$\mathbf{A} = \boldsymbol{\partial}\mathbf{f}\Big(0\,,\,-\sqrt{a/k}\Big) = \begin{pmatrix} -\sqrt{ka} & 0 \\ 0 & -2\sqrt{ka} \end{pmatrix}\,.$$

Because $\mathbf{A}$ is diagonal, we can read off that its eigenpairs are

$$\left(-\sqrt{ka}\,,\,\begin{pmatrix}1\\0\end{pmatrix}\right)\,, \qquad \left(-2\sqrt{ka}\,,\,\begin{pmatrix}0\\1\end{pmatrix}\right)\,.$$

Hence, the stationary point is a *nodal sink* and is *attracting.* There is one orbit that approaches $(0, -\sqrt{a/k})$ along each half of the v-axis. All other orbits approach $(0, -\sqrt{a/k})$ tangent to the line $v = -\sqrt{a/k}$.

The sketch resulting from the foregoing analysis is shown in Figure 9.5 below. □

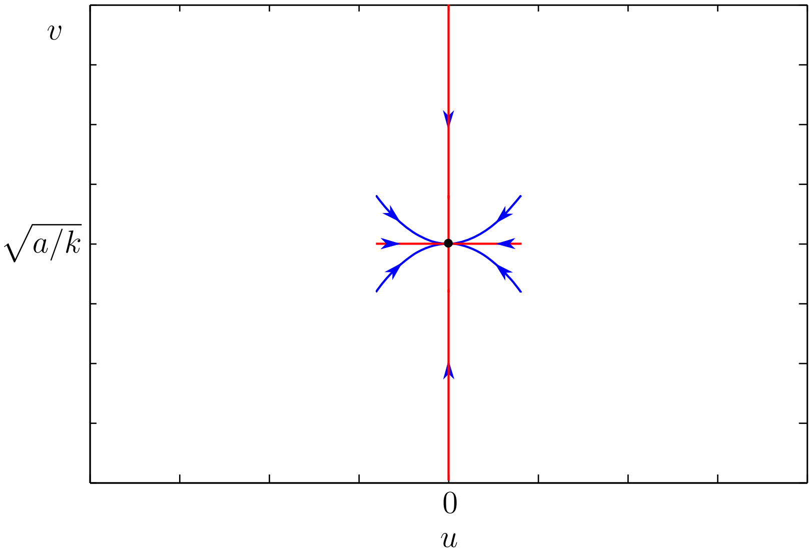

Figure 9.5. Sketch of the phase-plane portrait for Example 3 showing the semistationary solutions and the result of the linearized analysis near the stationary point.

Remark. Figure 9.5 should be compared with the phase-plane portait for Example 3 shown below, which was obtained in the previous chapter by plotting some level sets of the integral $H(u,v)$.

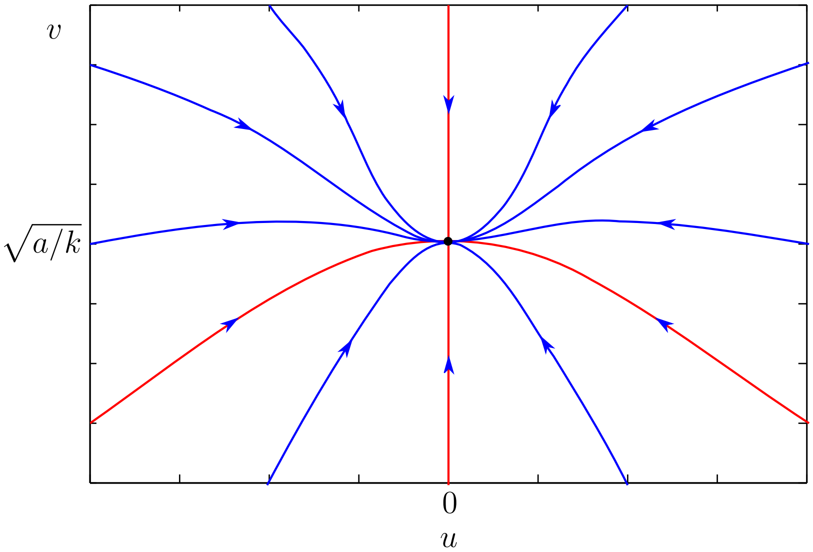

Figure 9.6. Sketch of a phase-plane portrait for Example 3 determined by level sets. Notice that this gives a more complete picture of the phase-plane portrait.

