Stiefel-Whitney Classes

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These notes are based on a reading course instructed by Dan Cristofaro-Gardiner where the author read from Hatcher's *Algebraic Topology* and Milnor and Stasheff's *Characteristic Classes*.

1 Review of Vector Bundles

We work in the category **Man** of topological manifolds (locally Euclidean paracompact Hausdorff spaces) and continuous maps. By locally Euclidean, we mean the space locally looks like R^A, where A is allowed to be infinite. This means that we permit infinite-dimensional manifolds, and we will specify that a manifold has dimension n when we want to talk about a finite-dimensional manifold. We define a smooth manifold as a topological manifold that has a smooth structure, though we use the word "manifold" to mean topological manifold.

Vector bundles over manifolds with bundle maps form the category **VB**. Recall that a bundle map from ξ to η is a continuous map $f : \xi \to \eta$ between the total spaces which carries fibers isomorphically onto fibers. A bundle map induces a map $\overline{f} : B(\xi) \to B(\eta)$ on the base spaces. We say that f covers $\overline{f}$. The assignment taking bundles to their base spaces $\xi \mapsto B(\xi)$ and bundle maps to their induced maps $f \mapsto \overline{f}$ is a covariant functor $\mathbf{VB} \to \mathbf{Man}$. A bundle isomorphism is a bijective bundle map, that is, a homeomorphism $f : \xi \to \eta$ that carries fibers isomorphically onto fibers.

We can also talk about the subcategory $\mathbf{VB}_B$ of vector bundles over a fixed base space B. The morphisms are bundle maps whose induced map on B is the identity.

The canonical line bundle γ_n^1 over real projective space $\mathbb{P}^n = S^n/\sim$ for $x \sim -x$ has total space

$$\gamma_n^1 = \{([x], v) : v = cx \text{ for some } c \in \mathbb{R}\} \subset \mathbb{P}^n \times \mathbb{R}^{n+1}$$

and projection $\pi([x], v) = [x]$. Alternatively, if we think of $\mathbb{P}^n$ as 1-dimensional linear subspaces of $\mathbb{R}^n$, then the total space is

$$\gamma_n^1 = \{(\text{line } L \subset \mathbb{R}^n, \text{vector in } L)\} \subset \mathbb{P}^n \times \mathbb{R}^{n+1}$$

and the projection is $\pi(L, v) = L$.

2 The Stiefel-Whitney Classes

Definition 2.1. The Stiefel-Whitney classes are the unique map

$$\xi \mapsto w(\xi) = w_0(\xi) + w_1(\xi) + \cdots \in H^*(B(\xi); \mathbb{Z}/2\mathbb{Z})$$

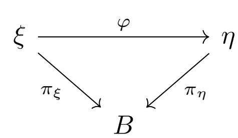
from vector bundles to cohomology classes of their base spaces such that the following four axioms hold:

1. $w_i(\xi) = 0$ for $i > n$ if ξ is an n-plane bundle, and $w_0(\xi) = 1 = (1, \ldots, 1)$, by which we mean it is the element $1 \in \mathbb{Z}/2\mathbb{Z}$ in each connected component.
2. Naturality: if $f : \xi \to \eta$ is a bundle map and $\overline{f} : B(\xi) \to B(\eta)$ is the induced map on the base spaces, then
$$w(\xi) = \overline{f}^* w(\eta).$$
3. The Whitney Product Theorem: If $B(\xi) = B(\eta)$, then
$$w(\xi \oplus \eta) = w(\xi)w(\eta).$$
4. $w_1(\gamma_1^1)$ is non-zero.

We'll prove existence and uniqueness of the Stiefel-Whitney classes later. Assuming it for now, we'll explore the consequences existence and uniqueness of the class gives us.

Proposition 2.1. *If $\xi \cong \eta$, then $w(\xi) = w(\eta)$.*

Proof. Suppose first that $\xi \cong \eta$ over the same base space B. If $\varphi : \xi \to \eta$ is the isomorphism, then id_B is covered by φ, so $w(\xi) = \mathrm{id}_B^* w(\eta) = w(\eta)$ by naturality.



