Spectral Density and Distribution Functions

Let $X_t, t = 0, \pm 1, \pm 2, ...$ be a real stationary (second order) process with $\mathrm{E(X_t)} = 0$, $R(k) = \mathrm{Cov(X_t X_{t+k})}$. Assume $R(0) = 1$. Then:

a. $R(k)$ is symmetric and non-negative definite: For real $\alpha_1, ..., \alpha_n$

$$\sum_{i=1}^{n} \sum_{j=1}^{n} \alpha_i \alpha_j R(i-j) \geq 0.$$

b. There is a probability distribution F on $[-\pi, \pi]$ such that

$$R(k) = \int_{-\pi}^{\pi} \cos(k\omega) dF(\omega), \quad k = 0, \pm 1, \pm 2, ...$$

Proof of (a):

$$0 \leq \mathrm{E} \left| \sum_{\mathrm{i=1}}^{\mathrm{n}} \alpha_{\mathrm{i}} \mathrm{X_{t+i}} \right|^2 = \sum_{\mathrm{i=1}}^{\mathrm{n}} \sum_{\mathrm{j=1}}^{\mathrm{n}} \alpha_{\mathrm{i}} \alpha_{\mathrm{j}} \mathrm{E(X_{t+i} X_{t+j})} = \sum_{\mathrm{i=1}}^{\mathrm{n}} \sum_{\mathrm{j=1}}^{\mathrm{n}} \alpha_{\mathrm{i}} \alpha_{\mathrm{j}} \mathrm{R(i-j)}.$$

Proof of (b):
Let $\alpha_i = \cos(i\omega)$, $\omega \in [-\pi, \pi]$. Then from (a) we have,

$$0 \leq \sum_{i=1}^{n} \sum_{j=1}^{n} R(i-j) \cos(i\omega) \cos(j\omega), \quad \omega \in [-\pi, \pi].$$

Similarly, for $\alpha_i = \sin(i\omega)$,

$$0 \leq \sum_{i=1}^{n} \sum_{j=1}^{n} R(i-j) \sin(i\omega) \sin(j\omega), \quad \omega \in [-\pi, \pi].$$

Summing the last two expressions we get,

$$\begin{aligned} 0 &\leq \frac{1}{2\pi n}\sum_{i=1}^{n}\sum_{j=1}^{n} R(i-j)[\cos(i\omega)\cos(j\omega)+\sin(i\omega)\sin(j\omega)] \\ &= \frac{1}{2\pi n}\sum_{i=1}^{n}\sum_{j=1}^{n} R(i-j)\cos((i-j)\omega) \\ &= \frac{1}{2\pi n}[nR(0)\cos((0)\omega)+2(n-1)R(1)\cos((1)\omega)+2(n-2)R(2)\cos((2)\omega)+ \\ &\qquad \cdots + 2R(n-1)\cos((n-1)\omega] \\ &= \frac{1}{2\pi n}\sum_{k=-(n-1)}^{n-1}(n-|k|)R(k)\cos(k\omega) \equiv f_n(\omega), \quad \omega\in[-\pi,\pi]. \end{aligned}$$

Now, $f_n(\omega) \geq 0$, and since

$$\int_{-\pi}^{\pi}\cos(k\omega)d\omega = \begin{cases} 2\pi, & \text{if } k=0 \\ 0, & \text{if } k \neq 0 \end{cases}$$

we have

$$\begin{aligned} \int_{-\pi}^{\pi} f_n(\omega)d\omega &= \frac{1}{2\pi n}\sum_{k=-(n-1)}^{n-1}(n-|k|)R(k)\int_{-\pi}^{\pi}\cos(k\omega)d\omega \\ &= \frac{1}{2\pi n} nR(0)\times 2\pi = R(0) = 1 \end{aligned}$$

Therefore,

- $f_n(\omega)$ is a pdf on $[-\pi,\pi]$
- $f_n(\omega) = f_n(-\omega)$
- $f_n(\omega)$ is continuous.

Let $F_n(\omega)$ be the cdf corresponding to $f_n(\omega)$,

$$F_n(\omega) = \int_{-\pi}^{\omega} f_n(\lambda) d\lambda,$$

and observe that,

$$\int_{-\pi}^{\pi} \cos(u\omega)\cos(v\omega) d\omega = \begin{cases} 2\pi, & \text{if } u = v \\ 0, & \text{if } u \neq v \end{cases}$$

Then,

$$\begin{aligned} \int_{-\pi}^{\pi} \cos(u\omega) dF_n(\omega) &= \int_{-\pi}^{\pi} \cos(u\omega) f_n(\omega) d\omega \\ &= \int_{-\pi}^{\pi} \cos(u\omega) \left[\frac{1}{2\pi n} \sum_{v=-(n-1)}^{n-1} (n - |v|) R(v) \cos(v\omega) \right] d\omega \\ &= \left(1 - \frac{|u|}{n}\right) R(u). \end{aligned}$$

Recall now the Helly-Bray lemma: There exists a distribution function F and a subsequence $\{F_{n_k}\}$ such that for any bounded continuous function h,

$$\lim_{k\to\infty}\int_{-\pi}^{\pi} h(\omega)dF_{n_k}(\omega) = \int_{-\pi}^{\pi} h(\omega)dF(\omega)$$

In particular, take $h(\omega) = \cos(u\omega)$, then

$$\lim_{k\to\infty}\int_{-\pi}^{\pi} \cos(u\omega)dF_{n_k}(\omega) = \int_{-\pi}^{\pi} \cos(u\omega)dF(\omega) = \lim_{k\to\infty}\left(1 - \frac{|u|}{n_k}\right) R(u).$$

It folows that,

$$R(u) = \int_{-\pi}^{\pi} \cos(u\omega)dF(\omega), \quad u = 0, \pm 1, \pm 2, ...$$

When F is absolutely continuous we have by symmetry

$$(\star) \quad R(u) = \int_{-\pi}^{\pi} \cos(u\omega)f(\omega)d\omega = \int_{-\pi}^{\pi} e^{iu\omega} f(\omega)d\omega, \quad u = 0, \pm 1, \pm 2, ...$$

and f is referred to as the *spectral density* of $\{X_t\}$.