Remark. The linearization shows that the stationary point $(0, -\sqrt{a/k})$ is not a saddle or a center, yet we saw in the last chapter that this system has an integral $H(u, v)$. However, the integral $H(u, v)$ is not defined on the v-axis where $u = 0$, so it only shows that the system is conservative off the v-axis. Hence, the stationary point on the v-axis might not be a saddle or center. Indeed, the foregoing example shows it is a nodal sink.

9.5. **Vector Fields and Direction Fields.** Vector fields and direction fields are other tools that can be used to indicate what orbits look like in the regions in the xy-plane that are separated by the orbits of the special solutions. These are the crudest tool in our toolbox. Their virtues are that they can be applied to almost any first-order autonomous planar system in the form (9.1) and that they are easy to apply using a computer with appropriate software. We assume that the functions $f(x, y)$ and $g(x, y)$ are defined over a set S in the xy-plane such that

- f and g are continuous over S,
- f and g are differentiable with respect to x and y over S,
- $\partial_x f$, $\partial_y f$, $\partial_x g$, and $\partial_y g$ are continuous over S.

Moreover, we assume that every point in a rectangle $[x_L, x_R] \times [y_L, y_R]$ is in the interior of S.

By Theorem 1.1 every point (x_I, y_I) in $[x_L, x_R]\times[y_L, y_R]$ has a unique curve $(X(t), Y(t))$ defined over a time interval $[a, b]$ such that $(X(t), Y(t))$ remains within the rectangle $[x_L, x_R] \times [y_L, y_R]$ and $(x, y) = (X(t), Y(t))$ solves the system (9.1) with initial condition $(X(0), Y(0)) = (x_I, y_I)$. This curve can be extended to the largest time interval $[a, b]$ such that $(X(t), Y(t))$ remains within the rectangle $[x_L, x_R] \times [y_L, y_R]$. It has a tangent vector given by

$$\frac{\mathrm{d}}{\mathrm{d}t}\big(X(t)\,,\,Y(t)\big) = \big(X'(t)\,,\,Y'(t)\big) = \big(f(X(t), Y(t))\,,\,g(X(t), Y(t))\big)\,.$$

In other words, the unique solution that goes through any point (x, y) in the rectangle $[x_L, x_R] \times [y_L, y_R]$ has the tangent vector $(f(x, y), g(x, y))$.

9.5.1. *Vector Fields.* A *vector field* for equation (9.1) over the rectangle $[x_L, x_R]\times[y_L, y_R]$ is a plot that shows the tangent vector $(f(x, y), g(x, y))$ with an arrow at each point of a grid in the rectangle $[x_L, x_R] \times [y_L, y_R]$. The idea is that these arrows might give us a correct picture of how the orbits move inside the rectangle.

We can produce such a vector field by using the MATLAB commands **meshgrid** and **quiver** as follows.

```
>> [X, Y] = meshgrid(x_L:h:x_R,y_L:k:y_R);
>> F = f(X, Y); G = g(X, Y);
>> quiver(X, Y, F, G, ℓ)
>> axis tight, xlabel ’x’, ylabel ’y’
>> title ’Vector Field for dx/dt = f(x, y), dy/dt = g(x, y)’
```

Here h and k are the grid spacings for the intervals $[x_L, x_R]$ and $[y_L, y_R]$ respectively, which should have values of the form

$$h = \frac{x_R - x_L}{m}\,, \qquad k = \frac{y_R - y_L}{n}\,, \qquad \text{where } m \text{ and } n \text{ are positive integers.}$$

The meshgrid command creates an array of *grid points* in the rectangle $[x_L, x_R] \times [y_L, y_R]$ given by (x_i, y_j) where

$$x_i = x_L + ih \text{ for } i = 0,\, 1,\, \cdots\, m\,, \qquad y_j = y_L + jk \text{ for } j = 0,\, 1,\, \cdots\, n\,.$$

More precisely, meshgrid creates two arrays; the array X contains x_i in its ij^{th}-entry while the array Y contains y_j in its ij^{th}-entry. Next, the arrays F and G are computed that contain $f(x_i, y_j)$ and $g(x_i, y_j)$ respectively in their ij^{th}-entry.

