Motivic Hilbert zeta functions of curves

Dori Bejleri

July 2017

Abstract

These are notes from a talk given at Brown University in February of 2016. After a historical overview, we review recent work on rationality of motivic Hilbert zeta functions of curves.

1 The Hasse-Weil zeta function

Let $X/\mathbb{F}_q$ be a variety over $\mathbb{F}_q$ and let

$$N_m := \#X(\mathbb{F}_{q^m})$$

be the number of $\mathbb{F}_{q^m}$ points of X. The (local) *Hasse-Weil zeta function* is the following power series.

$$Z_X^{\mathrm{HW}}(t) := \exp\left(\sum_{m\geq 1} \frac{N_m}{m} t^m\right) \in \mathbb{Q}[\![t]\!]$$

Theorem 1.1. *(Dwork, Grothendieck, et.al.) The Hasse-Weil zeta function of any variety is a rational function:*

$$Z_X^{\mathrm{HW}}(t) \in \mathbb{Q}(t).$$

Remark 1.2.

- The rationality of $Z_X^{\mathrm{HW}}(t)$ for X a smooth variety is part of the Weil conjectures.
- For a smooth X it follows from the formalism of ètale cohomology as developed by Grothendieck and others.
- Dwork used p-adic analytic methods to prove rationality for arbitrary X.

There is a well-known reformulation of the Hasse-Weil zeta function in terms of symmetric products of X. Recall that the symmetric product of X is defined as

$$\mathrm{Sym}^n(X) := X^n/\mathfrak{S}_n$$

where $\mathfrak{S}_n$ is the symmetric group acting by permuting the copies of X.

Proposition 1.3. *The Hasse-Weil zeta function is equal to the generating series for point counts over $\mathbb{F}_q$ of the symmetric product.*

$$Z_X^{\mathrm{HW}}(t) = \sum_{n\geq 0} \#\mathrm{Sym}^n(X)(\mathbb{F}_q)t^n \in \mathbb{Z}[\![t]\!]$$

Note in particular, the coefficients of the zeta function are non-negative integers, a fact which is not clear *a priori*.

2 The Motivic zeta function

Proposition 1 suggests that we can refine $Z_X^{\mathrm{HW}}(t)$ by taking other invariants of the symmetric product as coefficients rather than point counts. We pass to a sort of universal invariant – the class in the Grothendieck ring of varieties.

Let k be any field. The *Grothendieck ring of varieties over* k, denoted $K_0(\mathrm{Var}_k)$, is the ring whose underlying abelian group is generated by isomorphism classes $[X]$ of varietyes X/k subject to the relations

- $[X] = [U] + [X \setminus U]$ for $U \hookrightarrow X$ an open immersion;
- $[X \times Y] = [X][Y]$.

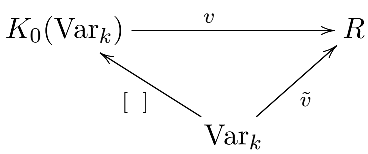
We denote by $\mathbb{L} := \mathbb{A}^1$ the class of the affine line. $K_0(\mathrm{Var}_k)$ satisfies the following universal property. For any ring R and any function

$$\tilde{v} : \mathrm{Var}_k \to R$$

satisfying the relations

- $\tilde{v}(X) = \tilde{v}(X')$ whenever $X \cong X'$,
- $\tilde{v}(X) = \tilde{v}(U) + \tilde{v}(X \setminus U)$ for $U \hookrightarrow X$ an open immersion,
- $\tilde{v}(X \times Y) = \tilde{v}(X)\tilde{v}(Y)$,

there is a unique ring homomorphism $v : K_0(Var_k) \to R$ such that the following diagram commutes.



Such homomorphism v are called *motivic measures*.

Example 2.1. (i) If $k = \mathbb{C}$, then the compactly supported euler characteristic

$$\chi_{top}(X) := \sum_i (-1)^i \dim_{\mathbb{Q}} H_c^i(X, \mathbb{Q}) \in \mathbb{Z}$$

is a motivic measure.

