Perfectoid rings and $\mathbb{A}_{\text{inf}}$

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1 Introduction

In this talk we will define and study Fontaine's ring $\mathbb{A}_{\text{inf}}$. The goal is to construct for a perfectoid ring S, morphisms $\theta_r, \tilde{\theta}_r : \mathbb{A}_{\text{inf}}(S) \to W_r(S)$ generalizing Fontaine's map $\theta : \mathbb{A}_{\text{inf}}(S) \to S$ and show that they behave like one parameter deformations.

Specialization along these maps is used in [BMS16] for constructing the comparison between the $\mathbb{A}_{\text{inf}}$ cohomology theory and the relative de Rham-Witt complex. We closely follow [BMS16, Section 3] as well as [Mor16, Section 3].

2 Witt vectors

We first review (p-typical) Witt vectors. For a detailed and general exposition, see [Rab14]. Fix p a prime number.

Let A be any ring and define $W_r(A) = A^r$ as a set. Consider the maps

$$w : W_r(A) \to A^r$$

given by

$$(x_0, x_1, \ldots, x_{r-1}) \mapsto (w_0(x_0), w_1(x_0, x_1), \ldots, w_{r-1}(x_0, \ldots, x_{r-1}))$$

where

$$w_n(x_0, \ldots, x_n) = \sum_{i=0}^{n} p^i x_i^{p^{n-i}} = x_0^{p^n} + p x_1^{p^{n-1}} + \ldots + p^n x_n.$$

w is called the ghost map and w_n the ghost components.

Theorem 2.1. *There exists a unique ring structure on $W_r(A)$ making*

$$W_r : \text{Ring} \to \text{Ring}$$

a functor such that

(i) $W_r(f)(x_0, \ldots, x_{r-1}) = (f(x_0), \ldots, f(x_{r-1}))$,

(ii) and $w : W_r(A) \to A^r$ is a natural transformation (and in particular a ring homomorphism).

Furthermore, W_r commutes with inverse limits.

$W_r(A)$ are called the length r Witt vectors of A. They come equipped with several maps:

(1) (Restriction) $R : W_r(A) \to W_{r-1}(A)$ is the ring homomorphism given by $R(x_0, \dots, x_{r-1}) = (x_0, \dots, x_{r-2})$.

(2) (Frobenius) $F : W_r(A) \to W_{r-1}(A)$ is the unique ring homomorphism making the diagram

$$\begin{array}{ccc} W_r(A) & \xrightarrow{F} & W_{r-1}(A) \\ \downarrow w & & \downarrow w \\ A^r & \longrightarrow & A^{r-1} \end{array}$$

commute where $A^r \to A^{r-1}$ is projection onto the first $r-1$ coordinates.

(3) (Verschiebung) $V : W_{r-1}(A) \to W_r(A)$ is the additive (but not multiplicative) map given by $(x_0, \dots, x_{r-2}) \mapsto (0, x_0, \dots, x_{r-2})$

(4) (Teichmuller representatives) $[\ ] : A \to W_r(A)$ is the multiplicative (but not additive) map $x \mapsto (x, 0, \dots, 0)$.

Lemma 2.2. *These maps satisfy the following relations:*

(a) $F[a] = [a^p]$,

(b) $(F \circ V)(x) = px$,

(c) $V(xF(y)) = V(x)y$,

(d) $x = \sum_{i=0}^{r} V^i[x_i]$.

Definition 2.3. The Witt vectors of A are defined as the inverse limit

$$W(A) := \varprojlim_{R} W_r(A)$$

over all restriction maps $R : W_{r+1}(A) \to W_r(A)$.

$W(A)$ inherits restriction maps $R : W(A) \to W_r(A)$ as well as Verschiebung and Frobenius maps $V, F : W(A) \to W(A)$ and a Teichmuller lift $[\ ] : A \to W(A)$ compatible with those at each level.

2.1 p-torsion free rings

Suppose that p is invertible in A so that A is a $\mathbb{Z}[\frac{1}{p}]$-algebra. Then the ghost map

$$w : W_r(A) \to A^r$$

is a ring isomorphism.

More generally, if A is p-torsion free, it embeds into the $\mathbb{Z}[\frac{1}{p}]$-algebra $A[\frac{1}{p}]$ and there is a commutative diagram

$$\begin{array}{ccc} W_r(A) & \hookrightarrow & W_r(A[\frac{1}{p}]) \\ \downarrow & & \downarrow \\ A^r & \hookrightarrow & A[\frac{1}{p}]^r \end{array}$$

of ring homomorphisms.