Finally, the quiver command plots an array of arrows with the ij^{th}-arrow being the vector $(f(x_i, y_j), g(x_i, y_j))$ scaled by ℓ and centered at the grid point (x_i, y_j). The problem with this approach is that the length of these vectors can vary greatly over the rectangle $[x_L, x_R] \times [y_L, y_R]$. In such cases either the smallest vectors might not be visible on the graph or the largest vectors might not fit on the graph.

9.5.2. *Direction Fields.* A *direction field* for equation (9.1) over the rectangle $[x_L, x_R] \times [y_L, y_R]$ is a plot that shows the direction of the tangent vector $(f(x, y), g(x, y))$ with an arrow of fixed length at each point of a grid in the rectangle $[x_L, x_R] \times [y_L, y_R]$. The idea is that these arrows might give us a good picture of how the orbits move inside the rectangle. Because these vectors all have the same length, all of them should be visible on the graph when this length is well chosen.

We can produce such a direction field by using the MATLAB commands **meshgrid** and **quiver** as follows.

```
>> [X, Y] = meshgrid(x_L:h:x_R,y_L:k:y_R);
>> F = f(X, Y); G = g(X, Y);
>> L = sqrt(F.^2 + G.^2);
>> quiver(X, Y, F./L, G./L, ℓ)
>> axis tight, xlabel ’x’, ylabel ’y’
>> title ’Direction Field for dx/dt = f(x, y), dy/dt = g(x, y)’
```

Here the meshgrid command generates the arrays X and Y as before, and then an additional array L is computed that contains the length of the tangent vector $(f(x_i, y_j), g(x_i, y_j))$ in its ij^{th}-entry. The quiver command plots an array of arrows of length ℓ so that the ij^{th}-arrow is centered at the grid point (x_i, y_j) and is pointing in the direction of the unit tangent vector

$$\left(\frac{f(x_i, y_j)}{\sqrt{f(x_i, y_j)^2 + g(x_i, y_j)^2}}\,, \frac{g(x_i, y_j)}{\sqrt{f(x_i, y_j)^2 + g(x_i, y_j)^2}} \right).$$

The length ℓ should be smaller than h or k so that the plotted arrows will not overlap. Typically m and n will be about 20 to insure there will be enough arrows to give a complete picture of the direction field, but not so many that the plot becomes cluttered.

Remark. Often it is hard to figure out how the orbits move solely from the arrows of a direction field. Directions fields are most effective when used in conjunction with plots of special solutions and some representative solutions.

9.6. **Nullclines and Isoclines.** Nullclines and isoclines are other tools that can be used to indicate what orbits look like in the regions in the xy-plane that are separated by the orbits of the special solutions. They do so by giving us a rough idea of what the direction field looks like. They can be applied to first-order autonomous planar system in the form (9.1) when $f(x, y)$ and $g(x, y)$ have fairly simple analytic forms.

9.6.1. *Nullclines.* The *x-nullcline* of system (9.1) is the set of points in the xy-plane that satisfy

$$f(x, y) = 0\,. \tag{9.4a}$$

The *y-nullcline* of system (9.1) is the set of points in the xy-plane that satisfy

$$g(x, y) = 0\,. \tag{9.4b}$$

A nullcline usually is comprised of one or more curves in the xy-plane. These curves can be easy to determine when $f(x, y)$ and $g(x, y)$ have simple enough forms. If so then they can be plotted in the xy-plane.

Remark. It should be clear from (9.4) that the points where the x-nullcline and y-nullcline intersect are exactly the stationary points of system (9.1).

