STAT430 Quiz 1

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Quiz 1

Prob. 1. Given the CDF's

$$a. \quad \textit{Logistic CDF} \quad F(x) = \frac{1}{1+\exp(-x)}$$

$$b. \quad \textit{Standard normal CDF} \quad F(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} \exp(-z^2/2)dz$$

Obtain the corresponding pdf's

Prob. 2. If $f(x) = \lambda \exp(-\lambda x)$ what are the cdf, mean, and variance?

Answer Prob. 1: Need to differentiate the CDF's

$$a. \quad f(x) = \frac{\exp(-x)}{(1+\exp(-x))^2}, \; x \in (-\infty, \infty)$$

$$b. \quad f(x) = \frac{1}{\sqrt{2\pi}} \exp(-x^2/2), \; x \in (-\infty, \infty)$$

Answer Prob. 2: Consider $X \sim Gamma(r, \lambda)$ defined for $x \geq 0$

$$f(x) = \frac{\lambda^r}{\Gamma(r)} x^{r-1} \exp(-\lambda x), \qquad \Gamma(r) = \int_0^{\infty} x^{r-1} \exp(-x)dx$$

$$E(x) = \frac{r}{\lambda}, \quad Var(X) = \frac{r}{\lambda^2}$$

Now take $r = 1$. Then we get the **exponential** distribution,

$$f(x) = \lambda \exp(-\lambda x), \qquad F(x) = 1 - \exp(-\lambda x)$$

with

$$E(X) = 1/\lambda, \quad Var(X) = 1/\lambda^2$$

Another special case: Chi square distribution with r degrees of freedom:

$$X \sim \chi^2_{(r)}$$

$$X \sim Gamma(\frac{r}{2}, \frac{1}{2})$$

$E(X) = r, \quad Var(X) = 2r$

This leads to the important distribution of the ratio,

$$F_{(r_1, r_2)} = \frac{\chi^2(r_1)/r_1}{\chi^2(r_2)/r_2}$$

where $\chi^2(r_1)$ and $\chi^2(r_2)$ are independent, and

$$E(F_{(r_1, r_2)}) = \frac{r_2}{r_2 - 2}$$

Note: if $X \sim N(\mu, \sigma^2)$ then

$$Z = \frac{X - \mu}{\sigma} \sim N(0, 1)$$

and

$$Z^2 \sim \chi^2_{(1)}$$

If $Z \sim N(0, 1)$ and $V \sim \chi^2_{(r)}$ are independent then,

$$\frac{Z}{\sqrt{V/r}} \sim t_{(r)}$$

A Property of the Gamma Distribution

Suppose X, Y are independent random variables where $X \sim Gamma(a, \lambda)$ and $Y \sim Gamma(b, \lambda)$. Then:

$$T = X + Y \sim Gamma(a + b, \lambda), \quad W = \frac{X}{X + Y} \sim Beta(a, b)$$

and T, W are independent. Then

$$E(X) = E\left[(X + Y)\frac{X}{X + Y}\right] = E(X + Y)E\left[\frac{X}{X + Y}\right]$$

and we get the unusual result about the expectation of a ratio,

$$\frac{E(X)}{E(X+Y)} = E\left[\frac{X}{X+Y}\right] = \frac{a/\lambda}{(a+b)/\lambda} = \frac{a}{a+b}$$

which is the mean of Beta(a,b). In general, however,

$$E\left[\frac{X}{Y}\right] \neq \frac{E(X)}{E(Y)}$$

Logistic Distribution vs Normal Distribution

Here is an R code to compare the logistic and normal pdf's and CDF's.

```
x <- seq(-5,5,0.01)
par(cex=1.25)

f <- function(x){exp(-x^2/2)/sqrt(2*pi)}
plot(x,f(x), type="l",col="blue",lwd=2, ylab="")
g <- function(x){exp(-x)/(1+exp(-x))^2}
lines(x,g(x),type="l",col="red", lwd=2)
legend("topleft",
c("Normal","Logistic"),
fill=c("blue","red")
)
title("Normal vs Logstic pdf's")

x <- seq(-5,5,0.01)
par(cex=1.25)

F <- function(x){pnorm(x)}
G <- function(x){1/(1+exp(-x))}
plot(x,F(x),type="l",col="blue",lwd=2,ylab="")
lines(x,G(x),type="l",col="red",lwd=2)
legend("topleft",
c("Normal","Logistic"),
fill=c("blue","red")
)
title("Normal vs Logstic CDF's")
```