If $\xi \cong \eta$ as vector bundles over different base spaces, strictly speaking, the Stiefel-Whitney classes are not equal because they are elements of different cohomology rings. But the next best thing is true: if $\varphi : \xi \to \eta$ is an isomorphism, the induced homeomorphism $\overline{\varphi} : B(\xi) \to B(\eta)$ on the base spaces induces an isomorphism in cohomology

$$\overline{\varphi}^* : H^*(B(\eta); \mathbb{Z}/2\mathbb{Z}) \to H^*(B(\xi); \mathbb{Z}/2\mathbb{Z})$$

for which $w(\xi)$ and $w(\eta)$ are isomorphic images:

$$w(\xi) = \overline{\varphi}^* w(\eta), \quad w(\eta) = (\overline{\varphi}^{-1})^* w(\xi).$$

We denote this situation as equality because the identification is as canonical as we could hope for. □

Let's compute the Stiefel-Whitney classes of some simple manifolds.

Example 2.1. If ε is a trivial n-plane bundle, then $\varepsilon \cong B(\varepsilon) \times \mathbb{R}^n$. But there is a bundle map from $B(\varepsilon) \times \mathbb{R}^n$ to a vector bundle over a point, which has trivial homology for $i > 0$. Therefore, $w(\varepsilon) = 1$.

$$\begin{array}{ccc} B(\varepsilon) \times \mathbb{R}^n & \xrightarrow{(x,v) \mapsto (p,v)} & \{p\} \times \mathbb{R}^n \\ \downarrow & & \downarrow \\ B(\varepsilon) & \xrightarrow[x \mapsto p]{} & \{p\} \end{array}$$

Example 2.2. Theorem A.1 states that $H^i(\mathbb{P}^n; \mathbb{Z}/2\mathbb{Z})$ is cyclic of order 2 for $0 \le i \le n$, and if a is the non-zero element of $H^1(\mathbb{P}^n; \mathbb{Z}/2\mathbb{Z})$, then a^i is the non-zero element of $H^i(\mathbb{P}^n; \mathbb{Z}/2\mathbb{Z})$. It is visually apparent that the inclusion

$$i : \mathbb{P}^1 \hookrightarrow \mathbb{P}^n, \quad i([x]) = [x, 0, \dots, 0]$$

is covered by a bundle map $\gamma_1^1 \to \gamma_n^1$. Thus

$$i^* w_1(\gamma_n^1) = w_1(\gamma_1^1) \neq 0,$$

so $w_1(\gamma_n^1) \neq 0$ and thus $w_1(\gamma_n^1) = a$. But γ_n^1 is a line bundle, so we may conclude that $w(\gamma_n^1) = 1 + a$.

Example 2.3. We will show $w(TS^n) = 1$ for $n \ge 1$. This follows from Example 2.1 for $n = 1$, since TS^1 is a trivial bundle, but for $n > 1$, this shows that the Stiefel-Whitney classes cannot differentiate the non-trivial bundles TS^n from the trivial bundles $S^n \times \mathbb{R}^n$.

Let $n > 1$. The ith cohomology of S^n is non-trivial only for $i = 0$ and n, so $w(TS^n) = 1 + w_n(TS^n)$ and it suffices to show $w_n(TS^n) = 0$. The map $f : S^n \to \mathbb{P}^n$ taking x to $[x]$ is a local diffeomorphism, hence $df : TS^n \to T\mathbb{P}^n$ is a bundle map covering f. By naturality, $w_n(TS^n) = f^* w_n(T\mathbb{P}^n)$. But $f^*(a^n) = (f^*a)^n = 0$ since $f^*a \in H^1(S^n; \mathbb{Z}/2\mathbb{Z}) = 0$. Therefore, $w_n(TS^n) = 0$.

The existence and uniqueness of the Stiefel-Whitney class enjoys a few deep consequences. In Chapter 4, Milnor and Stasheff use the existence and uniqueness of the Stiefel-Whitney class to show the following:

- $\mathbb{P}^n$ is parallelizable (has trivial tangent bundle) if and only if $n - 1$ is a power of 2.
- If there is a bilinear product without zero divisiors $\mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$, then n must be a power of 2. ($\mathbb{R}^n$ endowed with such a product is called a real division algebra.)
- The smallest n for which $\mathbb{P}^{2^k}$ can be immersed in $\mathbb{R}^n$ is $2^{k+1} - 1$.

We will focus on proving existence and uniqueness. To do this, we need to introduce more theory.

3 The Universal Bundle and Characteristic Classes

There is a map $t : M^1 \to S^k$ from a curve $M^1 \subset \mathbb{R}^{k+1}$ to its unit tangent vector. On the other hand, the Gauss map $n : M^k \to S^k$ takes a point in $M^k \subset \mathbb{R}^{k+1}$ to its unit normal.