Remark. Semistationary solutions of the form $(a, Y(t))$ lie on the x-nullcline. Conversely, vertical lines in the x-nullcline contain semistationary solutions of the form $(a, Y(t))$.

Remark. Semistationary solutions of the form $(X(t), b)$ lie on the y-nullcline. Conversely, horizontal lines in the y-nullcline contain semistationary solutions of the form $(X(t), b)$.

The x-nullcline partitions the xy-plane into regions where $f(x, y) < 0$ and regions where $f(x, y) > 0$. Similarly, the y-nullcline partitions the xy-plane into regions where $g(x, y) < 0$ and regions where $g(x, y) > 0$. We can then indicate the direction field along each nullcline as follows.

- Segments of the x-nullcline where $g(x, y) < 0$ should be marked with a $\downarrow$.
- Segments of the x-nullcline where $g(x, y) > 0$ should be marked with a $\uparrow$.
- Segments of the y-nullcline where $f(x, y) < 0$ should be marked with a $\leftarrow$.
- Segments of the y-nullcline where $f(x, y) > 0$ should be marked with a $\rightarrow$.

The arrows can only change direction at points where the x-nullcline and y-nullcline intersect — namely, at stationary points.

Remark. This is essentially the same way by which we determine the directions of semistationary solutions along their orbits. Indeed, that is exactly what we are doing when this method is applied to vertical lines in the x-nullcline or to horizontal lines in the y-nullcline.

Finally, we can give a rough indication of the direction field off the nullclines as follows.

- Regions of the xy-plane in which $f(x,y) < 0$ and $g(x,y) < 0$ should be labelled with a $(-,-)$.
- Regions of the xy-plane in which $f(x,y) < 0$ and $g(x,y) > 0$ should be labelled with a $(-,+)$.
- Regions of the xy-plane in which $f(x,y) < 0$ and $g(x,y) < 0$ should be labelled with a $(+,-)$.
- Regions of the xy-plane in which $f(x,y) > 0$ and $g(x,y) > 0$ should be labelled with a $(+,+)$.

When this information from the nullclines is combined with information obtained from stationary solutions, semistationary solutions, and linearization, we are often (but not always) able to get a fairly clear picture of the phase-portrait of system (9.1).

Example. Determine the nullclines for the system

$$x' = y \,, \qquad y' = 4x - x^3 \,.$$

Sketch the phase-plane portrait of the system that combines the information from its stationary points, the linearization about each stationary point, and its nullclines.

Solution. Recall that the stationary points of this system are

$$(-2,0)\,, \qquad (0,0)\,, \qquad (2,0)\,.$$

In an earlier example we used linearization to show the following.

- The stationary point $(0,0)$ is a *saddle* with associated eigenpairs

  $$\left(-2\,, \begin{pmatrix} 1 \\ -2 \end{pmatrix}\right)\,, \qquad \left(2\,, \begin{pmatrix} 1 \\ 2 \end{pmatrix}\right)\,.$$

  There is one orbit that approaches $(0,0)$ tangent to each half of the line $y = -2x$. There is also one orbit that emerges from $(0,0)$ tangent to each half of the line $y = 2x$.
- The stationary points $(-2,0)$ and $(2,0)$ are clockwise centers.

Points on the x-nullcline satisfy

$$0 = y \,.$$

Therefore the x-nullcline consists of the line $y = 0$, which is the x-axis. Along this line we have $g(x,0) = x(2+x)(2-x)$, so the arrows should be $\uparrow$ for $x < -2$, $\downarrow$ for $-2 < x < 0$, $\uparrow$ for $0 < x < 2$, and $\downarrow$ for $2 < x$.

Points on the y-nullcline satisfy

$$0 = 4x - x^3 = x(2+x)(2-x)\,.$$

Therefore the y-nullcline consists of the line $x = -2$, the line $x = 0$, and the line $x = 2$.

