Calculus 141, section 8.4 Partial Fractions, Group Work example

Evaluate $\int \frac{-2x+4}{(x^2+1)(x-1)^2}\,dx$.

(This integral has elements in common with Examples 7 and 8 in section 8.4 of your text.)

i) The degree of the numerator is already less than the degree of the denominator, so no synthetic (long) division is needed.

ii) Both numerator and denominator are already factored into linear and irreducible quadratic factors.

iii) Set up the partial fraction decomposition.

$$\frac{-2x+4}{(x^2+1)(x-1)^2} = \frac{Ax+B}{x^2+1} + \frac{C}{x-1} + \frac{D}{(x-1)^2}$$

Note that we needed ($Ax + B$) to have a numerator with degree one less than the degree of the denominator.
Note also that we needed to have both (x – 1) and $(x-1)^2$ as denominators in our partial fraction decomposition.

Multiply both sides by the lowest common dominator $(x^2+1)(x-1)^2$ and cancel.

$$-2x+4 = (Ax+B)(x-1)^2 + C(x^2+1)(x-1) + D(x^2+1)$$

$$-2x+4 = Ax(x^2-2x+1) + B(x^2-2x+1) + C(x^3-x^2+x-1) + D(x^2+1)$$

$$\begin{aligned} -2x+4 = {} & Ax^3 - 2Ax^2 + Ax \\ & + Bx^2 - 2Bx + B \\ & + Cx^3 - Cx^2 + Cx - C \\ & + Dx^2 + D \end{aligned}$$

$$-2x+4 = 0x^3 + 0x^2 - 2x + 4$$

Coefficients must match, thus we have a system of equations to solve.

$$\begin{aligned} A + C &= 0 && (1) \\ -2A + B - C + D &= 0 && (2) \\ A - 2B + C &= -2 && (3) \\ B - C + D &= 4 && (4) \end{aligned}$$

From (1) we get $C = -A$.
Substituting into (3) we get $A - 2B - A = -2 \quad \Rightarrow \quad B = 1$.
Substituting into (4) we get $1 - (-A) + D = 4 \quad \Rightarrow \quad D = 3 - A$.
Substituting into (2) we get $-2A + 1 - (-A) + (3 - A) = 0 \quad \Rightarrow \quad -2A + 4 = 0 \quad \Rightarrow \quad A = 2$.
Now, $C = -A = -2$.
Also, $D = 3 - A = 3 - 2 = 1$.

Final step: Integrate the partial fraction decomposition.

$$\int \frac{-2x+4}{(x^2+1)(x-1)^2}\,dx = \int \frac{2x}{x^2+1} + \frac{1}{x^2+1} - \frac{2}{x-1} + \frac{1}{(x-1)^2}\,dx$$

$$= \ln(x^2+1) + \tan^{-1} x - 2\ln|x-1| - \frac{1}{x-1} + C$$