TS 1: Stationary Processes

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1 A Brief Introduction to Stochastic Processes

Definition: A stochastic process is a collection of random variables defined on a common probability space

$$(\Omega, \mathcal{F}, P): \ \{X_t(\omega), t \in T\}, \ \omega \in \Omega$$

Examples of T: $\mathcal{R} = [0, \infty)$, $\mathcal{R} = \mathcal{Z} = \{\cdots - 1, 0, 1, 2, \cdots\}$, $\mathcal{R} = \mathcal{Z}^+ = \{0, 1, 2, \cdots\}$.

If $T = R^n$, $n \geq 2$, then $X_t(\omega)$ is called a *random field*.

The random variables need not be real valued. They could, for example, be complex-valued or vector-valued.

Note $X_t(\omega)$ is a function of two variables. For a *fixed* t $X_t(\cdot)$ is a random variable. For a *fixed* ω we get a function $X_t : T \to R^1$ (say) which is called a *sample path* or a *realization*.

An equivalent definition: A stochastic process is the ensemble of all realizations.

It is convenient to switch from one definition to the other as needed.

Let $t_1, .., t_n$ be in T, and consider the *finite dimensional distributions* of the form,

$$(*) \ P_{(t_1,..,t_n)}(C) = P(\omega \in \Omega : (X_{t_1}(\omega), ...X_{t_n}(\omega)) \in C)$$

Let $(X_{t_1}, ..., X_{t_n}) \sim F$. We can express the finite dimensional distributions by CDF's

$$F(b_1, ..., b_n; t_1, ..., t_n) = P(X_{t_1} \leq b_1, ..., X_{t_n} \leq b_n)$$

Assume $m = 1, .., n$, and let $(i_1, ..., i_n)$ be is a permutation of $(i, ..., n)$.

We assume the finite dimensional distributions obey the *consistency conditions*:

(i) $F(b_1, ..., b_m, \infty, ..., \infty; t_1, ..., t_m, t_{m+1}, ..., t_n) = F(b_1, ..., b_m; t_1, ..., t_m)$

(ii) $F(b_1, ..., b_n; t_1, ..., t_n) = F(b_{i_1}, ..., b_{i_n}; t_{i_1}, ..., t_{i_n})$

Then, the finite dim. distributions can be extended uniquely to a probability measure associated with a stochastic process $\{X_t(\omega), t \in T\}$ for which (*) are the finite dimensional distributions (Kolmogorov 1933).

So, if we wish to create a process we must supply a consistent family of distributions for every vector $(X_{t_1}, ..., X_{t_n})$.

However, the finite dim. distributions do NOT determine the distribution of every function of the process!!!

Example:
$U \sim Unif[0, 1]$. Define two processes X_t, Y_t, $t \in [0, 1]$:

$$X_t = \begin{cases} 1 & \text{if } t = U; \\ 0 & \text{Otherwise.} \end{cases}$$

$$Y_t \equiv 0$$

Then

$$P(X_t = 0) = P(U \neq t) = 1 = P(Y_t = 0)$$

Therefore the finite dimensional distributions are the same.
But

$$P(\sup_{0 \le t \le 1} X_t(\omega) \le 1/2) = 0$$

$$P(\sup_{0 \le t \le 1} Y_t(\omega) \le 1/2) = 1$$

Thus, the finite dimensional distributions do not determine uniquely the appearance! The appearance is more basic!

A complex random variable has the representation

$$X = X_1 + iX_2$$

where X_1, X_2 are real random variables. It is sometimes more convenient to deal with complex rv's in the context of Hilbert spaces.

$$E(x_t) = \int_\Omega X_t(\omega)P(d\omega)$$

$$E(X_t\bar{X}s) = \int_\Omega X_t(\omega)\bar{X}_s(\omega)P(d\omega)$$

1.1 Examples of Stochastic Processes

1. Let $\{X_n, n = 1, 2, 3, ...\}$ be the outcome of **die tossing**. Then a realization could be

$$2, 3, 4, 3, 1, 1, 1, 2, 4, 5, 5, 4, 6, 5, 6, 6, ...$$

2. Let $t_0 < t_1 < \cdots < t_n$ and suppose the increments of $\{X_t, t \geq 0\}$

$$X_{t_1} - X_{t_0}, X_{t_2} - X_{t_1}, ..., X_{t_n} - X_{t_{n-1}}$$

are independent. Assume
a. The distribution of $X_{t_2} - X_{t_1}$ depends on $t_2 - t_1$ only.
b. For $t > 0$, and b a positive constant, $X_t - X_s \sim N(0, b(t-s))$:

$$P(X_t - X_s \leq x) = \frac{1}{\sqrt{(2\pi b(t-s))}} \int_{-\infty}^{x} \exp\left(-\frac{u^2}{2b(t-s)}\right) du$$

c. $X_0 = 0$.