- Along the line $x = -2$ we have $f(-2,y) = y$, so the arrows should be $\leftarrow$ for $y < 0$ and $\rightarrow$ for $y > 0$.
- Along the line $x = 0$ we have $f(0,y) = y$, so the arrows should be $\leftarrow$ for $y < 0$ and $\rightarrow$ for $y > 0$.
- Along the line $x = 2$ we have $f(2,y) = y$, so the arrows should be $\leftarrow$ for $y < 0$ and $\rightarrow$ for $y > 0$.

Finally, we plot the stationary points, the local phase-plane portrait near each stationary point, and the nullclines marked with arrows. The regions separated by the nullclines in the xy-plane can then be labelled $(-,-)$, $(-,+)$, $(+,-)$, or $(+,+)$ consistent with the arrows on their bounding nullclines. The result is shown below in Figure 9.7. ☐

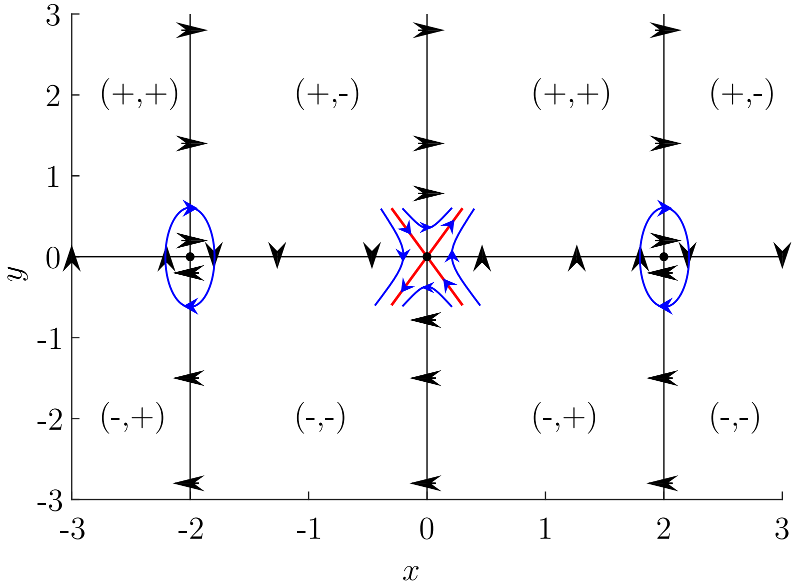

FIGURE 9.7. Phase portrait sketch with nullclines for the first example system.

Example. Determine the nullclines for the system

$$x' = (y - x)(x - 1)\,, \qquad y' = (3 + 2x - x^2)y\,.$$

Sketch the phase-plane portrait of the system that combines the information from its stationary points, the linearization about each stationary point, and its nullclines.

Solution. Recall that the stationary points of this system are

$$(-1,-1)\,, \qquad (0,0)\,, \qquad (1,0)\,, \qquad (3,3)\,.$$

In an earlier example we used linearization to show the following.

- The stationary point $(-1,-1)$ is a *saddle* with associated eigenpairs

$$\left(-2\,, \begin{pmatrix} 1 \\ 2 \end{pmatrix}\right)\,, \qquad \left(4\,, \begin{pmatrix} 1 \\ -1 \end{pmatrix}\right)\,.$$

  There is one orbit that approaches $(-1,-1)$ tangent to each half of the line $y = 2x + 1$. There is also one orbit that emerges from $(-1,-1)$ tangent to each half of the line $y = -x - 2$.

- The stationary point $(0,0)$ is a *nodal source* with associated eigenpairs

$$\left(1\,, \begin{pmatrix} 1 \\ 0 \end{pmatrix}\right)\,, \qquad \left(3\,, \begin{pmatrix} 1 \\ -2 \end{pmatrix}\right)\,.$$

There is one orbit that emerges from $(0,0)$ tangent to each half of the line $y = -2x$. All other orbits emerge from $(0,0)$ tangent to the x-axis.