(ii) If $k = \mathbb{F}_q$, the point counting function $\#X(\mathbb{F}_q) \in \mathbb{Z}$ is a motivic measure.

(iii) Let k be a field of characteristic zero. Then the function which sends a smooth projective variety X to its Hodge polynomial

$$\sum_{p,q} \dim_k H^q(X, \Omega_X^p) u^p v^q \in \mathbb{Z}[u, v]$$

extends to a unique motivic measure. This can be proved over $\mathbb{C}$ using Deligne's mixed hodge theory or over general k using Bittner's presentation of $K_0(\mathrm{Var}_k)$ and weak factorization of birational maps.

Inspired by Proposition 1.3, one can make the following definition.

Definition 2.2. (Kapranov) The motivic zeta function of X is defined as the generating series

$$Z_X^{\mathrm{Sym}}(t) := \sum_{n \geq 0} [\mathrm{Sym}^n(X)] t^n \in 1 + tK_0(\mathrm{Var}_k)[\![t]\!].$$

$Z_X^{\mathrm{Sym}}(t)$ satisfies the following basic properties:

Proposition 2.3. *(i) The motivic zeta function extends to a well defined map*

$$K_0(\mathrm{Var}_k) \to 1 + tK_0(\mathrm{Var}_k)[\![t]\!]$$

which is a monoid homomorphism where the right hand side is a monoid under multiplication.

$$Z_{[X]+[Y]}^{\mathrm{Sym}}(t) = Z_{[X]}^{\mathrm{Sym}}(t) Z_{[X]}^{\mathrm{Sym}}(t)$$

(ii) When $k = \mathbb{F}_q$, $Z_X^{\mathrm{Sym}}(t)$ specializes to $Z_X^{\mathrm{HW}}(t)$ under the point counting motivic measure.

The motivic zeta function is especially well behaved when X is a smooth curve.

Theorem 2.4. *(Kapranov '00, Litt '14) Let X be a smooth, projective, geometrically connected curve over k. Then*

$$Z_X^{\mathrm{Sym}}(t) \in K_0(\mathrm{Var}_k)(t),$$

that is, the motivic zeta function is a rational function with coeficients in $K_0(\mathrm{Var}_k)$.

Remark 2.5.

- Kapranov proved this theorem under the assumption that $X(k) \neq \emptyset$. We will discuss a generalization of his method to Hilbert schemes in the sequel. Litt extended the result to the general case using Severi-Brauer schemes.
- Using cut-and-paste relations one can extend this easily to show that $Z_X^{\mathrm{Sym}}(t)$ is rational whenever X is a curve such that the normalization X_{red}^{ν} is geometrically irreducible.
- Knowing that $Z_X^{\mathrm{HW}}(t)$ is a rational function for any variety X, one might expect that $Z_X^{\mathrm{Sym}}(t)$ is a rational function for any X. This is in fact false. Larsen and Luntz produced a counterexample where X is a smooth projective surface.

3 The motivic Hilbert zeta function

When X is a singular curve, the $Z_X^{\mathrm{Sym}}(t)$ does not capture much information about the singularities. In fact, $Z_X^{\mathrm{Sym}}(t)$ depends only on the number of branches at each singular point.

Example 3.1. Let X be a projective rational curve with a single cusp. Then the normalization $\mathbb{P}^1 \to X$ is a cut and paste isomorphism. Indeed $[X] = 1 + \mathbb{L} = [\mathbb{P}^1]$ so $Z_X^{\mathrm{Sym}}(t) = Z_{\mathbb{P}^1}^{\mathrm{Sym}}(t)$.

To remedy this we define a finer invariant that sees more about the singularities.