2.2 Perfect $\mathbb{F}_p$-algebras

The Witt vectors of $\mathbb{F}_p$ are computed by $W_r(\mathbb{F}_p) = \mathbb{Z}/p^r\mathbb{Z}$ (in fact this computation was the motivation for defining W_r!). It follows that $W(\mathbb{F}_p) = \mathbb{Z}_p$ the p-adic integers.

Now let K be a perfect $\mathbb{F}_p$-algebra and denote the Frobenius $\varphi : K \to K$. Then by functoriality of W_r there is a diagram

$$\begin{array}{ccc} W_r(K) & \longrightarrow & K \\ \uparrow & & \uparrow \\ \mathbb{Z}/p^r\mathbb{Z} & \longrightarrow & \mathbb{F}_p \end{array}$$

where the top map is surjective. This implies that $W_r(K)$ is a lift of A to $\mathbb{Z}/p^r\mathbb{Z}$-algebras. Taking the inverse limit, we get that $W(K)$ is a lift of K to $\mathbb{Z}_p$-algebras.

Proposition 2.4. *Let K be a perfect $\mathbb{F}_p$-algebra.*

(i) $W(A)$ is a p-adically complete, p-torsion free $\mathbb{Z}_p$-algebra such that $W(A)/p^r = W_r(A)$ for all r.

(ii) Conversely, if A is any p-adically complete, p-torsion free $\mathbb{Z}_p$ algebra with $A/p = K$, then there is a unique multiplicative lift $[\ ] : K \to A$ and the map

$$W(K) \to A$$

given by

$$x = (x_0, x_1, \ldots) \mapsto \sum_{i=0}^{\infty} V^i[x_i] = \sum_{i=0}^{\infty} p^i [x_i]^{1/p^i}$$

is an isomorphism.

Remark 2.5. This proposition characterizes the Witt vectors of a perfect $\mathbb{F}_p$ algebra K as the unique lift of K to a p-adically complete p-torsion free $\mathbb{Z}_p$-algebra. The existence and uniqueness of $W(K)$ in this case can also be argued formally. Indeed one computes using the Frobenius map $\varphi : K \to K$ that the cotangent complex $\mathbb{L}_{K/\mathbb{F}_p} \simeq 0$. Therefore there is a unique deformation of K over $\mathbb{Z}/p^r\mathbb{Z} \to \mathbb{F}_p$ which is necessarily $W_r(K)$.

The Frobenius isomorphism $\varphi : K \to K$ lifts to a Frobenius isomorphism $\varphi : W_r(K) \to W_r(K)$ for each r such that the diagram

$$\begin{array}{ccc} W_r(K) & \xrightarrow{\varphi} & W_r(K) \\ & \searrow^{F} & \downarrow R \\ & & W_{r-1}(K) \end{array}$$

commutes.

Lemma 2.6. *Let K be a perfect $\mathbb{F}_p$-algebra. Then the following relations are satisfied:*

1. $F[a] = [a]^p$,
2. $V[a] = p[a]^{1/p}$,
3. $FV = VF = p$,
4. $V^i[a]V^j[b] = p^i V^j[ab^{p^{j-i}}]$ *where* $j \geq i$.

3 Construction of $\mathbb{A}_{\inf}$

3.1 Tilting

Let A be a commutative ring that's π-adically complete and separated for some $\pi \in S$ with $\pi | p$ and let

$$\varphi : A/pA \to A/pA$$

be the Frobenius map.

The tilt of A is the perfect $\mathbb{F}_p$-algebra defined as

$$A^\flat := \varprojlim_{\varphi} A/pA$$

Lemma 3.1. *The natural maps*

$$\varprojlim_{x\mapsto x^p} A \to A^\flat \to \varprojlim_{\varphi} A/\pi A$$

are isomorphisms where the first is a map of monoids and the second is a map of rings.

Proof. Let $(x_i), (y_i) \in \varprojlim_{x\mapsto x^p} A$ be an inverse system of p^{th}-power roots mapping to the same element in $A^\flat$. Then $x_i = y_i \mod p$ and so by induction we deduce that for any n,

$$x_{i+n}^{p^n} = y_{i+n}^{p^n} \mod p^{n+1}.$$

It follows that $x_i = y_i \mod p^{n+1}$. Since n was arbitrary then $x_i = y_i$ by p-adic separatedness so the first map is injective.

Let $(y_i) \in A^\flat$ be an inverse system of p^{th}-power roots in A/pA and pick lifts $\tilde{y}_i \in A$. Then one can check that the limit

$$\lim_{n\to\infty} (\tilde{y}_{i+n}^{p^n}) = x_i \in A$$

exists and $(x_i) \in \varprojlim_{x\mapsto x^p} A$ gives an inverse system mapping to (y_i).

