STAT 700 Take home final exam December 6 - 11, 2012

1. [20+20+15] Let $(x_1, \ldots, x_n)$ be a sample from population with pdf

$$f(x;\theta) = \frac{\theta a^\theta}{x^{\theta+1}},\ x \geq a$$

with $\theta > 0$ as a parameter, $a > 0$ given.
(i) Find the method of moments (MME) and the maximum likelihood (MLE) estimators of θ and state the conditions on θ when the estimators are well defined.
(ii) Prove the consistency of the MME and MLE and find their non-degenerate limit distributions as $n \to \infty$.
(iii) Calculate the asymptotic efficiency of the MME.

2. [20+15+20] n identical coins with $P(\text{Head}) = p$ with p as a parameter are tossed independently. Let X_i be the number of tails in tossing the i-th coin before it falls a head, $i = 1, \ldots, n$.
(i) Find the minimal sufficient statistic of p and its distribution.
(ii) Prove that the minimal sufficient statistic is complete.
(**Hint**: Recall a property of infinite series.)
(iii) Find the UMVUE of $1/p$ and calculate its efficiency.

3. [20+15] Given θ, $X_1, \ldots, X_n$ are iid random variables having a negative binomial distribution NB(θ, r) with $\theta \in (0, 1)$ as a parameter and a positive integer r known.
(i) Construct a two-parameter family of conjugate priors for θ.
(ii) Assuming that the prior belongs to the conjugate family, find the Bayes estimator of θ from $X_1, \ldots, X_n$ for the quadratic loss function.

4. [20+15+20] Let $(x_1, \ldots, x_n)$ be a sample for a uniform population $U(0, \theta)$ with $\theta > 0$ as a parameter.
(i) Construct the UMP test of size α of $H_0 : \theta = 1$ versus alternative $H_1 : \theta > 1$.
(ii) Find the power function of the test.
(iii) Find the minimal value of n such that for $\alpha = .1$, the power at $\theta = 1.5$ be .9.