These maps depend on an orientation for M, however in the non-orientable case we still have maps into $\mathbb{P}^k$.

In general, if $M^n \subset \mathbb{R}^{n+k}$, we have a map $x \mapsto T_xM \subset \mathbb{R}^{n+k}$, called the generalized Gauss map $\overline{g}$. This map should have as its codomain n-dimensional linear subspaces of $\mathbb{R}^{n+k}$. For example, if $n = 1$, then the generalized Gauss map is t, and the codomain is 1-dimensional linear subspaces of $\mathbb{R}^{k+1}$, in other words, $\mathbb{P}^k$. And if $k = 1$, then this map is the ordinary Gauss map n, and the codomain is n-dimensional linear subspaces of $\mathbb{R}^{n+1}$. This is canonically $\mathbb{P}^k$ since a choice of n-dimensional linear subspace is the same as the choice of its 1-dimensional orthogonal complement. But in general, the codomain of the generalized Gauss map is not projective space, it is a generalization called the Grassmannian.

Definition 3.1. The Grassmannian is

$$\mathrm{Gr}_n(\mathbb{R}^{n+k}) = \{n\text{-dim linear subspaces of } \mathbb{R}^{n+k}\}$$

Proposition 3.1. *$Gr_n(\mathbb{R}^{n+k})$ is a compact smooth manifold of dimension nk. The map $x \mapsto x^\perp$ is a diffeomorphism between $Gr_n(\mathbb{R}^{n+k})$ and $Gr_k(\mathbb{R}^{n+k})$.*

If $M^n \subset \mathbb{R}^{n+k}$, we can now define the generalized Gauss map as

$$\overline{g} : M^n \to \mathrm{Gr}_n(\mathbb{R}^{n+k}), \quad \overline{g}(x) = T_xM.$$

Just as there is a canonical line bundle γ_n^1 over $\mathbb{P}^n$, there is a canonical n-plane bundle $\gamma^n(\mathbb{R}^{n+k})$ over $\mathrm{Gr}_n(\mathbb{R}^{n+k})$.

Definition 3.2. The canonical n-plane bundle $\gamma^n(\mathbb{R}^{n+k})$ over $\mathrm{Gr}_n(\mathbb{R}^{n+k})$ has total space

$$\gamma^n(\mathbb{R}^{n+k}) = \{(n\text{-plane } P \subset \mathbb{R}^{n+k}, \text{vector in } P)\} \subset \mathrm{Gr}_n(\mathbb{R}^{n+k}) \times \mathbb{R}^{n+k}$$

and projection $\pi(P, v) = P$.

If $M^n \subset \mathbb{R}^{n+k}$, then the generalized Gauss map $\overline{g} : M \to \mathrm{Gr}_n(\mathbb{R}^{n+k})$ is covered by a bundle map

$$g : TM \to \gamma^n(\mathbb{R}^{n+k}), \quad g(x, v) = (T_xM, v).$$

Therefore, by the Whitney embedding theorem, if M is an n-dimensional manifold, then the tangent bundle TM maps into $\gamma^n(\mathbb{R}^{n+k})$ for sufficiently large k. In fact, if ξ is any $\mathbb{R}^n$-bundle over a finite-dimensional base space, then there exists a bundle map $\xi \to \gamma^n(\mathbb{R}^{n+k})$ for sufficiently large k. This allows us to hope that if we let k "go to infinity", then we might get a "universal bundle" $\gamma^n(\mathbb{R}^\infty)$ that every $\mathbb{R}^n$-bundle maps into, even if it is over an infinite-dimensional base space. This will turn out to be true once everything has been defined.

Lemma 3.1. *The direct limit of compact spaces $K_1 \subset K_2 \subset K_3 \subset \cdots$ is paracompact.*

This lemma justifies that the following spaces are manifolds.

Definition 3.3. The infinite real space is the manifold

$$\mathbb{R}^\infty = \bigcup_k \mathbb{R}^k \quad \text{where } \mathbb{R}^k = \{(x_1, \ldots, x_k, 0, 0, \ldots)\} \subset \text{infinite sequences of reals.}$$

It is topologized as the direct limit of

$$\mathbb{R}^0 \subset \mathbb{R}^1 \subset \mathbb{R}^2 \subset \cdots,$$

which means that $U \subset \mathbb{R}^\infty$ if open iff $U \cap \mathbb{R}^k$ is open as a subset of $\mathbb{R}^k$ for all k.