This process is well defined since $X_0 = 0$, and for $t \geq 0$, $X_t \sim N(0, bt)$:

$$P(X_t \leq x) = \frac{1}{\sqrt{(2\pi bt)}} \int_{-\infty}^{x} \exp\left(-\frac{u^2}{2bt}\right) du$$

and the **joint characteristics function** is given by:

$$E\exp[i(u_1 X_{t_1} + \cdots u_n X_{t_n})] = \varphi_{x_{t_1}}(u_1 + \cdots + u_n)\varphi_{(x_{t_2} - x_{t_1})}(u_2 + \cdots u_n)\cdots\varphi_{(x_{t_n} - x_{t_{(n-1)}})}(u_n)$$

which is known as all the one-dimensional characteristic functions are known. Hence, we know the finite dimensional distributions.

Easy to see that

$$\varphi_{(x_t - x_s)}(u) = \exp[-\frac{1}{2}(u^2 b|t-s|)]$$

This process is called **Wiener Process** or **Brownian motion**.

3.**Point Process**.
Let $S = [0, \infty)$ and $\mathcal{A}$ a family of subsets of S. Define a point process

$$\{N(A), A \in \mathcal{A}\}$$

which counts the number of points scattered at random in A.

Assume:
1. $N(\phi) = 0$
2. $N(A_1 \cup A_2) = N(A_1) + N(A_2)$, $A_1 \cap A_2) = \phi$.
3. $N(A)$ hs a Poisson distribution with parameter $\lambda V(A)$, $V(A)$ the length of A.
4. $N(A_1), ..., N(A_n)$ are independent if the A_j are disjoint.

We see "time" is a **set**. So $N(A)$ is a random set function. Such functions are important in TS analysis.

4. **Gaussian Process**.
$\{X_t, -\infty < t < \infty\}$ where $(X_{t_1}, ..., X_{t_n})$, for any $(t_1, ..., t_n)$, has a multivariate normal distribution with

$$EX_t = m(t), \quad Cov(X_{t_i}, X_{t_j}) = \sigma(t_i, t_j)$$

where the matrix $(\sigma(t_i, t_j))$ is positive semi-definite. If the matrix $(\sigma(t_i, t_j))$ is positive definite then $(X_{t_1}, ..., X_{t_n})$ has a pdf. The process always has a characteristic function.

Many procedures assume the TS is Gaussian.

5. **White Noise**.

$\{u_t, \cdots - 1, 0, 1, \cdots\}$ where the u_t are uncorrelated. We could let the finite dimensional distributions be Gaussian. WN is a fundamental process in TS analysis.

What is the origin of the name "white noise"?

2 Stationary Stochastic Processes

A stochastic process is stationary if its probability laws are invariant with respect to time shifts.

Definition: X_t is *strictly stationary* if $(X_{t_1}, ..., X_{t_n})$ and $(X_{t_1+h}, ..., X_{t_n+h})$ have the same distribution for all $t_1, ..., t_n$ and h.

Assume second order moments exist. Then strict stationarity implies:

a. $m(t) = EX_t = EX_{t+h} = m(t+h)$ for all h. Take $h = -t$.
Then $m(t)$ is a constant:

$$m(t) = m(0)$$

b. (X_{t_1}, X_{t_2}) has the same distribution as (X_{t_1+h}, X_{t_2+h}).
Therefore, for all h,

$$\sigma(t_1, t_2) = \sigma(t_1 + h, t_2 + h)$$

In particular, take $h = -t_1$. Then, only the time shift counts:

$$\sigma(t_1, t_2) = \sigma(0, t_2 - t_1)$$

This motivates the notion of *weakly stationary* processes.

Definition: If $E|x_t|^2 < \infty$, and $EX_t = constant$, and $Cov(X_t, X_s) = \sigma(t - s)$, then X_t is weakly stationary.

Other names: Covariance stationary, $2nd$ order stationary, wide sense stationary.

Fact:
1. A strictly stationary process with finite second order moments is weakly stationary as well.
2. A weakly stationary Gaussian process is also strictly stationary.

We shall be dealing mainly with weakly stationary processes.