- The stationary point $(1,0)$ is a *saddle* with associated eigenpairs

$$\left(-1\,,\,\begin{pmatrix}1\\0\end{pmatrix}\right)\,,\qquad \left(4\,,\,\begin{pmatrix}0\\1\end{pmatrix}\right)\,.$$

  There is one orbit that approaches $(1,0)$ tangent to each half of the x-axis. There is also one orbit that emerges from $(1,0)$ tangent to each half of the line $x = 1$.
- The stationary point $(3,3)$ is a *clockwise spiral sink.*

Points on the x-nullcline satisfy

$$0 = (y-x)(x-1)\,.$$

Therefore the x-nullcline consists of the line $y = x$ and the line $x = 1$.

- Along the line $y = x$ we have $g(x,x) = (x+1)(3-x)x$, so the arrows should be $\uparrow$ for $x < -1$, $\downarrow$ for $-1 < x < 0$, $\uparrow$ for $0 < x < 3$, and $\downarrow$ for $3 < x$.
- Along the line $x = 1$ we have $g(1,y) = 4y$, so the arrows should be $\downarrow$ for $y < 0$ and $\uparrow$ for $y > 0$.

Remark. The line $x = 1$ is vertical, and thereby contains semistationary solutions of the form $(1, Y(t))$.

Points on the y-nullcline satisfy

$$0 = (3 + 2x - x^2)y = (x+1)(3-x)y\,.$$

Therefore the y-nullcline consists of the line $x = -1$, the line $x = 3$, and the line $y = 0$.

- Along the line $x = -1$ we have $f(-1,y) = -2(y+1)$, so the arrows should be $\rightarrow$ for $y < -1$ and $\leftarrow$ for $y > -1$.
- Along the line $x = 3$ we have $f(3,y) = 2(y-3)$, so the arrows should be $\leftarrow$ for $y < 3$ and $\rightarrow$ for $y > 3$.
- Along the line $y = 0$ we have $f(x,0) = -x(x-1)$, so the arrows should be $\leftarrow$ for $x < 0$, $\rightarrow$ for $0 < x < 1$, and $\leftarrow$ for $x > 1$.

Remark. The line $y = 0$ is horizontal, and thereby contains semistationary solutions of the form $(X(t), 0)$.

Finally, we plot the stationary points, the local phase-plane portrait near each stationary point, and the nullclines marked with arrows. The regions separated by the nullclines in the xy-plane can then be labelled $(-,-)$, $(-,+)$, $(+,-)$, or $(+,+)$ consistent with the arrows on their bounding nullclines. The result is shown below in Figure 9.8. $\square$

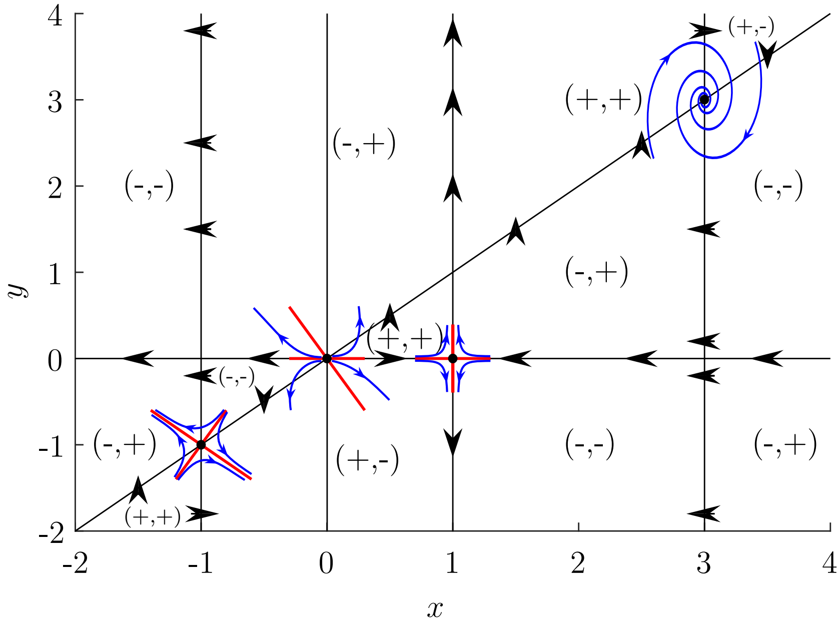

FIGURE 9.8. Phase portrait sketch with nullclines for the second example system.

