Integration on Curves and Surfaces: MATH 340, Fall 2013

This handout is a supplement to Section 6.10 in our textbook and describes some alternative notation and concepts that may be useful for doing the homework problems from that section.

1 Integration on curves

Throughout this section, C is a smooth curve in $\mathbb{R}^n$ parametrized by a C^1 function $\vec{\gamma} : [a, b] \to \mathbb{R}^n$.

Tangent vectors. The unit tangent vector $\vec{T}(\vec{\mathbf{x}})$ to C at a point $\vec{\mathbf{x}} = \vec{\gamma}(t)$ is

$$\vec{T}(\vec{\mathbf{x}}) = \frac{\vec{\gamma}\,'(t)}{|\vec{\gamma}\,'(t)|}.$$

(Here $\vec{\gamma}\,'(t)$ means the same thing as $D\vec{\gamma}(t)$.) The tangent vector can point in one of two directions, depending on the direction in which we parametrize C. If C is a closed curve in $\mathbb{R}^2$, we conventionally assume that C is parametrized in the counterclockwise direction. (This is the case if the unit circle is parametrized by the polar angle θ.)

Work. To express an integral over C as an integral with respect to t, we write

$$|d^1\vec{\mathbf{x}}| = \sqrt{\det(D\vec{\gamma}(t)^\top D\vec{\gamma}(t))}\,dt = |\vec{\gamma}\,'(t)|dt.$$

The "work" done by a vector field $\vec{F} : \mathbb{R}^n \to \mathbb{R}^n$ along the curve C is

$$\int_C \vec{F}(\vec{\mathbf{x}}) \cdot \vec{T}(\vec{\mathbf{x}})|d^1\vec{\mathbf{x}}| = \int_a^b \vec{F}(\vec{\gamma}(t)) \cdot \vec{\gamma}\,'(t)dt.$$

In our textbook, the work is written $\int_C W_{\vec{F}}$; the form $W_{\vec{F}}$ is equivalent to $\vec{F}(\vec{\mathbf{x}}) \cdot \vec{T}(\vec{\mathbf{x}})|d^1\vec{\mathbf{x}}|$ (when the curve is oriented by the direction of $\vec{T}(\vec{\mathbf{x}})$).

In physics, if $\vec{F}(\vec{\mathbf{x}})$ represents the total force on a particle at position $\vec{\mathbf{x}}$, and $\vec{\gamma}(t)$ is the position of a particle with mass m at time t, then by Newton's second law $\vec{F}(\vec{\gamma}(t)) = m\vec{\gamma}\,''(t)$, and the work done by $\vec{F}$ between time a and time b is

$$\int_a^b m\vec{\gamma}\,''(t) \cdot \vec{\gamma}\,'(t)dt = \frac{1}{2}m\vec{\gamma}\,'(t) \cdot \vec{\gamma}\,'(t)\Big|_a^b = \frac{1}{2}m|\vec{\gamma}\,'(b)|^2 - \frac{1}{2}m|\vec{\gamma}\,'(a)|^2.$$

Since $\vec{\gamma}\,'(t)$ is the velocity of the particle at time t, this says that the work done by $\vec{F}$ is the change in the particle's kinetic energy.

The fundamental theorem of calculus on curves. If $f : \mathbb{R}^n \to \mathbb{R}$ is C^1, then by the chain rule and the fundamental theorem of calculus, the work done by $\vec{\nabla} f$ along the curve C is

$$\int_C \vec{\nabla} f(\vec{\mathbf{x}}) \cdot \vec{T}(\vec{\mathbf{x}})|d^1\vec{\mathbf{x}}| = \int_a^b \vec{\nabla} f(\vec{\gamma}(t)) \cdot \vec{\gamma}\,'(t)dt = f(\vec{\gamma}(t))|_a^b = f(\vec{\gamma}(b)) - f(\vec{\gamma}(a)).$$

This is equivalent to Theorem 6.10.1 in our textbook; the form $\mathbf{d}f$ is the same as $W_{\vec{\nabla} f} = \vec{\nabla} f(\vec{\mathbf{x}})\cdot\vec{T}(\vec{\mathbf{x}})|d^1\vec{\mathbf{x}}|$.

