(a) Mean Estimation ← in TS_Mean - Est.
(b) Estimation of R_k, ρ_k
(c) Sample Spectral Density
(d) Estimation of $f(\omega)$
(e) Distribution of Spectral Estimates

(1)

(a) MEAN ESTIMATION

$Z_1, \cdots, Z_N$ stat TS, $EZ=0$

FROM ANDERSON 1971

$\bar{Z} = \frac{1}{N}\sum_{i=1}^{N} Z_i$, $E\bar{Z} = 0$

$$\mathrm{Var}(\bar{Z}) = \frac{1}{N^2}\sum_{i=1}^{N}\sum_{j=1}^{N} EZ_iZ_j = \frac{1}{N^2}\sum_{i=1}^{N}\sum_{j=1}^{N} R_{i-j}$$

$$= \frac{1}{N^2}\sum_{r=-(N-1)}^{N-1} (N-|r|)R_r = \frac{1}{N}\sum_{r=-(N-1)}^{N-1}\left(1-\frac{|r|}{N}\right)R_r$$

$$\Longrightarrow N\,\mathrm{Var}(\bar{Z}) = \sum_{r=-(N-1)}^{N-1}\left(1-\frac{|r|}{N}\right)R_r \longrightarrow \sum_{r=-\infty}^{\infty} R_r \text{ if sum converges}$$

$$\boxed{\text{Fact: } \sum_{r=-\infty}^{\infty} R_r < \infty \Rightarrow \lim_{N\to\infty} N\,\mathrm{Var}(\bar{Z}) = \sum_{r=-\infty}^{\infty} R_r}$$

Now assume $f(\lambda)$ exists. Then

$$f(\lambda) = \frac{1}{2\pi}\sum_{-\infty}^{\infty}\cos(k\lambda)R_k \Rightarrow \sum_{-\infty}^{\infty} R_r = 2\pi f(0)$$

$$\boxed{\text{Fact: } \lim_{N\to\infty} N\,\mathrm{Var}(\bar{Z}) = 2\pi f(0)}$$

So: $\mathrm{Var}(\bar{Z}) \approx \frac{1}{N}\times 2\pi f(0) \to 0$

In general: $\frac{1}{2\pi}\int_{-T}^{T} x(t)\,dt = E[x(t)] + \xi(\{0\})$

cont. time

More precisely

$$N \operatorname{Var}(\bar{Z}) = \sum_{r=-(N-1)}^{N-1} \left(1 - \frac{|r|}{N}\right) R_r = \sum \left(1 - \frac{|r|}{N}\right) \int_{-\pi}^{\pi} e^{irλ} f(λ)\,dλ$$

$\boxed{HW}$

$$= \int_{-\pi}^{\pi} \frac{1}{N} \frac{\sin^2 \frac{1}{2}λN}{\sin^2 \frac{1}{2}λ} f(λ)\,dλ$$

Define: $W_N(λ) \equiv \dfrac{1}{2\pi N} \dfrac{\sin^2 \frac{1}{2}λN}{\sin^2 \frac{1}{2}λ}$ ← FEJÉR KERNEL

Then: $\int_{-\pi}^{\pi} W_N(λ)\,dλ = 1$

and $W_N(λ)$ behaves as δ-function as $N \to \infty$

∴

$$N \operatorname{Var}(\bar{Z}) = 2\pi \int_{-\pi}^{\pi} f(λ) W_N(λ)\,dλ \to 2\pi f(0)$$

Note: $\delta(x)$ ← Delta function

$$\int_{-\infty}^{\infty} f(x)\,\delta(x-a)\,dx = f(a)$$

$$\int_{-\infty}^{\infty} f(x)\,\delta(x)\,dx = f(0)$$

$$\int_{-\infty}^{\infty} \delta(x)\,dx = 1$$

$\boxed{\text{HW: plot } W_N(λ) \text{ for } N = 10, 50, 100}$

(b) ESTIMATION of R_k for $\{z_t\}$ stationary

$z_1, \cdots, z_n$ ts, $E z_t = \mu$, $\mathrm{Cov}(z_t, z_{t+h}) = R_h$

There are several estimates all asymptotically same

1. $C_h = C_{-h} = \frac{1}{n-h} \sum_{t=1}^{n-h} (z_t - \mu)(z_{t+h} - \mu)$, $h = 0, 1, \cdots, n-1$

