TS 4: Spectral Representation

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We'll show that a stationary process is essentially a superposition of random harmonic oscillations.

1 Random Orthogonal Set Function

Let Λ be an interval of λ points. Let $\mathcal{B}$ be the Borel field of subintervals, and on this field define a measure F. We define a stochastic process $Z(\Delta) = Z(\Delta, \omega)$, $\Delta \in \mathcal{B}$, $\omega \in \Omega$, such that,

1. $Z(\Delta_1 \cup \Delta_2) = Z(\Delta_1) + Z(\Delta_2),\ \Delta_1 \cap \Delta_2 = \emptyset$
2. $EZ(\Delta) = 0,\ \Delta \in \mathcal{B}$
3. $EZ(\Delta_1)\overline{Z(\Delta_2)} = F(\Delta_1 \cap \Delta_2)$

Therefore,

1. When $\Delta_1 \cap \Delta_2 = \emptyset$ then

$$EZ(\Delta_1)\overline{Z(\Delta_2)} = F(\Delta_1 \cap \Delta_2) = F(\emptyset) = 0$$

2. $E|Z(\Delta)|^2 = F(\Delta)$
3. For disjoint $\Delta_1, ..., \Delta_n$ such that $\cup_{j=1}^n \Delta_j = \Lambda$,

$$Z(\Lambda) = Z(\cup_{k=1}^n \Delta_j) = \sum_{k=1}^n Z(\Delta_k)$$

Thus, Z is an orthogonal random measure.

2 Definition of a Stochastic Integral

Let $f(\lambda)$ be a complex valued simple function define over disjoint intervals Δ_k:

$$f(\lambda) = \varphi_k,\ \lambda \in \Delta_k,\ k = 1, 2, ...,\ \cup\Delta_k = \Lambda.$$

We say $f(\lambda)$ is integrable if

$$\sum_{k=1}^{\infty} \varphi_k Z(\Delta_k)$$

converges in mean square. In that case we write:

$$\int_\Lambda f(\lambda) Z(d\lambda) = \sum_{k=1}^{\infty} \varphi_k Z(\Delta_k)$$

Therefore,

$$\begin{aligned} E|\int_\Lambda f(\lambda)Z(d\lambda)|^2 &= E|\sum_{k=1}^{\infty} \varphi_k Z(\Delta_k)|^2 \\ &= E\sum_{k=1}^{\infty} \varphi_k Z(\Delta_k) \sum_{j=1}^{\infty} \overline{\varphi_j} \overline{Z(\Delta_j)} \\ &= \sum_{k=1}^{\infty} |\varphi_k|^2 E|Z(\Delta_k)|^2 \\ &= \sum_{k=1}^{\infty} |\varphi_k|^2 F(\Delta_k) \\ &\equiv \int_\Lambda |f(\lambda)|^2 F(d\lambda) \end{aligned}$$

We see the integral exists iff $\int_\Lambda |f(\lambda)|^2 F(d\lambda) < \infty$

(*) In the previous manipulation we used that fact $X_n \to X$, $Y_n \to Y$ then $EX_nY_n \to EXY$.

2.1 An Extension

Now let $\{f_n\}$ be a sequence of integrable simple functions. Assume there is f such that

$$\int_\Lambda |f_n(\lambda) - f(\lambda)|^2 F(d\lambda) \to 0, \; n \to \infty$$

That is, $f_n \to f$ in mean square. Therefore,

$$\int_\Lambda |f_n(\lambda) - f_m(\lambda)|^2 F(d\lambda) \to 0, \; m, n \to \infty$$

and

$$\begin{aligned} E|\int_\Lambda f_n(\lambda)Z(d\lambda) - \int_\Lambda f_m(\lambda)Z(d\lambda)|^2 &= E|\int_\Lambda (f_n(\lambda) - f_m(\lambda))Z(d\lambda)|^2 \\ = \int_\Lambda |f_n(\lambda) - f_m(\lambda)|^2 F(d\lambda) \to 0, \; m, n \to \infty \end{aligned}$$

Hence, we see that the sequence $\{\int_\Lambda f_n(\lambda)Z(d\lambda)\}$ has a limit in mean square! We denote this limit by

$$\int_\Lambda f(\lambda)Z(d\lambda)$$

In particular, if $\int |f|^2 F(d\lambda) < \infty$ and $\int |g|^2 F(d\lambda) < \infty$ then

1. $E\int_\Lambda f(\lambda)Z(d\lambda)\overline{\int_\Lambda g(\omega)Z(d\omega)} = \int_\Lambda f(\lambda)\overline{g(\lambda)}F(d\lambda)$
which follows from (*).

2. $\int_\Lambda [af(\lambda) + bg(\lambda)]Z(d\lambda) = a\int_\Lambda f(\lambda)Z(d\lambda) + b\int_\Lambda g(\lambda)Z(d\lambda)$

3 The Spectral Representation of Weakly Stationary Processes

Assume a zero-mean complex process X_t admits the representation,

$$(\star)\ X_t = \int_\Lambda f(t,\lambda)Z(d\lambda),\ t = 0, \pm 1, ...$$

such that $\int_\Lambda |f(t,\lambda)|^2 F(d\lambda) < \infty$ for all t. Then

$$EX_s\bar{X}_t = R(s,t) = E\int_\Lambda f(s,\lambda)Z(d\lambda)\overline{\int_\Lambda f(t,\omega)Z(d\omega)} =$$
$$(\star\star)\ \int_\Lambda f(s,\lambda)\overline{f(t,\lambda)}F(d\lambda)$$

Thus, if X_t admits the representation $(\star)$ then its covariance function has the representation $(\star\star)$. Is it also true that $(\star\star)$ implies $(\star)$?