2.1 Example of Stationary Processes

1. **An iid sequence** $\{y_t\}$ is strictly stationary. If $Var(y_t) = \sigma^2 < \infty$, then

$$R(v) = Cov(y_t, y_{t+v}) = \begin{cases} \sigma^2 & \text{if } v = 0; \\ 0 & \text{if } v \neq 0. \end{cases}$$

and by LLN

$$\frac{y_1 + \cdots y_n}{n} \xrightarrow{a.s.} m = constant$$

2. Let $Z_t \equiv Z$. Thus, the realizations are straight lines. For $z_1 < \cdots < z_n$,

$$P(Z_{t_1} \leq z_1, ..., Z_{t_n} \leq z_n) = P(Z_{t_1+h} \leq z_1, ..., Z_{t_n+h} \leq z_n) = P(Z \leq z_1)$$

Thus, Z_t is strictly stationary. If $EZ^2 < \infty$ then $EZ = m$, and $R(v) = \sigma^2$ for all v.

$$\frac{Z_1 + \cdots Z_n}{n} = Z$$

which is a random variable. Thus, Z_t is not "ergodic".

3. **A Sinusoid**

$$X_t = A\cos(\omega t) + B\sin(\omega t),\ t = 0, \pm 1, \pm 2, ..., \omega \in [0, \pi].$$

where A, B are constant amplitudes. Then $EX_t \neq EX_s$ for some t, s. It follows that X_t is not stationary.

However, suppose A, B are iid $(0, \sigma^2)$. Then $EX_t = 0$ for all t, and:

$$\begin{aligned} EX_t X_{t+v} &= E[A\cos(\omega t) + B\sin(\omega t)][A\cos(\omega(t+v)) + B\sin(\omega(t+v))] \\ &= \sigma^2[\cos(\omega t)\cos(\omega(t+v) + \sin(\omega t)\sin(\omega(t+v)] \\ &= \sigma^2\cos(\omega v) = R(v) \end{aligned}$$

Therefore, X_t is weakly stationary. If A, B are normal, then X_t is a stationary Gaussian process.

Note: $\cos(A - B) = \cos A \cos B + \sin A \sin B$

4. **Sum of Sinusoids**

$$X_t = \sum_{k=0}^{m} [A_k \cos(\omega_k t) + B_k \sin(\omega_k t)], \quad t = 0, \pm 1, \pm 2 \cdots$$

a. $\omega_0, \omega_1, ..., \omega_m$ are distinct frequencies in $[0, \pi]$.
b. $A_0, ..., A_m, B_0, ..., B_m$ are uncorrelated.
c. $EA_j = EB_j = 0$, $EA_j^2 = EB_j^2 = \sigma_j^2$.

Then $EX_t = 0$, and, doing trig as in the previous example, we see that X_t is weakly stationary:

$$R(v) = EX_t X_{t+v} = \sum_{k=0}^{m} \sigma_k^2 \cos(\omega_k v)$$

Let $\sigma^2 = \sigma_0^2 + \sigma_1^2 + \cdots + \sigma_m^2$, and define $p_k = \sigma_k^2 / \sigma^2$.
Then $\{p_k\}$ is a probability distribution, and:

$$R(v) = \sigma^2 \sum_{k=0}^{m} p_k \cos(\omega_k v) = \sigma^2 \int_0^{\pi} \cos(\omega v) dF(\omega)$$

where $F(\omega)$ is a CDF with jumps of size p_k at ω_k.

Does $R(v)$ have such a representation in general? As we shall see the answer is "yes".

5. Stationary Process on a Circle
Let U, V be iid N(0,1), and let $T = [0, 2\pi]$. For $t \in T$, define the BIVARIATE process

$$X(t) = (Y(t), Z(t))$$

Where
$Y(t) = U \sin t + V \cos t$
$Z(t) = -U \cos t + V \sin t$
Then
$EY(t) = EZ(t) = 0, \quad EY^2(t) = EZ^2(t) = 1$
and in particular

$$EY(t)Z(t) = \sin t \cos t - \sin t \cos t = 0$$

Therefore $Y(t), Z(t)$ are independent since they are jointly normal.

$EY(t)Y(t+v) = \cos(v)$
$EZ(t)Z(t+v) = \cos(v)$
$EZ(t)Y(t+v) = \sin(-v)$
We can check that $(X(t_1), ..., X(t_n))$ and $(X(t_1 + h), ..., X(t_n + h))$ have the same multivariate normal distribution, and hence $X(t)$ is a strictly stationary

bivariate process, as well a weakly stationary.