2. $C_h^* = C_{-h}^* = \frac{1}{n-h} \sum_{t=1}^{n-h} (z_t - \bar{z})(z_{t+h} - \bar{z})$, $h = 0, 1, \cdots, n-1$

3. $\overset{LC}{c_h} = \frac{1}{n} \sum_{t=1}^{n-h} (z_t - \mu)(z_{t+h} - \mu)$

4. $\overset{LC}{c_h^*} = \frac{1}{n} \sum_{t=1}^{n-h} (z_t - \bar{z})(z_{t+h} - \bar{z})$ ← Recommeded by Parzen & used in Box and Jenkins

$E[C_h] = C_h$ unbiased

$E[c_h] = (1 - \frac{1}{n}) R_h \to R_h$

$E[C_h^*] = R_h + \frac{1}{n} \times$ [Convergent series]

Assume 4'th order moments exist:

$$E(z_t-\mu)(z_{t+h}-\mu)(z_{t+r}-\mu)(z_{t+s}-\mu)$$

$$= E(z_0-\mu)(z_h-\mu)(z_r-\mu)(z_s-\mu) \equiv m(h,r,s)$$

Fact (Isserlis 1918): If $\{z_t\}$ is Gaussian then

$$m(h,r,s) = R_h R_{r-s} + R_r R_{h-s} + R_s R_{h-r}$$

Definition: The 4'th order CUMULANT FUNCTION is

$$K(h,r,s) = m(h,r,s) - [R_h R_{r-s} + R_r R_{h-s} + R_s R_{h-r}]$$

Fact: If $\{z_t\}$ is Gaussian $\Rightarrow K(h,r,s) = 0$

Fact (Bartlett 1946)

$$(n-h)\,\mathrm{Var}[C_h] = \sum_{r=-(n-h-1)}^{n-h-1} \left(1-\frac{|r|}{n-h}\right)\left[R_r^2 + R_{r+h}R_{r-h} + K(h,-r,h-r)\right]$$

$\therefore$

If $\sum R_r^2 < \infty$, and $\left|\sum K(h,-r,h-r)\right| < \infty$

then C_h is consistent

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Let $r_k = \dfrac{\overset{\text{LC}}{\downarrow} C_k^*}{C_0^*} \approx \rho_k$

Fact (Bartlett 1946): For Gaussian $\{z_t\}$

$$\mathrm{Var}(r_k) \approx \frac{1}{n} \sum_{j=-\infty}^{\infty} \left[\rho_j^2 + \rho_{j+k}\, \rho_{j-k} - 4\rho_k\, \rho_j\, \rho_{j-k} + 2\rho_j^2 \rho_k^2 \right]$$

$\therefore$ If $\rho_k = 0$, $k > q$, then

$$\mathrm{Var}(r_k) \approx \frac{1}{n}\left[1 + 2\sum_{j=1}^{q} \rho_j^2\right], \quad k > q$$

If $q = 0$ we take $\mathrm{Var}(r_k) \approx \frac{1}{n}$, $\forall\, k \neq 0$

Assume r_k is approx normal. In practice we reject $H_0: \rho_k = 0,\ k > q$ if

$$|r_k| > 1.96 \cdot \frac{1}{\sqrt{n}}\left[1 + 2\sum_{j=1}^{q} r_j^2\right]^{1/2}$$

See Box & Jenkins pp. 34–35

(5) SAMPLE SPECTRAL DENSITY

$$I(\lambda) = \frac{1}{2\pi n} \left| \sum_{t=1}^{n} (z_t - \mu) e^{it\lambda} \right|^2, \quad \mu \text{ known}$$

$$\boxed{-\pi \le \lambda \le \pi}$$

$$I^*(\lambda) = \frac{1}{2\pi n} \left| \sum_{t=1}^{n} (z_t - \bar{z}) e^{it\lambda} \right|^2, \quad \mu \text{ unknown}$$

Fact!

$$I(\lambda) = \frac{1}{2\pi} \sum_{r=-(n-1)}^{n-1} \overset{\text{LC}}{c_r} \cos(\lambda r), \quad -\pi \le \lambda \le \pi$$

$$I^*(\lambda) = \frac{1}{2\pi} \sum_{r=-(n-1)}^{n-1} \overset{\text{LC}}{c_r^*} \cos(\lambda r), \quad -\pi \le \lambda \le \pi$$

Proof: See Anderson (1971)