Definition 3.2. The motivic Hilbert zeta function of X is the generating series

$$Z_X^{\mathrm{Hilb}}(t) := \sum_{d \geq 0} [\mathrm{Hilb}^d(X)] t^d \in 1 + tK_0(\mathrm{Var}_k)[\![t]\!].$$

Recall that the Hilbert scheme of points $\mathrm{Hilb}^d(X)$ is the moduli space of zero-dimension length d subschemes of X.

$$\begin{aligned} \mathrm{Hilb}^d(X) &= \{Z \subset X \mid \dim_k \mathcal{O}_Z = d\} \\ &= \{\mathcal{I} \subset \mathcal{O}_X \mid \dim_k \mathcal{O}_X/\mathcal{I} = d\} \end{aligned}$$

Remark 3.3.

- When X is a smooth curve, $\mathrm{Hilb}^d(X) = \mathrm{Sym}^d(X)$ so $Z_X^{\mathrm{Sym}}(t) = Z_X^{\mathrm{Hilb}}(t)$.
- More generally, for $Y \subset X$ a closed subscheme, we define

$$\begin{aligned}\mathrm{Hilb}^d_Y(X) &\subset \mathrm{Hilb}^d(X)\\ &= \{\text{length } d \text{ subschemes of } X \text{ supported on } Y\}\end{aligned}$$

and

$$Z^{\mathrm{Hilb}}_{Y\subset X}(t) := \sum_{d\geq 1} [\mathrm{Hilb}^d_Y(X)] t^d.$$

Lemma 3.4. *Let $U \subset X$ be open with $Y = X \setminus U$ the closed complement. Then*

$$Z_X^{\mathrm{Hilb}}(t)(t) = Z_U^{\mathrm{Hilb}}(t) Z^{\mathrm{Hilb}}_{Y\subset X}(t).$$

Corollary 3.5. *If X is a reduced curve such that all the singular points $p_1, \ldots, p_m \in X$ are defined over k. Then*

$$\begin{aligned}Z_X^{\mathrm{Hilb}}(t) &= Z^{\mathrm{Hilb}}_{X^{sm}}(t) \prod_{i=1}^m Z^{\mathrm{Hilb}}_{p_i\subset X}(t)\\ &= Z^{\mathrm{Sym}}_{X^{sm}}(t) \prod_{i=1}^m Z^{\mathrm{Hilb}}_{p_i\subset X}(t)\end{aligned}$$

In particular, this reduces the study of $Z_X^{\mathrm{Hilb}}(t)$ for a reduced curve X to the motivic zeta function of the smooth locus and the local Hilbert zeta functions of the singular points

$$Z^{\mathrm{Hilb}}_{p\subset X}(t) = Z^{\mathrm{Hilb}}_{\mathrm{Spec}\widehat{\mathcal{O}}_{X,p}}(t)$$

which depends only on the analytic type of the singularity. In particular, this produces a family of analytic invariants of curve singularities.

Example 3.6. (i) Let $R = k[\![x,y]\!]/(y^2 = x^3) = k[\![t^2, t^3]\!] \subset k[\![t]\!]$ so that $X = \mathrm{Spec} R$ is a cusp with normalization $\mathrm{Spec} k[\![t]\!]$. Then one can compute

$$\begin{array}{ll}\mathrm{Hilb}^0(X) = \{\mathcal{O}_X\} & [\mathrm{Hilb}^0(X)] = 1\\ \mathrm{Hilb}^1(X) = \{\mathfrak{m} = (t^2, t^3)\} & [\mathrm{Hilb}^1(X)] = 1\\ \mathrm{Hilb}^2(X) = \{(\alpha t^3 + \beta t^2, t^4)\} & [\mathrm{Hilb}^2(X)] = \mathbb{L} + 1\\ \quad\vdots & \quad\vdots\\ \mathrm{Hilb}^d(X) = \{(\alpha t^{d+1} + \beta t^d, t^{d+2})\} & [\mathrm{Hilb}^d(X)] = \mathbb{L}+1\end{array}$$

From this we can compute the Hilbert zeta function:

$$Z_X^{\mathrm{Hilb}}(t) = 1 + t + \frac{(\mathbb{L}+1)t^2}{1-t} = \frac{1 + \mathbb{L}t^2}{1-t}$$