The isomorphism $\varprojlim_{x\mapsto x^p} A \to \varprojlim_{\varphi} A/\pi A$ follows by the same argument and the fact that the induced map $A^\flat \to \varprojlim_{\varphi} A/\pi A$ is a ring homomorphism follows since both rings are characteristic p.

□

Under the isomorphism in Lemma 3.1, we will identify $x = (x_0, x_1, \ldots) \in A^\flat$ with

$$(x^{(0)}, x^{(1)}, \ldots) \in \varprojlim_{x\mapsto x^p} A$$

. Note that $x_{i+1}^p = x_i$ and $(x^{(i+1)})^p = x^{(i)}$.

Definition 3.2. Let A be as above. The ring $\mathbb{A}_{\inf}(A)$ is defined by

$$\mathbb{A}_{\inf}(A) := W(A^\flat).$$

3.2 The maps $\theta_r, \tilde{\theta}_r$

Let A as above be a π-adically complete ring for some $\pi \in A$ with $\pi | p$.

Proposition 3.3. *There are isomorphisms*

$$W(A^\flat) \xleftarrow[(i)]{\varphi^\infty} \varprojlim_F W_r(A^\flat) \xrightarrow[(ii)]{} \varprojlim_F W_r(A/\pi A) \xleftarrow[(iii)]{} \varprojlim_F W_r(A)$$

where

(i) φ^∞ *is induced by* $\varphi^r : W_r(A^\flat) \to W_r(A^\flat)$ *for each* r*,*

(ii) the second arrow is induced by the natural map $A^\flat \to A/pA \to A/\pi A$*,*

(iii) and the third arrow is induced by $A \to A/\pi A$*.*

Proof. (i) Since $A^\flat$ is a perfect ring of characteristic p, we have a commutative diagram

$$\begin{array}{ccc} W_{r+1}(A^\flat) & \xrightarrow{\varphi} & W_{r+1}(A^\flat) \\ & \searrow^{F} & \downarrow R \\ & & W_r(A^\flat) \end{array}$$

where φ is an isomorphism. Taking the inverse limit and using that φ and R commute, we obtain that $\varphi^\infty : \varprojlim_F W_r(A^\flat) \to W_r(A^\flat)$ is an isomorphism.

(ii) We have the following maps

$$\varprojlim_F W_r(A^\flat) \cong \varprojlim_F \varprojlim_\varphi W_r(A/\pi A) \cong \varprojlim_\varphi \varprojlim_F W_r(A/\pi A) \to \varprojlim_F W_r(A/\pi A).$$

where the final map is projection onto the first factor. Here we have used that W_r commutes with limits, limits commute with limits and $A^\flat \cong \varprojlim_\varphi A/\pi A$ by Lemma 3.1 to show the first three isomorphisms. Finally note that φ is an isomorphism on $\varprojlim_F W_r(A/\pi A)$ since $R\varphi = \varphi R = F$ for Witt vectors of a characteristic p ring. Thus the final projection is also an isomorphism.

(iii) First we claim that for any s,

$$\varprojlim_F W_r(A/\pi^s A) \to \varprojlim_F W_r(A/\pi A)$$

induced by $A/\pi^s A \to A/\pi A$ is an isomorphism.

Indeed it is level-wise surjective so we need to check that the kernel is zero in the limit. At each level the kernel is given by

$$W_r(\pi A/\pi^s A)$$

which is generated by elements of the form $V^i[\pi a_i]$. For some c, consider the Frobenius map

$$F^{s+c} : W_{r+s+c}(A/\pi^s A) \to W_r(A/\pi^s A).$$

Using Lemma 2.6, we compute

$$F^{s+c}V^i[\pi a_i] = p^i[\pi a_i]^{p^{s+c-i}}.$$

For $i < c$ this is vanishes since $\pi^{s+c-i} = 0$. By Lemma 3.4, we can pick c large enough so that $p^i = 0$ in $W_r(A/\pi^s A)$ for any $i \geq c$. This shows that at each level, the kernel $W_r(piA/pi^s A)$ is killed by a large enough Frobenius map so its 0 in the inverse limit proving the claim.

Now we may take the limit over s to obtain

$$\varprojlim_F W_r(A/\pi A) \cong \varprojlim_s \varprojlim_F W_r(A/\pi^s A) = \varprojlim_F W_r(\varprojlim_s A/\pi^s A) = \varprojlim_F W_r(A)$$

where the last equality is by π-adic completeness of A.

