Solution of Problem 3 from Midterm 3

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There are several ways to do this problem. Here are two methods:

METHOD 1:

Use that the sine is a bounded function

$$-1 \le \sin(n) \le 1$$
$$1 \le 2 + \sin(n) \le 3 \quad \textit{(5 points)}$$

In particular, this implies this is a positive series.

$$\frac{1}{n^2-5} \le \frac{2+\sin(n)}{n^2-5} \le \frac{3}{n^2-5},$$

by the Comparison Theorem, convergence of

$$\sum_{n=4}^{\infty} \frac{3}{n^2-5} \quad \text{implies convergence of} \quad \sum_{n=4}^{\infty} \frac{2+\sin(n)}{n^2-5}. \quad \textit{(5 points)}$$

Use the Limit Comparison Theorem to compare the first series to $\sum_{n=4}^{\infty} \frac{1}{n^2}$.

$$\lim_{n\to\infty} \frac{\frac{3}{n^2-5}}{\frac{1}{n^2}} = \lim_{n\to\infty} \frac{3}{1-\frac{5}{n^2}} = 3 \neq 0.$$

$$\text{So } \sum_{n=4}^{\infty} \frac{3}{n^2-5} \text{ converges} \iff \sum_{n=4}^{\infty} \frac{1}{n^2} \text{ converges.} \quad \textit{(5 points)}$$

$\sum_{n=4}^{\infty} \frac{1}{n^2}$ converges by p-Series Theorem for $p = 2 > 1$. *(5 points)*

$$\text{The series } \sum_{n=4}^{\infty} \frac{2+\sin(n)}{n^2-5} \text{ converges.}$$

METHOD 2: Use the Limit Comparison Theorem with $\sum_{n=4}^{\infty} \frac{1}{n^\alpha}$, for $1 < \alpha < 2$. Take for example $\alpha = 3/2$.

$$\lim_{n\to\infty} \frac{\frac{2+\sin(n)}{n^2-5}}{\frac{1}{n^{3/2}}} = \lim_{n\to\infty} \frac{2+\sin(n)}{\frac{n^2-5}{n^{3/2}}} = \lim_{n\to\infty} \frac{2+\sin(n)}{\sqrt{n}-\frac{5}{n^{3/2}}}. \quad \textit{(5 points)}$$

Since $2+\sin(n)$ is bounded and $\lim_{n\to\infty} \sqrt{n} - \frac{5}{n^{3/2}} = \infty$,

$$\lim_{n\to\infty} \frac{2+\sin(n)}{\sqrt{n}-\frac{5}{n^{3/2}}} = 0. \quad \textit{(5 points)}$$

By the Limit Comparison Theorem,

$$\text{Convergence of } \sum_{n=4}^{\infty} \frac{1}{n^{3/2}} \Rightarrow \text{ convergence of } \sum_{n=4}^{\infty} \frac{2+\sin(n)}{n^2-5}. \quad \textit{(5 points)}$$

Finally, $\sum_{n=4}^{\infty} \frac{1}{n^{3/2}}$ converges because of the p-Series Theorem for $p = 3/2 > 1$. *(5 points)*