# MATH 734, SPRING 2024: FINAL EXAM

The exam contains 5 regular problems and 3 bonus problems. You may use notes and textbooks during the exam. Collaborations, internet resources, and electronic devices are not allowed.

Theorems from the textbook and results we proved in class can be cited without proof.

**Problem 1. (40 pts)** Assume $M$ is a compact, connected, oriented $n$-dimensional manifold with a non-empty boundary.

(1) Show that $H_c^i(M - \partial M) \cong H^i(M, \partial M)$ for all $i$.
(2) Show that $H_n(M, \partial M; \mathbb{Z}) \cong \mathbb{Z}$.
(3) Show that the boundary map $H_n(M, \partial M; \mathbb{Z}) \to H_{n-1}(\partial M; \mathbb{Z})$ is injective.
(4) Show that there is no retraction from $M$ to $\partial M$.

For this problem, you may use the fact (or take it as an additional assumption) that $M$ is homeomorphic to the manifold obtained by gluing $M$ with $[0,1] \times \partial M$ via identifying $\partial M \subset M$ with $\{0\} \times \partial M \subset [0,1] \times \partial M$. (This is called the collar neighborhood theorem.)

**Problem 2. (15 pts)** Assume $S^3 \to E \to S^5$ is a fibration, compute $H_3(E; \mathbb{Z})$.

Hint: You may need the Hurewicz theorem.

**Problem 3. (40 pts)** Suppose $m, n$ are non-negative integers. Assume $X$ and $Y$ are connected CW complexes such that $\pi_i(X) = 0$ for all $i = 1, \ldots, m$ and $\pi_i(Y) = 0$ for all $i = 1, \ldots, n$.

(1) Show that
$$\pi_i(X \vee Y) \cong \pi_i(X) \times \pi_i(Y)$$
for all $i = 1, \ldots, m + n$.
(2) Find an example of $m, n, X, Y$ satisfying the conditions above such that $m + n \geq 1$ and
$$\pi_{m+n+1}(X \vee Y) \ncong \pi_{m+n+1}(X) \times \pi_{m+n+1}(Y).$$

**Problem 4. (25 pts)** Suppose $G$ is an abelian group, $N$ is a subgroup of $G$, and $H = G/N$. Assume $n$ is a positive integer. Show that there is a fibration $K(G, n) \to K(H, n)$ such that the fiber is (weakly homotopy equivalent to) $K(N, n)$.

**Problem 5. (30 pts)**

(1) Construct a CW complex $X$ that is an Eilenberg–MacLane space of type $K(\mathbb{Z}/2, 1)$. Compute the homology groups of $X$ in $\mathbb{Z}/2$ coefficients.
(2) Assume $G$ is an abelian group that contains a subgroup isomorphic to $\mathbb{Z}/2$, show that $K(G, 1)$ cannot be realized by any finite-dimensional manifold.

**Bonus Problem 1. (30 pts)** Generalize Problem 5 from $\mathbb{Z}/2$ to $\mathbb{Z}/n$.

**Bonus Problem 2. (30 pts)** Assume $n \geq 2$. Construct a path connected subset $X$ of $\mathbb{R}^n$ such that $\pi_{n-1}(X)$ is uncountable. Prove the desired property.

**Bonus Problem 3. (30 pts)** Let $f : S^2 \to S^2$ be a map of degree 2. Compute the cohomology groups of the homotopy fiber of $f$.

Hint: You need the Serre spectral sequence for Bonus Problem 3.