Fact:

$$\overset{\text{LC}}{c_h} = \int_{-\pi}^{\pi} \cos(\lambda h) I(\lambda)\, d\lambda, \quad h = 0, \pm 1, \cdots, \pm(n-1)$$

$$c_h^* = \int_{-\pi}^{\pi} \cos(\lambda h) I^*(\lambda)\, d\lambda, \quad h = 0, \pm 1, \cdots, \pm(n-1)$$

$$I(\lambda) = \frac{1}{2\pi n} \sum_{t=1}^{n} \sum_{s=1}^{n} (z_t - \mu)(z_s - \mu) e^{i\lambda(t-s)}$$

$$\therefore \quad E[I(\lambda)] \overset{\text{HW}}{=} \frac{1}{2\pi n} \int_{-\pi}^{\pi} \left| \sum_{t=1}^{n} e^{it(\nu-\lambda)} \right|^2 f(\nu)\, d\nu$$

$$= \frac{1}{2\pi n} \int_{-\pi}^{\pi} \frac{\sin^2 \frac{1}{2}(\nu-\lambda) n}{\sin^2 \frac{1}{2}(\nu-\lambda)} f(\nu)\, d\nu$$

$$= \int_{-\pi}^{\pi} W_n(\nu-\lambda) f(\nu)\, d\nu \rightarrow f(\lambda), \quad n \to \infty$$

BECOMES δ FUNCTION

Get similar result for $I^*(\lambda)$

<u>Note</u>:
$$\boxed{\frac{1}{n} \sum_{t=1}^{n} \sum_{s=1}^{n} g(t-s) = \sum_{r=-(n-1)}^{n-1} \left(1 - \frac{|r|}{n}\right) g(r)}$$

So, $I(\lambda)$ is asymptotically unbiased.

Fact: If $\frac{1}{n}\sum_{r,u,v=-(n-1)}^{n-1} |K(r,u,v)| \to 0, \; n \to \infty$

Then

$$\lim_{n\to\infty} \mathrm{Var}[I(0)] = 2f^2(0)$$

$$\lim_{n\to\infty} \mathrm{Var}[I(\pm\pi)] = 2f^2(\pi)$$

$$\lim_{n\to\infty} \mathrm{Var}[I(\lambda)] = f^2(\lambda), \qquad \lambda \neq 0, \pm\pi$$

$$\lim_{n\to\infty} \mathrm{Cov}(I(\lambda), I(\omega)) = 0, \quad \lambda \neq \pm\omega$$

$\therefore$ $I(\lambda)$ not a consistent estimate of $f(\lambda)$!!!

But $I(\lambda)$, $I(\omega)$, $\lambda \neq \omega$, are uncorrelated asymptotically!!!

This suggests averaging $I(\lambda)$ over some values in a neighborhood of λ to get a better estimates of $f(\lambda)$:

$$\int_{-\pi}^{\pi} W_n(\nu-\lambda)\, I(\lambda)\, d\lambda \approx f(\nu)$$

(d) Estimation of $f(\omega)$

So consider:

$$\hat{f}(\lambda) \equiv \frac{1}{2\pi} \sum_{r=-(n-1)}^{n-1} e^{-ir\lambda} \, c_r \underbrace{w_r}_{\left(1-\frac{|r|}{n}\right)}$$

$$= \frac{1}{2\pi} \sum_{r=-(n-1)}^{n-1} e^{-ir\lambda} \left[\int_{-\pi}^{\pi} e^{ir\omega} I(\omega) d\omega \right] w_r$$

$$= \int_{-\pi}^{\pi} \left(\frac{1}{2\pi} \sum_{r=-(n-1)}^{n-1} e^{ir(\omega-\lambda)} w_r \right) I(\omega) d\omega$$

$$= \int_{-\pi}^{\pi} \underbrace{\left[\frac{1}{2\pi} \sum_{r=-(n-1)}^{n-1} \left(1-\frac{|r|}{n}\right) e^{ir(\omega-\lambda)} \right]}_{\equiv\, W_n(\omega-\lambda)} I(\omega) d\omega$$

$$\longrightarrow f(\lambda), \quad n \to \infty$$

Then $E[\hat{f}(\lambda)] \to f(\lambda)$

and $Var[\hat{f}(\lambda)] = ?$

More generally:

$$\hat{f}(\lambda) \equiv \frac{1}{2\pi} \sum_{r=-M_n}^{M_n} e^{-ir\lambda} C_r w_r \qquad w_r = k\left(\frac{r}{M_n}\right)$$

$$= \int_{-\pi}^{\pi} \underbrace{\left[\frac{1}{2\pi} \sum_{r=-M_n}^{M_n} e^{ir(\omega-\lambda)} w_r \right]}_{= W(\omega-\lambda)} I(\omega)\, d\omega$$