Note: We could have defined

$$f_n(\omega) = \sum_i \sum_j R(i-j)[\alpha_i(\omega)\alpha_j(\omega) + \beta_i(\omega)\beta_j(\omega)]$$

for some α's and β's, but the choice of cosines and sines is useful due to their orthogonality relationships.

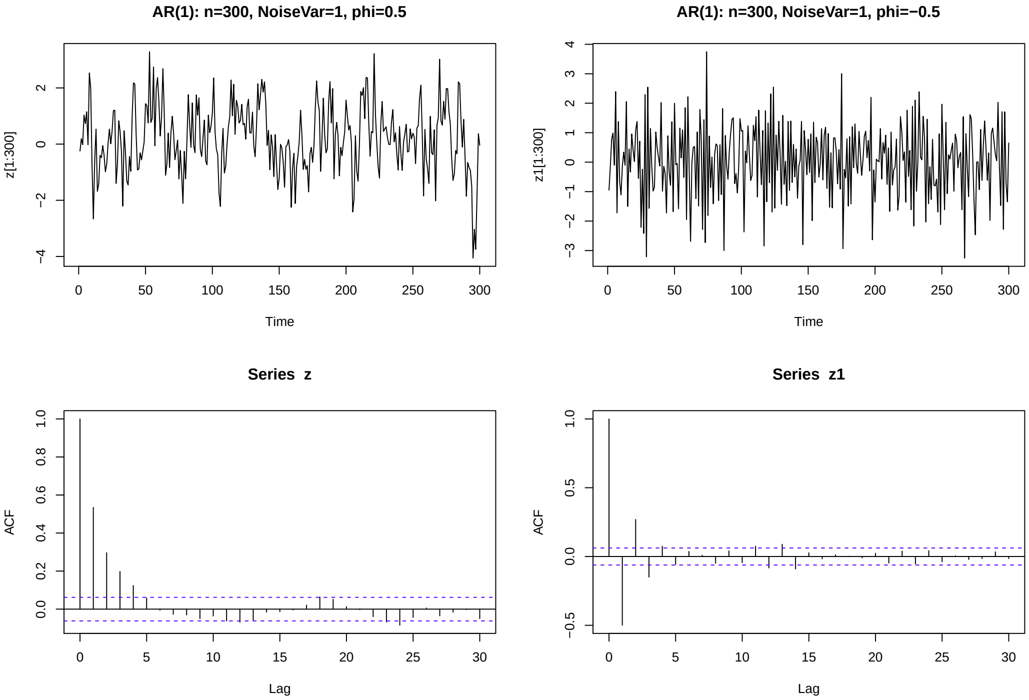

Figure 1: AR(1): $X_t = \phi X_{t-1} + \epsilon_t$, $\phi = \pm 0.5$, $\epsilon_t \sim N(0, 1)$.

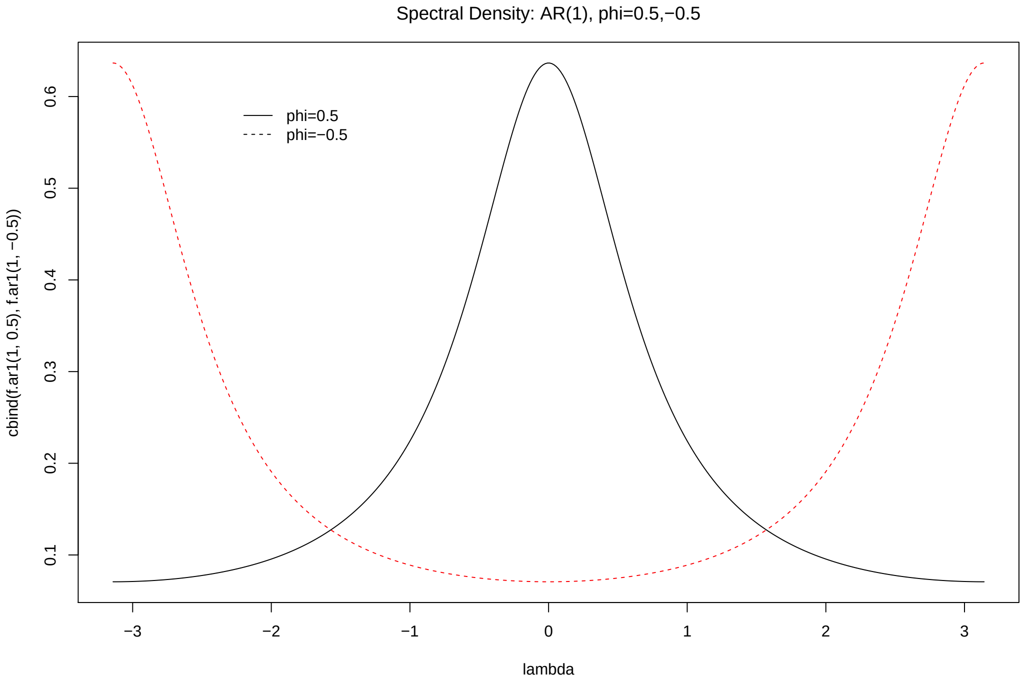

Figure 2: AR(1): $X_t = \phi X_{t-1} + \epsilon_t$, $\phi = \pm 0.5$, $\epsilon_t \sim N(0, 1)$.