Definition 3.4. The infinite Grassmannian is the manifold

$$\mathrm{Gr}_n = \mathrm{Gr}_n(\mathbb{R}^\infty) = \{n\text{-dim linear subspaces of } \mathbb{R}^\infty\} = \bigcup_k \mathrm{Gr}_n(\mathbb{R}^{n+k})$$

where $\mathbb{R}^{n+k}$ is as in Definition 3.3. It is topologized as the direct limit of

$$\mathrm{Gr}_n(\mathbb{R}^n) \subset \mathrm{Gr}_n(\mathbb{R}^{n+1}) \subset \mathrm{Gr}_n(\mathbb{R}^{n+2}) \subset \cdots.$$

We also define the infinite projective space as the manifold $\mathbb{P}^\infty = \mathrm{Gr}_1$.

Definition 3.5. The universal n-plane bundle γ^n over Gr_n has total space

$$\gamma^n = \{(n\text{-plane } P \subset \mathbb{R}^\infty, \text{vector in } P)\} \subset \mathrm{Gr}_n \times \mathbb{R}^\infty$$

and projection $\pi(P, v) = P$.

Definition 3.6. Two bundle maps $f, g : \xi \to \eta$ are called bundle-homotopic if there exists a family of bundle maps $h_t : \xi \to \eta$ with $H(x,t) = h_t(x)$ continuous such that $h_0 = f$ and $h_1 = g$.

Theorem 3.1. *Any $\mathbb{R}^n$-bundle ξ admits a bundle map $\xi \to \gamma^n$. This map is unique up to bundle-homotopy.*

This theorem would fail were we to work with non-paracompact manifolds.

We can now definite the concept of a characteristic class. First, note that by Theorem 3.1, an $\mathbb{R}^n$-bundle ξ determines a unique homotopy class of maps $f_\xi : \xi \to \gamma^n$, and this determines a unique homotopy class of maps $\overline{f}_\xi : B(\xi) \to \mathrm{Gr}_n$.

Next, pick a coefficient group or ring Λ and choose any cohomology class $c \in H^i(\mathrm{Gr}_n; \Lambda)$. Then c and ξ uniquely determine the "characteristic class of ξ determined by c"

$$c(\xi) := \overline{f}_\xi^*(c) \in H^i(B(\xi); \Lambda).$$

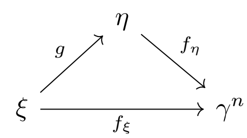
Note that for this to be well-defined, we have used the fact that cohomology is a homotopy invariant. A map

$$\xi \mapsto c(\xi) \in H^i(B(\xi); \Lambda)$$

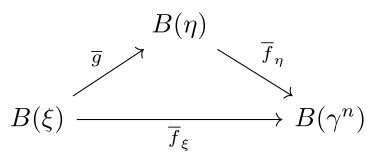
arising in this way is called a characteristic class. Notice that we can add and multiply characteristic classes, and in fact they form a ring. The ring of characteristic classes for $\mathbb{R}^n$ bundles with Λ coefficients is canonically isomorphic to $H^*(\mathrm{Gr}_n; \Lambda)$.

Any natural correspondence $\xi \mapsto c(\xi) \in H^i(B(\xi); \Lambda)$ is a characteristic class, because naturality implies $c(\xi) = \overline{f}_\xi^* c(\gamma^n)$. This implies that the Stiefel-Whitney classes $\xi \mapsto w_i(\xi)$ are characteristic classes, justifying their name.

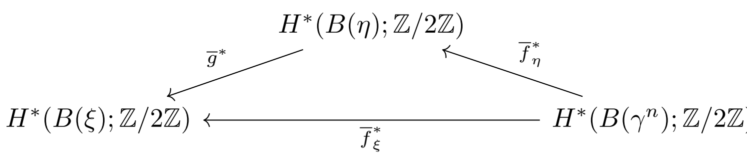
On the other hand, every characteristic class is natural. To prove this, suppose that $g : \xi \to \eta$. We want to show that $c(\xi) = \overline{g}^* c(\eta)$. By Theorem 3.1, the diagram



commutes up to bundle-homotopy. Applying the base space/induced map functor, we obtain



which commutes up to homotopy. Applying the cohomology functor, we finally have



which is commutative since cohomology is a homotopy invariant. By definition, $c(\xi) = \overline{f}^*_\xi c$ and $c(\eta) = \overline{f}^*_\eta c$. Therefore,

$$c(\xi) = \overline{f}^*_\xi c = \overline{g}^*(\overline{f}^*_\eta c) = \overline{g}^* c(\eta),$$

as desired.