Under some conditions including

$M_n \to \infty$ as $n \to \infty$ s.t. $\frac{M_n}{n} \to 0$

and $\sum_{k=-\infty}^{\infty} |k| \, |R_k| < \infty$

Then

$$E(\hat{f}(\lambda) - f(\lambda))^2 = \underbrace{Var(\hat{f}(\lambda))}_{O\left(\frac{M_n}{n}\right)} + \underbrace{E^2(\hat{f}(\lambda) - f(\lambda))}_{O\left(\frac{1}{M_n^2}\right)}$$

$\therefore$ TRADEOFF

e.g. See Kedem (1980), Binary TS, p. 82

pp 75–83

So, spectral estimates have the form

$$(*)\quad \hat{f}(\omega) = \frac{1}{2\pi} \sum_{s=-M_n}^{M_n \leftarrow \text{"BANDWIDTH"}} k\left(\frac{s}{M_n}\right) \underbrace{\hat{R}(s)}_{C^*_s \text{ or } C^{*LC}_s} e^{-is\omega}$$

$$\overset{\text{symmetry}}{=} \frac{1}{2\pi} \sum_{s=-M_n}^{M_n} k\left(\frac{s}{M_n}\right) \hat{R}(s) \cos(s\omega)$$

So, we need $k(\cdot) \leftarrow$ "lag window"

Bartlett: $k(u) = \begin{cases} 1-|u|, & |u| \le 1 \\ 0, & |u| > 1 \end{cases}$

Tukey: $k(u) = \frac{1}{2}\left[1 + \cos(\pi u)\right]$

Parzen: $k(u) = \begin{cases} 1 - 6u^2 + 6|u|^3, & |u| \le \frac{1}{2} \\ 2(1-|u|)^3, & \frac{1}{2} \le |u| \le 1 \\ 0, & |u| > 1 \end{cases}$

Priestley: $k(u) = \frac{3}{(\pi u)^2}\left[\frac{\sin(\pi u)}{\pi u} - \cos(\pi u)\right]$

(e) Distribution of Spectral Estimates

$$\omega_j \in \left\{ \frac{2\pi j}{N},\ j = 0, 1, \ldots, \left[\frac{N}{2}\right] \right\}$$

Recall the periodogram

$$I(\omega_j) = \frac{2}{N} \left| \sum_{t=1}^{N} x_t e^{-i\omega_j t} \right|^2 .$$

Fact: If $\{x_t\}$ Gaussian $WN(0, \sigma_x^2)$
← i.i.d

Assume N even.

Then:

$$I(\omega_j) \sim \sigma_x^2 \chi^2_{(2)}, \quad j \neq 0, N/2$$

$$\sim 2\sigma_x^2 \chi^2_{(1)}, \quad j = 0, N/2$$

$\therefore$

$$E[I(\omega_j)] = 2\sigma_x^2, \quad \forall j$$

$$Var[I(\omega_j)] = \begin{cases} 4\sigma_x^4, & j \neq 0, N/2 \\ 8\sigma_x^4, & j = 0, N/2 \end{cases}$$

and $I(\omega_j)$, $I(\omega_i)$ independent $i \neq j$

$\{X_t\}$ stationary, $E X_t = 0$, spectral density $f(\lambda)$

Define: $J_x(\omega) \equiv \frac{1}{\sqrt{2\pi N}} \sum_{t=1}^{N} X_t e^{-i\omega t}$ (13)

But $X_t = \int_{-\pi}^{\pi} e^{it\lambda} \xi_x(d\lambda) \Longrightarrow$

$$J_x(\underset{\frac{2\pi j}{N}}{\omega_j}) \simeq \sqrt{\frac{N}{2\pi}}\ \xi_x(d\omega_j) \equiv \frac{\xi_x(d\omega_j)}{\sqrt{d\omega_j}} \quad (*)$$

$\lambda = \omega_j$

$$\boxed{\text{Use: } \frac{1}{N}\sum_{t=1}^{N} e^{i\lambda t} \to \begin{cases} 0, & \lambda \neq 0 \\ 1, & \lambda = 0 \end{cases}, \quad N \to \infty}$$

