Solution for Problem 4

4) Find the interval of convergence of the series

$$\sum_{n=1}^{\infty} \frac{2^{n+1}}{n^n} x^n.$$

Solution 1:

Use the formula for the radius of convergence $r = \lim_{n\to\infty} \frac{|a_n|}{|a_{n+1}|}$.

$$r = \lim_{n\to\infty} \frac{2^{n+1}}{n^n} \cdot \frac{(n+1)^{n+1}}{2^{n+2}} = \lim_{n\to\infty} \frac{(n+1)\cdot(n+1)^{n+1}}{2\cdot n^n}$$

$$= \left(\lim_{n\to\infty} \frac{n+1}{2}\right) \cdot \underbrace{\lim_{n\to\infty} \left(1+\frac{1}{n}\right)^n}_{=e} = \infty.$$

The series converges absolutely over $\mathbb{R}$.

Solution 2:

Use the formula for the radius of convergence $r = \lim_{n\to\infty} \sqrt[n]{|a_n|}$.

$$\frac{1}{r} = \lim_{n\to\infty} \sqrt[n]{\frac{2^{n+1}}{n^n}} = \underbrace{\lim_{n\to\infty} \sqrt[n]{2}}_{=1} \cdot \lim_{n\to\infty} \frac{2}{n}.$$

$$r = \lim_{n\to\infty} \frac{n}{2} = \infty.$$

The series converges absolutely over $\mathbb{R}$.

Scoring Key:

- 5 points for correctly writing the terms in the radius of convergence formulae, the ratio or the root test.
- 5 points for computing either $\lim_{n\to\infty} \left(\frac{n}{n+1}\right)^n = e^{-1}$ or $\lim_{n\to\infty} \sqrt[n]{2} = 1$.
- 5 points for computing either $\lim_{n\to\infty} \frac{2|x|}{n} = 0$ or $\lim_{n\to\infty} \frac{2|x|}{n+1} = 0$.
- 5 points for correctly interpreting that the series converges absolutely everywhere.