9.6.2. *Isoclines.* If the stationary solutions, linearization, and nullclines do not give us a clear picture of the phase-plane portrait then sometimes the addition of a few isoclines can clarify things. Isoclines are sets in the xy-plane where the orbits through each point have the same (iso) slope (cline).

The *isocline for slope m* of system (9.1) is the set of points in the xy-plane that satisfy

$$g(x,y) = mf(x,y)\,. \tag{9.5}$$

An isocline usually is comprised of one or more curves in the xy-plane. These curves can be easy to determine for a few values of m when $f(x,y)$ and $g(x,y)$ have simple enough forms. If so then they can be plotted in the xy-plane.

Remark. In practice the values of m we choose might be guided by the complexity of equation (9.5). For example, we might pick values of m that make equation (9.5) easier to solve.

Remark. It is clear from (9.5) that the isocline for slope 0 is the y-nullcline, which we have already plotted. Therefore we will require $m \neq 0$. Because for $m \neq 0$ we can recast (9.5) as $f(x,y) = \frac{1}{m}g(x,y)$, we can view the isocline for slope ∞ as the x-nullcline.

Remark. It should be clear from (9.5) that the points where an isocline and a nullcline intersect are exactly the stationary points of system (9.1). It might be less clear that the points where two different isoclines intersect also are exactly the stationary points of system (9.1). This is because if $m_1 \neq m_2$ then

$$g(x,y) = m_1 f(x,y) \quad \text{and} \quad g(x,y) = m_2 f(x,y)$$

implies $f(x, y) = 0$ and $g(x, y) = 0$. (The implication in the other direction is obvious.)

We can then indicate the direction field along the isocline for slope m as follows.

- Segments of the isocline where $f(x, y) < 0$ should be marked with an arrow having slope m that points left.
- Segments of the isocline where $f(x, y) > 0$ should be marked with an arrow having slope m that points right.

The arrows can only change direction at points where the isocline and the x-nullcline intersect — namely, at stationary points.

Exercises on Nonintegral Methods

Compute the coefficient matrix of the linearization of the system of differential equations at each stationary point of the system.

(1) $x' = x + y^2 \qquad y' = x + y$

Solution

(2) $x' = x - xy, \qquad y' = y + 2xy$

Solution

(3) $x' = (1 + x)\sin(y) \qquad y' = 2 - x - \cos(y)$

Solution

(4) $x' = x - y^2 \qquad y' = y - x^2$

Solution

(5) $x' = 1 - xy \qquad y' = x - y^2$

Solution

(6) $x' = y \qquad y' = x + 2x^3$

Solution

For 7-18, determine the stability and classify each stationary point of the system

(7) $x' = x, \qquad y' = -2y + x^3$

Solution

(8) $x' = 2 - y, \qquad y' = 3 - x^2$

Solution

(9) $x' = -(2 + y)y, \qquad y' = x(1 - y)$

Solution

(10) $x' = 1 - y \qquad y' = x^2 - y^2$

Solution

(11) $x' = -x + 2xy \qquad y' = y - x^2 - y^2$

Solution

(12) $x' = 4x - y \qquad y' = 5x - x^2$

Solution

(13) $x' = -x - x^2, \qquad y' = y + 2xy$

Solution

(14) $x' = (1 + x)(y - x) \qquad y' = (2 - x)(y + x)$

Solution

(15) $x' = 1 + 2y, \qquad y' = 1 - 3x^2$

Solution

(16) $x' = -x + y + x^2, \qquad y' = y - 2xy$

Solution

(17) $x' = (12 - 2x - 3y)x, \qquad y' = (5x - 15)y$

Solution

(18) $x' = x - x^2 - xy, \qquad y' = \frac{1}{2}y - \frac{1}{4}y^2 - \frac{3}{4}xy$