(ii) Let $R = k[\![x,y]\!]$ so that $\mathrm{Spec} R$ is a node. Then the first few Hilbert schemes are as follows.

$$\begin{aligned}\mathrm{Hilb}^0(X) &= pt\\ \mathrm{Hilb}^1(X) &= pt\\ \mathrm{Hilb}^2(X) &= \{(\alpha x + \beta y, x^2, y^2)\} \cong \mathbb{P}^1\end{aligned}$$

More generally, inside $\mathrm{Hilb}^d(X)$ there are monomial subschemes cut out by ideals of the form (x^a, y^b) with $a + b = d + 1$ that are connected by rational curves of the form $(\alpha x^a + \beta y^{b-1}, x^{a+1}, y^b)$. It follows that $\mathrm{Hilb}^d(X)$ is a chain of $d-1$ rational curves and the class $[\mathrm{Hilb}^d(X)] = (d-1)\mathbb{L} + 1$. Putting this together, the Hilbert zeta function is

$$Z_X^{\mathrm{Hilb}}(t) = \frac{1 - t + \mathbb{L}t^2}{(1-t)^2}$$

4 Rationality of $Z_X^{\mathrm{Hilb}}(t)$ for reduced curves

Given the above computations, it is natural to ask when is $Z_X^{\mathrm{Hilb}}(t)$ rational for curves?

Theorem 4.1. *(Maulik-Yun, Migliorini-Shende)*[1] *Suppose X is a projective, reduced and irreducible curve with only planar singularities and a smooth point defined over k. Then $Z_X^{\mathrm{Hilb}}(t)$ is a rational function.*

Proof sketch. One strategy is to extend Kapranov's proof to this setting using the fact that X is Gorenstein. Let $p \in X$ be a smooth point defined over k. Then there is an Abel-Jacobi map

$$AJ^d : \mathrm{Hilb}^d(X) \to \overline{\mathrm{Jac}}(X) = \{\text{rank 1, degree 0 torsion free sheaves}\}$$

to the compactified Jacobian which in this case is an irreducible compactification of $\mathrm{Jac}(X)$ by a theorem of Altman, Kleiman and Iarrobino. The map is defined by

$$AJ^d(I) = I \otimes \mathcal{O}_X(dp)$$

and the fiber above a torsion free sheaf $J = I \otimes \mathcal{O}_X(dp)$ is given by

$$(AJ^d)^{-1}(J) = \mathbb{P}(\mathrm{Hom}_{\mathcal{O}_X}(J \otimes \mathcal{O}_X(-dp), \mathcal{O}_X)) = \mathbb{P}(H^0(I^\vee)).$$

By Riemann-Roch and base change, for $d \geq 2g-1$ the dimensions $h^0(I^\vee)$ are constant equal to $d-g+1$ and so AJ^d is a $\mathbb{P}^{d-g}$-bundle. Thus $[\mathrm{Hilb}^d] = [\overline{\mathrm{Jac}}(X)][\mathbb{P}^{d-g}]$ in this range from which it follows that $Z_X^{\mathrm{Hilb}}(t)$ is a rational function. □

What happens for worse singularities?

Theorem 4.2. *(Bejleri-Ranganathan-Vakil) Let X be any reduced curve with singular points defined over k. Then $Z_X^{\mathrm{Hilb}}(t)$ is a rational function.*

One reason this is surprising is that one might expect $\mathrm{Hilb}^d(X)$ to have arbitrary large irreducible components when X has large embedding dimension since this happens for $\mathrm{Hilb}^d(\mathbb{A}^r)$.

4.1 The proof of Theorem 4.2

The considerations from Section 3 reduce the problem to understanding $Z_X^{\mathrm{Hilb}}(t)$ for $X = \mathrm{Spec} R$ for $(R, \mathfrak{m})$ a complete local ring of a curve singularity. In this case we prove rationality by generalizing a method of Pfister and Steenbrink for understanding Hilbert schemes of points on such germs in the unibranch case.

