TS3: Mean Square Convergence

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Definition: $X_n \to X$ in mean square if $E(X_n - X)^2 \to 0$

Most of our modes of convergence will be in mean square.

Fact: $X_n \to X$ iff $E(X_m - X_n)^2 \to 0$, $m, n \to \infty$.

Proof: $\Leftarrow$
$P(|X_m - X_n| > \epsilon) \leq \frac{E(X_m - X_n)^2}{\epsilon^2} \to 0, \ m, n \to \infty$
Therefore $\exists$ X such that $X_n \xrightarrow{p} X$, $n \to \infty$
Therefore $\exists$ a subsequence X_{n_k} such that $X_{n_k} \to X$, w.p. 1, $k \to \infty$
Therefore for a fixed m, $X_{n_k} - X_m \to X - X_m$, w.p. 1, $k \to \infty$
Therefore, by Fatou's lemma (see below)

$$E(X - X_m)^2 \leq \liminf_{k \to \infty} E(Xn_k - X_m)^2 \to 0, \ n_k, m \to \infty$$

$\Rightarrow$
By Cauchy-Schawrtz inequality,

$$E(X_m - X_n)^2 = E((X_m - X) - (X_n - X))^2$$
$$\leq E(X_n - X)^2 + 2\sqrt{E(X_m - X)^2 E(X_n - X)^2} + E(X_m - X)^2 \to 0$$

Fatou's Lemma:
For $f_n > 0$,

$$\int \liminf f_n d\mu \leq \liminf \int f_n d\mu$$

Fact: If $X_n \to X$, $Y_n \to Y$ then $EX_nY_n \to EXY$

Corollary:
$EX_n \to EX$
$EX_n^2 \to EX^2$
$EXY_n \to EXY$

Fact: $X_n \to X$ iff $EX_mX_n \to c$, $c > 0$ is a finite constant.

Note: In case of complex rv's

$E|X|^2 = EX\bar{X}$
$E|X_m - X_n|^2 = E(X_m - X_n)\overline{(X_m - X_n)}$