TS Mean Estimation

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March 2021

A Note about Summation

Recall $R_k = R_{-k}$. Therefore:

$$
\begin{aligned}
\sum_{s=1}^{N}\sum_{t=1}^{N} R_{s-t} = NR_0 \quad & + \quad (N-1)R_1 + (N-1)R_{-1} \\
& + \quad (N-2)R_2 + (N-2)R_{-2} \\
& + \quad (N-3)R_3 + (N-3)R_{-3} \\
& . \\
& . \\
& . \\
& + \quad [N-(N-1)]R_{N-1} + [N-(N-1)]R_{-(N-1)} \\
& = \quad \sum_{r=-(N-1)}^{N-1} (N-|r|)R_r
\end{aligned}
$$

Therefore

$$
\frac{1}{N}\sum_{s=1}^{N}\sum_{t=1}^{N} R_{s-t} = \sum_{r=-(N-1)}^{N-1}\left(1-\frac{|r|}{N}\right)R_r
$$

Therefore

$$
\begin{aligned}
\frac{1}{N}\sum_{s=1}^{N}\sum_{t=1}^{N} e^{i\lambda(s-t)} &= \sum_{r=-(N-1)}^{N-1}\left(1-\frac{|r|}{N}\right)e^{i\lambda r} \\
&= \frac{1}{N}\left|\sum_{t=1}^{N} e^{it\lambda}\right|^2
\end{aligned}
$$

and observe that

$$
\sum_{t=1}^{N} e^{it\lambda} = e^{i\lambda}\frac{1-e^{i\lambda N}}{1-e^{i\lambda}} = e^{i\lambda}\frac{e^{i\lambda N/2}e^{-i\lambda N/2}-e^{i\lambda N}}{e^{i\lambda/2}e^{-i\lambda/2}-e^{i\lambda}} = e^{i\frac{1}{2}\lambda(N+1)}\frac{\sin\frac{1}{2}\lambda N}{\sin\frac{1}{2}\lambda}
$$

It follows that

$$\left|\sum_{t=1}^{N} e^{it\lambda}\right|^2 = \frac{\sin^2 \frac{1}{2}\lambda N}{\sin^2 \frac{1}{2}\lambda}$$

On Variance of a Time Series Average

Given a zero-mean stationary time series $z_1, ..., z_N$, a natural estimate of the mean (here 0) is

$$\bar{Z} = \frac{1}{N}\sum_{i=1}^{N} Z_i$$

Clearly $E\bar{Z} = 0$, and

$$\begin{aligned} Var(\bar{Z}) &= \frac{1}{N^2}\sum_{i=1}^{N}\sum_{j=1}^{N} EZ_iZ_j = \frac{1}{N^2}\sum_{i=1}^{N}\sum_{j=1}^{N} R_{i-j} \\ &= \frac{1}{N^2}\sum_{r=-(N-1)}^{N-1}(N-|r|)R_r = \frac{1}{N}\sum_{r=-(N-1)}^{N-1}\left(1-\frac{|r|}{N}\right)R_r \end{aligned}$$

Fact: Cesáro summation of a convergent sum is the sum: If $\sum_{r=1}^{\infty} a_r < \infty$, then

$$\lim_{N\to\infty}\sum_{r=1}^{N-1}\left(1-\frac{r}{N}\right)a_r = \sum_{r=1}^{\infty} a_r$$

It follows that if

$$\sum_{r=-\infty}^{\infty} R_r < \infty$$

then

$$N\ Var(\bar{Z}) \to \sum_{r=-\infty}^{\infty} R_r$$

Fact: If $f(\lambda)$ is continuous then

$$\lim_{N\to\infty} N\ Var(\bar{Z}) = 2\pi f(0)$$

Proof

$$\begin{aligned} N\ Var(\bar{Z}) &= \sum_{r=-(N-1)}^{N-1}\left(1-\frac{|r|}{N}\right)R_r = \sum_{r=-(N-1)}^{N-1}\left(1-\frac{|r|}{N}\right)\int_{-\pi}^{\pi} e^{ir\lambda} f(\lambda)d\lambda \\ &= \int_{-\pi}^{\pi}\frac{1}{N}\frac{\sin^2\frac{1}{2}\lambda N}{\sin^2\frac{1}{2}\lambda} f(\lambda)d\lambda \end{aligned}$$