Note:

$$I_N(\omega) \equiv \frac{1}{2\pi N} \left| \sum_{t=1}^{N} X_t e^{-i\omega t} \right|^2 = J_x(\omega)\overline{J_x(\omega)}$$

SLIGHT CHANGE

Assume: $X_t = \sum_u g_u \varepsilon_{t-u}$ ← i.i.d $N(0, \sigma_\varepsilon^2)$

$$\therefore \boxed{\xi_x(d\omega) = G(\omega)\xi_\varepsilon(d\omega)}, \quad G(\omega) = \sum_u g_u e^{-i\omega u}$$

TRANSFER FUNCTION

$\therefore$ From (*)

$$J_x(\omega_j) \simeq G(\omega_j) J_\varepsilon(\omega_j)$$

But $I_N = J\bar{J} \Longrightarrow$

$$\boxed{I_{N,x}(\omega_j) \simeq |G(\omega_j)|^2 I_{N,\varepsilon}(\omega_j)}$$

Now

$$f_\varepsilon(\omega) = \frac{\sigma_\varepsilon^2}{2\pi}, \quad f_x(\omega) = \frac{\sigma_\varepsilon^2}{2\pi}\left|G(\omega)\right|^2$$

WN

$$\therefore \quad |G(\omega)|^2 = \frac{2\pi}{\sigma_\varepsilon^2} f_x(\omega)$$

$\therefore$

$$\boxed{I_{N,x}(\omega_j) \simeq 2\pi f_x(\omega_j) \times I_{N,\varepsilon}(\omega_j) / \sigma_\varepsilon^2}$$

$\therefore$ From above

$$I_{N,\varepsilon}(\omega_j) = \frac{1}{4\pi} I(\omega_j) \sim \begin{cases} \dfrac{\sigma_\varepsilon^2 \chi^2_{(2)}}{4\pi}, & j \neq 0, N/2 \\[2ex] \dfrac{2\sigma_\varepsilon^2 \chi^2_{(1)}}{4\pi}, & j = 0, N/2 \end{cases}$$

$$\therefore \quad I_{N,x}(\omega_j) \overset{j \neq 0, N/2}{\simeq} \frac{2\pi f_x(\omega_j)}{\sigma_\varepsilon^2} \times \frac{\sigma_\varepsilon^2 \chi^2_{(2)}}{4\pi} = \frac{1}{2} f_x(\omega_j) \chi^2_{(2)}$$

$\therefore$

$$\boxed{E\left[I_{N,x}(\omega_j)\right] \sim f_x(\omega_j), \quad \mathrm{Var}\left(I_{N,x}(\omega_j)\right) \overset{\omega_j \neq 0, \pi}{\sim} f_x^2(\omega_j)}$$

Recall:

$$\hat{f}(\omega) = \frac{1}{2\pi} \sum_{s=-M_N}^{M_N} k\left(\frac{s}{M_N}\right) \hat{R}(s) \cos(s\omega)$$

$$= \int_{-\pi}^{\pi} W_N(\lambda - \omega) I_{N,x}(\lambda) \, d\lambda$$

$$\simeq \frac{2\pi}{N} \sum_j W_N(\omega - \omega_j) I_{N,x}(\omega_j) \quad ☆$$

But $I_{N,x}(\omega_j) \sim \frac{1}{2} f(\omega_j) \chi^2_{(2)}, \quad j \neq 0, N/2$

$\sim f(\omega_j) \chi^2_{(1)}, \quad j = 0, N/2$

$\therefore$ ☆ is a weighted ave of independent χ^2 variables

With a well known idea, we approx the dist of

$$\hat{f}(\omega)/f(\omega)$$

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$$\boxed{\text{by } a\chi^2_{(\nu)}}$$

$\therefore$ $$E\left[\frac{\hat{f}(\omega)}{f(\omega)}\right] = 1 = a\nu \Longrightarrow a = 1/\nu$$

WORK WITH THIS

$$\text{Var}\left[\frac{\hat{f}(\omega)}{f(\omega)}\right] \simeq \frac{1}{N}\sum_{s=-M_N}^{M_N} k^2\left(\frac{s}{M_N}\right) = \begin{cases} 2a^2\nu, & \omega \neq 0, \pm\pi \\ a^2\nu, & \omega = 0, \pm\pi \end{cases}$$