4 Existence and Uniqueness

We are now ready to prove the existence and uniqueness of Stiefel-Whitney classes. Recall Theorem A.2, which say that $H^*(\mathrm{Gr}_n; \mathbb{Z}/2\mathbb{Z})$ is a polynomial algebra over $\mathbb{Z}/2\mathbb{Z}$ freely generated by $x_1, \ldots, x_n$.

Theorem 4.1. *There exists a correspondence $\xi \mapsto w(\xi)$ satisfying the Stiefel-Whitney axioms.*

Proof. Following the last section, the Stiefel-Whitney classes of an arbitrary $\mathbb{R}^n$-bundle can be specified by a choice of $w(\gamma^n)$, for then the Stiefel-Whitney classes of ξ are $w(\xi) = \overline{f}^*_\xi w(\gamma^n)$ We choose

$$w(\gamma^n) = 1 + x_1 + \cdots + x_n.$$

By the last section, we have naturality, so it remains to verify axioms 1, 3, and 4.

Axiom 1: Let ξ be an $\mathbb{R}^n$-bundle. Then

$$w_0(\xi) = \overline{f}^*_\xi w_0(\gamma^n) = \overline{f}^*_\xi(1) = 1.$$

Also, for $i > n$,

$$w_i(\xi) = \overline{f}_\xi^* w_i(\gamma^n) = \overline{f}_\xi^*(0) = 0.$$

Axiom 3: Omitted.

Axiom 4: As in Example 2.2, the inclusion

$$i : \mathbb{P}^1 \hookrightarrow \mathbb{P}^\infty = \mathrm{Gr}_1, \quad i([x]) = [x, 0, 0, \ldots]$$

is covered by a bundle map $\gamma_1^1 \to \gamma^1$. The injectivity of i implies i^* is surjective. Therefore,

$$w_1(\gamma_1^1) = i^* w_1(\gamma^1) = i^*(x_1) = a \neq 0$$

for a as in Theorem A.1 because $i^*(0) = 0$ and we need something to hit a for surjectivity. □

We now forget about the choice of w made in the proof of existence and once again let w be anything satisfying the Stiefel-Whitney axioms. It turns out that the choice we made was the only possible one. The proof of uniqueness uses the following lemma.

Lemma 4.1. *$H^*(Gr_n; \mathbb{Z}/2\mathbb{Z})$ is a polynomial algebra over $\mathbb{Z}/2\mathbb{Z}$ freely generated by $w_1(\gamma^n), \ldots, w_n(\gamma^n)$.*

Proof. We'll show that $H^*(\mathrm{Gr}_n; \mathbb{Z}/2\mathbb{Z})$ contains a polynomial algebra freely generated by $w_1(\gamma^n), \ldots, w_n(\gamma^n)$, and then by Theorem A.2, $H^*(\mathrm{Gr}_n; \mathbb{Z}/2\mathbb{Z})$ cannot contain anything else.

Clearly $H^*(\mathrm{Gr}_n; \mathbb{Z}/2\mathbb{Z})$ contains $w_i(\gamma^n)$, but we need to show there are not any polynomial relations among them. So suppose there were a polynomial $p \neq 0$ in n variables in $\mathbb{Z}/2\mathbb{Z}$ coefficients such that

$$p(w_1(\gamma^n), \ldots, w_n(\gamma^n)) = 0.$$

Then for any n-plane bundle ξ, we have $w(\xi) = \overline{f}_\xi^* w(\gamma^n)$. Hence

$$p(w_1(\xi), \ldots, w_n(\xi)) = \overline{f}_\xi^* p(w_1(\gamma^n), \ldots, w_n(\gamma^n)) = 0.$$

Therefore, to find a contradiction, we just need to find one n-plane bundle ξ for which there are no polynomial relations among the $w_i(\xi)$'s.

Form the n-fold product $X = \mathbb{P}^\infty \times \cdots \times \mathbb{P}^\infty$ with projection onto the ith factor $\pi_i : X \to \mathbb{P}^\infty$. Recalling that a is the generator of $H^*(\mathbb{P}^\infty; \mathbb{Z}/2\mathbb{Z})$, we define $a_i = \pi_i^*(a)$. By the Künneth formula, $H^*(X; \mathbb{Z}/2\mathbb{Z})$ is a polynomial algebra generated by $a_1, \ldots, a_n$.