[1] Actually, they do more; they compute the cohomological realization of $Z_X^{\mathrm{Hilb}}(t)$ in terms of a filtration on the cohomology of of the compactified Jacobian of X.

Suppose X is unibranch and let δ is the δ-invariant of R and $\tilde{R} = k[\![t]\!]$ be the normalization. Then for any $I \in \mathrm{Hilb}^d(X)$, there are inclusions

$$(t^{2\delta+d}) \cap R \subset I \subset (t^d) \cap R \subset \tilde{R}$$

where (t^a) is an ideal of $\tilde{R}$. Furthermore,

$$\dim_k I/(t^{2\delta+d}) = \dim_k t^{-d} I/(t^{2\delta}) = \delta$$

so we get an embedding

$$\phi^d : \mathrm{Hilb}^d(X) \hookrightarrow \mathcal{M} \subset \mathrm{Gr}(\delta, \tilde{R}/(t^{2\delta}))$$

given by $\phi^d(I) = t^{-d} I/(t^{2\delta})$ onto the subvariety $\mathcal{M}$ of the Grassmannian consisting of those subspaces which are R-modules. Pfister and Steenbrink show that the image ϕ^d stabalizes for large d so that $[\mathrm{Hilb}^d(X)]$ is eventually constant from which rationality follows.

We generalize this to singularities with an arbitrary number of branches as follows. If X has s branches, then the normalization

$$\nu : \tilde{X} \to X$$

is given by $\tilde{X} = \mathrm{Spec} \prod_{i=1}^s k[\![t_i]\!]$. Now let $Z \subset X \times \mathrm{Hilb}^d(X)$ be the universal family and take the pullback $\nu^* Z \to \tilde{X} \times \mathrm{Hilb}^d(X)$. Consider the flattening stratification for the composition $\nu^* Z \to \mathrm{Hilb}^d(X)$. This stratifies $\mathrm{Hilb}^d(X)$ into a union

$$\mathrm{Hilb}^d(X) = \bigsqcup \mathrm{Hilb}^{d,a_1,\ldots,a_s}(X)$$

where the a_i record the length of the pullback of a subscheme to the i^{th} branch of the normalization.

Using a degeneration of X to a toric variety, we prove the following proposition.

Proposition 4.3. *For the values of d, a_i for which $\mathrm{Hilb}^{d,a_1,\ldots,a_s}(X) \neq \emptyset$, the difference*

$$d - \sum_{i=1}^s a_i$$

is uniformly bounded from above and below independent of d and a_i.

Using the above proposition we construct a generalization of the Pfister-Steenbrink embedding ϕ^d for the strata $\mathrm{Hilb}^{d,a_1,\ldots,a_s}(X)$ and use it to prove the following stabalization result.

Proposition 4.4. *If $a_s \gg 0$, then*

$$[\mathrm{Hilb}^{d,a_1,\ldots,a_s}(X)] = [\mathrm{Hilb}^{d+1,a_1,\ldots,a_s+1}(X)]$$

in the Grothendieck ring.

Intuitively, these results say that the classes $[\mathrm{Hilb}^d(X)]$ for all d are determined by a finite amount of data. Theorem 4.2 follows from these propositions using some generating functionology.

4.2 Future questions

One interpretation of rationality is that the classes $[\mathrm{Hilb}^d(X)]$ are determined by a finite amount of data. A natural question then is how could one characterize this finite amount of data intrinsically? How is $Z_X^{\mathrm{Hilb}}(t)$, at least after passing to various motivic measures, related to other invariants of the curve singularity? How does $Z_X^{\mathrm{Hilb}}(t)$ vary in flat families?

The answers to some of these questions are known (or conjectured) in the planar case and involve to connections with knot theory, mathematical physics, the geometric Langlands program, and other parts of math. However, very little is known about singularities with higher embedding dimension and we hope that the techniques involved in proving rationality will open the door to answering some of these questions more generally.