$\therefore$ Solving for ν:

$$\frac{2\nu}{\nu^2} = \frac{1}{N}\sum_{s=-M_N}^{M_N} k^2\left(\frac{s}{M_N}\right)$$

$\therefore$

$$\nu = \frac{2N}{\sum_{s=-M_N}^{M_N} k^2\left(\frac{s}{M_N}\right)} \sim \frac{2N}{M_N\int_{-\infty}^{\infty} k^2(u)\,du}$$

Finally:

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$$\Rightarrow \frac{\hat{f}(\omega)}{f(\omega)} \sim \chi^2(\nu)$$

$$\nu = \frac{2N}{M_N \int_{-\infty}^{\infty} k^2(u)\,du}$$

$$\therefore \operatorname{Var}[\hat{f}(\omega)] = O\left(\frac{M_N}{N}\right)$$

where $M_N/N \to 0$, $N \to \infty$

NOTE!

(18)

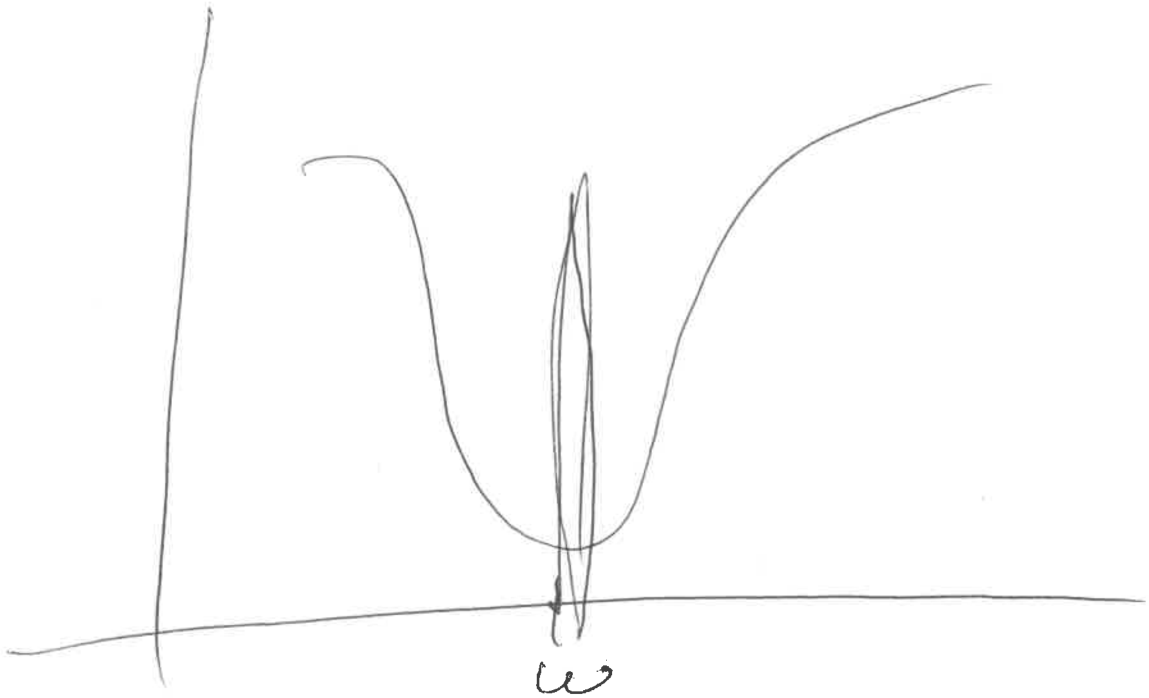
$$\sum_{i=1}^{N}\sum_{j=1}^{N} R_{i-j} = NR_0 + (N-1)R_1 + (N-1)R_{-1}$$
$$+ (N-2)R_2 + (N-2)R_{-2}$$
$$\vdots$$
$$[N-(N-1)]R_{N-1} + [N-(N-1)]R_{-(N-1)}$$

$$= \sum_{n=-(N-1)}^{N-1} (N-|n|)R_n$$

$\therefore$

$$\frac{1}{N}\sum_{s=1}^{N}\sum_{t=1}^{N} R_{s-t} = \frac{1}{N}\sum_{n=-(N-1)}^{N-1} (N-|n|)R_n$$

$$= \sum_{n=-(N-1)}^{N-1} \left(1-\frac{|n|}{N}\right) R_n$$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