We choose

$$\xi = \underbrace{\gamma^1 \times \cdots \times \gamma^1}_{n} \cong \underbrace{(\pi_1^* \gamma^1) \oplus \cdots \oplus (\pi_n^* \gamma^1)}_{n}$$

which has base space X. Since $w(\gamma^1) = 1 + a$, by the Whitney Product Theorem,

$$w(\xi) = (1 + a_1) \cdots (1 + a_n).$$

Expanding out the product, we see

$$\begin{aligned} w_1(\xi) &= a_1 + \cdots + a_n \\ w_2(\xi) &= a_1a_2 + a_1a_3 + \cdots + a_{n-1}a_n \\ &\vdots \\ w_n(\xi) &= a_1 \cdots a_n. \end{aligned}$$

These are the elementary symmetric functions of n variables over a field, and one can show that there are no polynomial relations among them. Hence there are no polynomial relations among the $w_i(\gamma^n)$'s. □

Theorem 4.2. *There exists at most one correspondence $\xi \mapsto w(\xi)$ satisfying the Stiefel-Whitney axioms.*

Proof. Now suppose that $\xi \mapsto w(\xi)$ and $\xi \mapsto \tilde{w}(\xi)$ both satisfy the Stiefel-Whitney axioms. It suffices to show that $w(\gamma^n) = \tilde{w}(\gamma^n)$, for then if η is an $\mathbb{R}^n$-bundle,

$$w(\eta) = \overline{f}_\eta^* w(\gamma^n) = \overline{f}_\eta^* \tilde{w}(\gamma^n) = \tilde{w}(\eta).$$

Recalling that a is the generator of $H^*(\mathbb{P}^\infty; \mathbb{Z}/2\mathbb{Z})$, we have

$$w(\gamma_1^1) = \tilde{w}(\gamma_1^1) = 1 + a.$$

Embedding γ_1^1 in γ^1 gives

$$w(\gamma^1) = \tilde{w}(\gamma^1) = 1 + a.$$

As in the proof of the previous lemma, let

$$\xi = \underbrace{\gamma^1 \times \cdots \times \gamma^1}_{n} \cong \underbrace{(\pi_1^*\gamma^1) \oplus \cdots \oplus (\pi_n^*\gamma^1)}_{n}$$

over $\mathbb{P}^\infty \times \cdots \times \mathbb{P}^\infty$. Then,

$$\begin{aligned} w(\xi) = \tilde{w}(\xi) &= (1 + a_1) \cdots (1 + a_n), \\ \overline{f}_\xi^* w(\gamma^n) = w(\xi) &= \tilde{w}(\xi) = \overline{f}_\xi^* \tilde{w}(\gamma^n). \end{aligned}$$

But $\overline{f}_\xi^* : H^*(\mathrm{Gr}_n; \mathbb{Z}/2\mathbb{Z}) \to H^*(\mathbb{P}^\infty \times \cdots \times \mathbb{P}^\infty; \mathbb{Z}/2\mathbb{Z})$ is injective, so $w(\gamma^n) = \tilde{w}(\gamma^n)$. □

A Cohomology Results in $\mathbb{Z}/2\mathbb{Z}$

Theorem A.1. *$H^i(\mathbb{P}^n; \mathbb{Z}/2\mathbb{Z})$ is cyclic of order 2 for $0 \leq i \leq n$. If a is the non-zero element of $H^1(\mathbb{P}^n; \mathbb{Z}/2\mathbb{Z})$, then a^i is the non-zero element of $H^i(\mathbb{P}^n; \mathbb{Z}/2\mathbb{Z})$.*

Theorem A.2. *$H^*(Gr_n; \mathbb{Z}/2\mathbb{Z})$ is a polynomial algebra over $\mathbb{Z}/2\mathbb{Z}$ freely generated by $x_1, \ldots, x_n$.*

The proof of Theorem A.2 can be done using spectral sequences. Note that $x_i^2 = -x_i^2$ sometimes due to the graded commutativity of the cup product, and hence in characteristics other than two, $x_i^2 = 0$ is a polynomial relation among the x_i's. This is why we use $\mathbb{Z}/2\mathbb{Z}$ coefficients.