Define the *Fejér kernel*

$$W_N(\lambda) = \frac{1}{2\pi N}\frac{\sin^2\frac{1}{2}\lambda N}{\sin^2\frac{1}{2}\lambda}$$

Then

$$\begin{aligned} \int_{-a}^{a} W_N(\lambda)d\lambda &= \int_{-a}^{a}\frac{1}{2\pi N}\frac{\sin^2\frac{1}{2}\lambda N}{\sin^2\frac{1}{2}\lambda}d\lambda \\ &= \int_{-a}^{a}\frac{1}{2\pi N}\sum_{s=1}^{N}\sum_{t=1}^{N} e^{i\lambda(s-t)}d\lambda \\ &= \int_{-a}^{a}\frac{1}{2\pi}\sum_{r=-(N-1)}^{N-1}\left(1-\frac{|r|}{N}\right)\cos(\lambda r)d\lambda \\ &= \frac{1}{\pi}\left[a + 2\sum_{r=1}^{N-1}\frac{N-r}{Nr}\sin(ar)\right] \end{aligned}$$

Therefore, with $a=\pi$

$$\int_{-\pi}^{\pi} W_N(\lambda)d\lambda = 1$$

Since $|\sin x| \le 1$

$$W_N(\lambda) = \frac{1}{2\pi N}\frac{\sin^2\frac{1}{2}\lambda N}{\sin^2\frac{1}{2}\lambda} \le \frac{1}{2\pi N\sin^2\frac{1}{2}\lambda} \to 0, \quad \lambda \neq 0$$

More precisely, since $\sin\frac{1}{2}\lambda$ is increasing in $[0,\pi)$, we have for $a>0$

$$\int_{a}^{\pi} W_N(\lambda)d\lambda \le \int_{a}^{\pi}\frac{d\lambda}{2\pi N\sin^2\frac{1}{2}a} = \frac{\pi - a}{2\pi N\sin^2\frac{1}{2}a} \to 0,\ N\to\infty$$

By symmetry we have:

$$\lim_{N\to\infty}\int_{-\pi}^{-a} W_N(\lambda)d\lambda = \lim_{N\to\infty}\int_{a}^{\pi} W_N(\lambda)d\lambda = 0,\ 0 < a \le \pi$$

So, we have

$$1 = \int_{-\pi}^{\pi} W_N(\lambda) d\lambda = \int_{-\pi}^{-a} + \int_{-a}^{a} + \int_{a}^{\pi} \rightarrow \int_{-a}^{a}$$

Now, by continuity of $f(\lambda)$ at 0, we have for $\delta > 0$:

$$\left| \int_{-\delta}^{\delta} f(\lambda) W_N(\lambda) d\lambda - f(0) \int_{-\delta}^{\delta} W_N(\lambda) d\lambda \right| = \left| \int_{-\delta}^{\delta} [f(\lambda) - f(0)] W_N(\lambda) d\lambda \right|$$

$$\leq \int_{-\delta}^{\delta} |f(\lambda) - f(0)| \, W_N(\lambda) d\lambda \leq \epsilon \int_{-\delta}^{\delta} W_N(\lambda) d\lambda \rightarrow \epsilon$$

and by continuity of $f(\lambda)$

$$\left| \int_{-\pi}^{-\delta} f(\lambda) W_N(\lambda) d\lambda + \int_{\delta}^{\pi} f(\lambda) W_N(\lambda) d\lambda \right| \leq K \left| \int_{-\pi}^{-\delta} W_N(\lambda) d\lambda + \int_{\delta}^{\pi} W_N(\lambda) d\lambda \right| \rightarrow 0$$

Therefore, as N increases

$$N \; Var(\bar{Z}) = 2\pi \int_{-\pi}^{\pi} W_N(\lambda) f(\lambda) d\lambda \rightarrow 2\pi f(0)$$

Thus, we have shown rigorously that $W_N(\lambda)$ approaches a Dirac delta function as $N \rightarrow \infty$.