5. *Random Telegraph*
HW1:
$P(X(t)=1)=P(X(t)=-1)=1/2,\ 0\le t<\infty$. The times at which sign changes occur constitute a Poisson process with parameter λ: $\{N(t),\ t\ge 0\}$,

$$p(k)=P(N(t)=k)=\frac{e^{-\lambda t}(\lambda t)^k}{k!}$$

Show: $X(t)$ is weakly stationary with covariance function $R(u)=e^{-2\lambda|u|}$.

Recall $N(t)$ has independent increments.

6. *Random Sinusoid*
$X_t=\sin(2\pi\alpha t)$, $\alpha\sim Unif(0,1)$, $t=1,2,....$

$$EX_t=\int_0^1\sin(2\pi\alpha t)d\alpha=0$$

$$EX_sX_t=\int_0^1\sin(2\pi\alpha s)\sin(2\pi\alpha t)d\alpha=\frac{1}{2}\delta_{s-t}$$

Thus, $X(t)$ is weakly stationary but NOT strictly stationary; e.g. X_1 and X_2 have different distributions.

7. **Random Walk**
$Z_t=Z_{t-1}+u_t$, u_t WN, Assume $Z_0=0$ w.p. 1. Then

$$Z_t=\sum_{j=1}^{t}u_j$$

Therefore $EZ_t=0$, and $Var(Z_t)=\sigma^2t$. Hence Z_t is not weakly stationary. But what is??? The answer brought about a watershed in TSA.

8. **AR(1)**
HW1:
Consider the stochastic difference equation

$$z_t=\phi z_{t-1}+\epsilon_t,\quad t=0,\pm1,\pm2,....$$

ϵ_t is WN.

If $|\phi|<1$, is z_t stationary?
If $|\phi|>1$, is z_t stationary?

Better ask: When does AR(1) have a stationary solution?

9. **MA(1)**

$$z_t = u_t + \theta u_{t-1}, \quad t = 0, \pm 1, \pm 2, ...$$

u_t is WN. $Ez_t = 0$, $\rho(0) = 1$, and:

$$\rho(k) = R(k)/R(0) = \begin{cases} \frac{\theta}{1+\theta^2}, & \text{if } k = \pm 1; \\ 0, & \text{if } |k| > 1. \end{cases}$$

9. **MA(q)**
$z_t = u_t + \theta_1 u_{t-1} + \cdots + \theta_q u_{t-q}, \quad t = 0, \pm 1, \pm 2, ..., u_t$ WN.

$$\rho(k) = R(k)/R(0) = \begin{cases} \frac{\sum_{i=0}^{q-k} \theta_i \theta_{i+k}}{\sum_{i=0}^{q} \theta_i^2}, & \text{if } k = 0, \pm 1, ..., \pm q; \\ 0, & \text{if } |k| > q. \end{cases}$$

10. **Cointegration** (Nobel laureates Robert Engle and Clive Granger, in 1987). Two time series X_t and Y_t are cointegrated if a linear combination $aX_t + bY_t$ is stationary. For example, assume X_t and Y_t are not stationary, but

$$X_t = \beta Y_t + \epsilon_t$$

where ϵ_t is WN. Hence

$$X_t - \beta Y_t = \epsilon_t$$

IS stationary.

2.2 Complex Valued Processes

$\{X_t\}$, $\{Y_t\}$ are real-valued. Define a complex-valued process $\{Z_t\}$ by

$$Z_t = X_t + iY_t$$

1. $EZ_t = EX_t + iEY_t$
2. $\overline{EZ_t} = E\bar{Z}_t = EX_t - iEY_t$
3. $Cov(Z_t, Z_{t+h}) \equiv E(Z_t - EZ_t)(\bar{Z}_{t+h} - E(\bar{Z}_{t+h}))$
4. In weakly stationary case with mean 0, $\overline{R_z(h)} = R_z(-h)$

Example: Sum of Complex Sinusoids
$EZ_j = 0$, $E|Z_j|^2 = F_j < \infty$, $EZ_j\bar{Z}_k = 0, \ j \neq k$
Define

$$X_t = \sum_{j=1}^{m} \exp(i\lambda_j t) Z_j$$

Then $EX_t = 0$, and

$$EX_t\bar{X}_s = \sum_{j=1}^{m} \exp(i\lambda_j (t-s)) F_j$$

Hence, X_t is weakly stationary.

This is an important example as it points to a relationship between a covariance function and a “spectral distribution”. Do you think this implies a spectral representation for the process?