In physics, if we can write a force field $\vec{F}(\vec{\mathbf{x}}) = -\vec{\nabla} f(\vec{\mathbf{x}})$, then we call f the potential energy associated with $\vec{F}$, and the formula above says the work done by $\vec{F}$ on a moving particle is *minus* the change in the particle's potential energy.

Green's theorem. For curves in $\mathbb{R}^2$, one often sees integrals like $\int_C f dx$, where f is a function of coordinates x and y. Here dx is equivalent to the form $\mathbf{d}g = W_{\vec{\nabla} g}$ for the function $g\begin{pmatrix} x \\ y \end{pmatrix} = x$. If we write $\vec{\gamma}(t) = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$, then since $\vec{\nabla} g \begin{pmatrix} x \\ y \end{pmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$,

$$\int_C f dx = \int_C f\begin{pmatrix} x \\ y \end{pmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \cdot \vec{T}\begin{pmatrix} x \\ y \end{pmatrix} \left| d^1 \begin{pmatrix} x \\ y \end{pmatrix} \right| = \int_a^b f\begin{pmatrix} x \\ y \end{pmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \cdot \vec{\gamma}'(t) dt = \int_a^b f(\vec{\gamma}(t)) x'(t) dt.$$

Similarly,

$$\int_C f dy = \int_a^b f(\vec{\gamma}(t)) y'(t) dt.$$

In this notation, if $\vec{F} = \begin{bmatrix} F_1 \\ F_2 \end{bmatrix}$ is a vector field in $\mathbb{R}^2$, the work done by $\vec{F}$ along C is

$$\int_a^b \vec{F}(\vec{\gamma}(t)) \cdot \vec{\gamma}'(t) dt = \int_C (F_1 dx + F_2 dy).$$

Green's theorem (Theorem 6.10.2) then says that the work done by $\vec{F}$ around a closed curve C (going counterclockwise) is the integral of $D_1 F_2 - D_2 F_1$ over the domain bounded by C. In $\mathbb{R}^2$, the form $\mathbf{d}W_{\vec{F}}$ is equivalent to $(D_1 F_2(\vec{\mathbf{x}}) - D_2 F_1(\vec{\mathbf{x}}))|d^2\vec{\mathbf{x}}|$.

Green's theorem is sometimes used to compute the area inside a closed curve C. Since the area is the integral of $|d^2\vec{\mathbf{x}}|$, we can use Green's theorem with any $\vec{F}$ for which $D_1 F_2 - D_2 F_1 = 1$. Two common choices of $\vec{F}$ are $\vec{F}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{bmatrix} 0 \\ x \end{bmatrix}$ and $\vec{F}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{bmatrix} -y \\ 0 \end{bmatrix}$. The area inside C is then

$$\int_C x dy = -\int_C y dx.$$

As before, the integrals should be done counterclockwise. (This choice of direction determines which of the two integrals is positive; going clockwise, the signs here and in Green's theorem would be reversed.)

2 Integration on "surfaces"

Here "surface" means the boundary of a domain in $\mathbb{R}^n$, or a piece thereof. Throughout this section, S is a subset of smooth $(n-1)$-dimensional manifold in $\mathbb{R}^n$ defined by an equation $h(\vec{\mathbf{x}}) = 0$, where $h : \mathbb{R}^n \to \mathbb{R}$ is C^1. We assume that the gradient of h is nonzero on S (this condition along with Theorem 3.1.10 ensures that we have a smooth manifold).

Normal vectors. The unit normal vector $\vec{N}(\vec{\mathbf{x}})$ to S at a point $\vec{\mathbf{x}} \in S$ is

$$\vec{N}(\vec{\mathbf{x}}) = \frac{Dh(\vec{\mathbf{x}})}{|Dh(\vec{\mathbf{x}})|}.$$

The normal vector lies along the line perpendicular to S and points in the direction that h increases. If S is the boundary of a domain in R^n, we conventionally assume that h increases outside the domain, and $\vec{N}$ is called the outward unit normal to S.