Fact: (Cramer 1951, Grenander and Rosenblatt 1957). $(\star\star)$ implies $(\star)$ if the family $\{f(t,\lambda),\ t = 0, \pm 1, ...\}$ forms a basis for $L_2(\Lambda) = \{f : \int_\Lambda |f|^2 F(d\lambda) < \infty\}$.

That is, every function in $L_2(\Lambda)$ can be represented in mean square F by $\sum \nu a_\nu f(t_\nu, \lambda)$.

So, if X_t is stationary then

$$EX_s\bar{X}_t = \int e^{i(s-t)\lambda}F(d\lambda) = \int e^{is\lambda}\overline{e^{it\lambda}}F(d\lambda)$$

therefore,

$$X_t = \int e^{it\lambda}Z(d\lambda)$$

where $E|Z(d\lambda)|^2 = F(d\lambda)$.

3.1 Proof of Spectral Representation.

We follow Cramér (1951) as given in Koopmans (1974).

Definition: A Hilbert space is a vector space with an inner product, which contains limits in mean square.

If x, y are random elements of a Hilbert space, we take $Ex\bar{y}$ as the inner product.

An example of an infinite-dimensional Hilbert space is $L_2(F)$, the set of all complex-valued functions such that the integral of $|f|^2$ over the whole real line w.r.t. F is finite. In this case, the inner product is

$$< f, g >= \int_{-\infty}^{\infty} f(x)\overline{g(x)}F(dx)$$

A technical note:
If $E|x|^2 = Ex\bar{x} = 0$, it does not follow that $x(\omega) = 0$ for all ω, it only means that $P(x = 0) = 1$. Also, we say that x, y are equivalent if $P(x = y) = 1$. This creates equivalent classes such that any two random variables in the same equivalent class are equal w.p. 1.

Let $\{X_t\}$, $-\infty < t < \infty$ be a complex-valued weakly stationary process on $(\Omega, \mathcal{F}, \mathcal{P})$, where

$$EX_t = 0, \quad (*)\ EX_t\overline{X}_s = R(t-s) = \int_{-\infty}^{\infty} e^{i\lambda(t-s)}F(d\lambda)$$

We now form tow Hilbert spaces.
1. $\mathcal{M}_{\mathcal{X}}$ of all linear combinations

$$\sum_j a_j X_{t_j},\ a_j\ complex\ coefficients$$

together with their limits in mean square.

2. $L_2(F)$ of all complex-valued functions $g(\lambda)$ such that

$$\int_{-\infty}^{\infty} |g(\lambda)|^2 F(d\lambda) < \infty$$

We can establish a correspondence $\mathcal{M}_{\mathcal{X}}$ and $L_2(F)$ as follows:

$$X_t \longleftrightarrow e^{i\lambda t}$$

$$\sum_j a_j X_{t_j} \longleftrightarrow \sum_j a_j e^{i\lambda t_j}$$

The correspondence preserves the inner products for the generating elements:

$$EX_t\overline{X_s} = \int e^{i\lambda(t-s)}F(d\lambda) = \int e^{i\lambda t}\overline{e^{i\lambda s}}F(d\lambda)$$

Using linear combinations and passage to the limit, this property is extended for $h(\lambda), g(\lambda) \in L_2(F)$ and $G, H \in \mathcal{M}_{\mathcal{X}}$ such that

$$G \longleftrightarrow g, \quad H \longleftrightarrow h$$

Then

$$EG\overline{H} = \int g(\lambda)\overline{h(\lambda)}F(d\lambda)$$

Now, let A be a Borel set and I_A its indicator:

$$I_A(\lambda) = \begin{cases} 1, & \text{if } \lambda \in A; \\ 0, & \text{if } \lambda \notin A. \end{cases}$$

Since X_t is stationary, $Var(X_t) < \infty$. Therefore $I_A \in L_2(F)$. Then $\exists$ a random variable $Z(A) \in \mathcal{M}_{\mathcal{X}}$ such that,

$$(\clubsuit)\ Z(A) \longleftrightarrow I_A$$

and

$$EZ(A)\overline{Z(B)} = \int I_A(\lambda)\overline{I_B(\lambda)}F(d\lambda) = F(A \cap B)$$

Therefore $Z(A)$ is an orthogonal set function!!

Now, if $g(\lambda) \in L_2(F)$ then it can be represented as a limit in mean square of

$$\sum_j a_j I_{A_j}(\lambda),\ A_j \cap A_k = \emptyset,\ j \neq k$$

Hence, by the correspondence $\clubsuit$,

$$\sum_j a_j I_{A_j}(\lambda) \to g(\lambda) \longleftrightarrow \sum_j a_j Z(A_j) \to \int g(\lambda)Z(d\lambda)$$

So,

$$g(\lambda) \longleftrightarrow \int g(\lambda)Z(d\lambda)$$

Therefore for $g(\lambda) = e^{it\lambda}$ we have

$$e^{it\lambda} \longleftrightarrow \int e^{it\lambda}Z(d\lambda)$$

But also

$$e^{it\lambda} \longleftrightarrow X_t$$

Therefore, finally, we obtain the spectral representation of X_t:

$$\boxed{X_t = \int e^{it\lambda}Z(d\lambda)}$$