Solution

(19) Derive and explain the linearization of a system of differential equations, $x' = f(x,y)$ and $y' = g(x,y)$. That is, show how

$$\frac{d}{dt}\begin{pmatrix}\tilde{x}\\ \tilde{y}\end{pmatrix} = \begin{pmatrix}\partial_x f(x_o,y_o) & \partial_y f(x_o,y_o)\\ \partial_x g(x_o,y_o) & \partial_y g(x_o,y_o)\end{pmatrix}\begin{pmatrix}\tilde{x}\\ \tilde{y}\end{pmatrix}$$

Solution

(20) Sketch the phase portrait of $x' = y$, $y' = \sin(x)$ using the methods of this chapter. Compare it the the phase portrait of the methods used in the previous chapter, “Integral Methods to Autonomous Planar Systems”.

Solution

(21) Consider the nonlinear planar system

$$\begin{aligned} x' &= 1 - y \\ y' &= x^2 - y^2 \end{aligned}$$

Determine the stability and classify each stationary point of the system if possible. Sketch a phase portrait.

Solution

(22) Consider the system

$$x' = -y + x^3 + xy^2, \quad y' = x + y^3 + yx^2.$$

(a) Linearize the system about $(0,0)$ and classify this stationary points. What can you say about the stability of $(0,0)$?

(b) Show that $V(x(t),y(t)) = x(t)^2 + y(t)^2$ is strictly increasing in t away from $(x,y) = (0,0)$. What can you deduce about the stability of $(0,0)$?

Solution

(23) A damped pendulum of mass m and lenght ℓ moving in a plane can be described by the equation

$$m\ell\theta'' = -\gamma\theta' - ma\sin(\theta).$$

where $\gamma > 0$ is the damping coefficient of the pendulum. This can be written as a system

$$x' = y, \quad y' = -2\mu y - \omega^2\sin(x),$$

where $x = \theta$, $y = \theta'$, $\mu = \gamma/(2m\ell)$ and $\omega^2 = a/\ell$. Answer the following

(a) Find all stationary points and write the Jacobian matrix at each of the stationary points.

(b) Determine the type and stability of each stationary point when $\mu^2 > \omega^2$.

(c) Determine the type and stability of each stationary point when $\mu^2 < \omega^2$.

(d) Determine the type and stability of each stationary point when $\mu^2 = \omega^2$.

Solution

The following questions are related to a special type of differential system of the form

$$\begin{aligned} x' &= -\partial_x U(x,y) \\ y' &= -\partial_y U(x,y) \end{aligned}$$

referred to as a gradient system.

(24) Show that a planar system

$$x' = f(x,y), \quad y' = g(x,y)$$

is a gradient system if and only if

$$\partial_y f(x,y) = \partial_x g(x,y).$$

Solution

(25) Show that a linear planar system $\mathbf{x}' = \mathbf{A}\mathbf{x}$ is a gradient system if and only if $\mathbf{A} = \mathbf{A}^T$.

Solution

(26) Suppose that (x_0, y_0) is a stationary solution of the gradient system $x' = -\partial_x U(x,y), \quad y' = -\partial_y U(x,y)$, then clearly (x_0, y_0) is also a critical point of $U(x,y)$. Show the following

  (a) If (x_0, y_0) is a strict local minimum of $U(x,y)$, then (x_0, y_0) is either a node or radial sink.
  (b) If (x_0, y_0) is a strict local maximum of $U(x,y)$, then (x_0, y_0) is a node or radial source.
  (c) If (x_0, y_0) is a saddle point of $U(x,y)$, then (x_0, y_0) corresponds to a saddle.
  (d) Can a gradient system have critical points that correspond to spirals or centers? Give an example or show why not.

Solution

(27) If $(x(t), y(t))$ is a solution to a gradient system. Show that $U(x(t), y(t))$ is always stricly decreasing in t except at stationary points.

Solution

(28) Show that if

$$x' = f(x,y), \quad y' = g(x,y)$$

is Hamiltonian, then

$$x' = g(x,y), \quad y' = -f(x,y)$$

is a gradient system.

Solution

Navigation to Other Chapters

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