Flux. The "flux" of a vector field $\vec{F} : \mathbb{R}^n \to \mathbb{R}^n$ across S in the direction of increasing h is

$$\int_S \vec{F}(\vec{\mathbf{x}}) \cdot \vec{N}(\vec{\mathbf{x}}) |d^{n-1}\vec{\mathbf{x}}|.$$

In our textbook, the flux is written $\int_S \Phi_{\vec{F}}$; the form $\Phi_{\vec{F}}$ is equivalent to $\vec{F}(\vec{\mathbf{x}}) \cdot \vec{N}(\vec{\mathbf{x}})|d^{n-1}\vec{\mathbf{x}}|$ (when the surface is oriented by the direction of $\vec{N}(\vec{\mathbf{x}})$).

In physics, if $\vec{F}(\vec{\mathbf{x}})$ represents the velocity of a fluid at position $\vec{\mathbf{x}}$ at a particular time, the flux of $\vec{F}$ across S is the net volume of fluid per unit time that is crossing S in the direction of $\vec{N}(\vec{\mathbf{x}})$.

The divergence theorem. For a C^1 vector field $\vec{F} : \mathbb{R}^n \to \mathbb{R}^n$, the divergence theorem says that if S is the boundary of a bounded domain $M \subset \mathbb{R}^n$, then the flux of $\vec{F}$ across S in the outward direction is equal to the integral over M of the divergence of $\vec{F}$:

$$\int_S \vec{F}(\vec{\mathbf{x}}) \cdot \vec{N}(\vec{\mathbf{x}}) |d^{n-1}\vec{\mathbf{x}}| = \int_M \vec{\nabla} \cdot \vec{F}(\vec{\mathbf{x}}) |d^n\vec{\mathbf{x}}|.$$

In our textbook, this theorem is stated only for $n = 3$ (Theorem 6.10.6), but it is in fact true for all n. The form $\mathbf{d}\Phi_{\vec{F}} = M_{\vec{\nabla}\cdot\vec{F}}$ there is equivalent to $\vec{\nabla} \cdot \vec{F}(\vec{\mathbf{x}})|d^n\vec{\mathbf{x}}|$.

Again if $\vec{F}$ is the velocity field of a fluid, the divergence of $\vec{F}$ represents the rate at which the fluid is expanding (positive divergence) or contracting (negative divergence) at a given point. In these terms, for an expanding fluid (such as a gas at the time of an explosion), the divergence theorem says that the amount of gas leaving a domain per unit time is given by the integral over the domain of the expansion rate. Some fluids, such as water, are essentially "incompressible", meaning that they don't expand or contract. The divergence of the velocity field of such a fluid is always zero, and hence the flux of the fluid across the boundary of any bounded domain is also zero (equal amounts of the fluid move into and out of the domain at any given time).

More generally, the divergence theorem relates the integral of the partial derivative of any function over a bounded domain in $\mathbb{R}^n$ to an integral involving the function on the boundary of the domain. To integrate $D_j f$ for some C^1 function $f : \mathbb{R}^n \to \mathbb{R}$ and $1 \le j \le n$, let $\vec{F}$ be the vector field whose jth coordinate is f and all of whose other coordinates are zero; then $\vec{\nabla} \cdot \vec{F} = D_j f$ and we can apply the divergence theorem to $\vec{F}$.

Stokes's theorem. For a surface S in $\mathbb{R}^3$ bounded by a curve C, Stokes's theorem (Theorem 6.10.4) says that the work done by a vector field $\vec{F}$ around C is equal to the flux of the curl of $\vec{F}$ across S:

$$\int_C \vec{F}(\vec{\mathbf{x}}) \cdot \vec{T}(\vec{\mathbf{x}}) |d^1\vec{\mathbf{x}}| = \int_S (\vec{\nabla} \times \vec{F}(\vec{\mathbf{x}})) \cdot \vec{N}(\vec{\mathbf{x}}) |d^2\vec{\mathbf{x}}|.$$

In order for the integrals to have the same sign, we must traverse C in the counterclockwise direction looking from the side of S into which $\vec{N}$ points. In $\mathbb{R}^3$, the form $\mathbf{d}W_{\vec{F}} = \Phi_{\vec{\nabla}\times\vec{F}}$ in our textbook is equivalent to $(\vec{\nabla} \times \vec{F}(\vec{\mathbf{x}})) \cdot \vec{N}(\vec{\mathbf{x}})|d^2\vec{\mathbf{x}}|$ (when the surface is oriented by the direction of $\vec{N}(\vec{\mathbf{x}})$).