□

Lemma 3.4. *There exists a large enough $c \gg 0$ (depending on r and s) such that $p^c = 0 \in W_r(A/\pi^s A)$.*

Proof. $W_r(A/\pi A)$ is a $W_r(\mathbb{F}_p) = \mathbb{Z}/p^r\mathbb{Z}$ algebra so $p^r = 0$. Thus p^r is in the kernel of the map $W_r(A/\pi^s A) \to W_r(A/\pi A)$ so it can be written as

$$p^r = \sum V^i[a_i\pi] \in W_r(A/\pi^s A).$$

Now expand

$$(p^r)^t = \left(\sum V^i[a_i\pi]\right)^t$$

and note that

$$V^j[a_j\pi]V^i[a_i\pi] = p^j V^i[\pi a_j(\pi a_i)^{p^{i-j}}]$$

for $i \geq j$ by Lemma 2.6. So for large enough t, the power of π in the product of terms $V^i[a_i\pi]$ will be 0 since $\pi^s = 0$, completing the proof. □

Now using the isomorphisms in Proposition 3.3, we obtain an isomorphism

$$\mathbb{A}_{\text{inf}}(A) = W(A^\flat) \cong \varprojlim_F W_r(A).$$

Definition 3.5. The map

$$\tilde{\theta}_r : \mathbb{A}_{\text{inf}}(A) \to W_r(A)$$

is defined as the composition of the isomorphism above with the projection onto $W_r(A)$. The map θ_r is defined as

$$\theta_r := \tilde{\theta}_r \circ \varphi^r : \mathbb{A}_{\text{inf}}(A) \to W_r(A).$$

Lemma 3.6. *Let $x \in A^\flat$ so that $[x] \in W(A^\flat) = \mathbb{A}_{\text{inf}}(A)$. Then $\theta_r([x]) = [x^{(0)}]$ and $\tilde{\theta}_r([x]) = [x^{(r)}]$ in $W_r(A)$.*

Proof. The Teichmuller representative $[x] \in W(A^\flat)$ maps to the inverse system

$$([x], [x]^{1/p}, [x]^{1/p^2}, \ldots) \in \varprojlim_F W_R(A^\flat)$$

under $(\varphi^\infty)^{-1}$. Then commuting the limits and projecting in construction of the second map of Proposition 3.3 gives us

$$([x], [x]^{1/p}, [x]^{1/p^2}, \ldots) \mapsto ([x_0], [x_0]^{1/p}, [x_0]^{1/p^2}, \ldots) = ([x_0], [x_1], [x_2], \ldots)$$

for $x = (x_0, x_1, \ldots) \in A^\flat$ which maps to $([x^{(0)}], [x^{(1)}], \ldots)$ under the lift to $\varprojlim_F W(A)$ in the third isomorphism of Proposition 3.3 which completes the proof.

□

Corollary 3.7. *There are commutative diagrams*

$$\begin{array}{ccc} \mathbb{A}_{\mathrm{inf}}(A) & \xrightarrow{\theta_r} & W_r(A) \\ {\scriptstyle R}\downarrow & & \downarrow \\ W_r(A^\flat) & \longrightarrow & W_r(A/pA) \end{array}$$

where the right and bottom mpas are induced by the projections $A \to A/pA$ and $A^\flat \to A/pA$.

Proof. This follows from Lemma 3.6 and the fact that under the identification $x = (x_0, x_1, \ldots) \in A^\flat$ with $(x^{(0)}, x^{(1)}, \ldots) \in \varprojlim_{x \to x^p} A$, $x^{(i)} = x_i \mod p$.

□

Remark 3.8. The above diagram when $r = 1$ shows that $\mathbb{A}_{\mathrm{inf}}(A)$ interpolates between characteristic 0 geometry of A and characteristic p geometry of $A^\flat$. In particular, it is crucial that $\mathbb{A}_{\mathrm{inf}}(A)$ has a Frobenius automorphism φ. This will produce a Frobenius action on the $\mathbb{A}_{\mathrm{inf}}$ cohomology despite the fact that A itself doesn't necessarily have a Frobenius.

Finally we state the compatibilities of θ_r, $\tilde{\theta}_r$ with the usual Witt vector maps.

Lemma 3.9. *(a) There are commutative diagrams*

$$\begin{array}{ccc} \mathbb{A}_{\mathrm{inf}}(A) & \xrightarrow{\theta_{r+1}} & W_{r+1}(A) \\ \| & & \downarrow{\scriptstyle R} \\ \mathbb{A}_{\mathrm{inf}}(A) & \xrightarrow{\theta_r} & W_r(A) \end{array} \quad \begin{array}{ccc} \mathbb{A}_{\mathrm{inf}}(A) & \xrightarrow{\theta_{r+1}} & W_{r+1}(A) \\ {\scriptstyle \varphi}\downarrow & & \downarrow{\scriptstyle F} \\ \mathbb{A}_{\mathrm{inf}}(A) & \xrightarrow{\theta_r} & W_r(A) \end{array} \quad \begin{array}{ccc} \mathbb{A}_{\mathrm{inf}}(A) & \xrightarrow{\theta_{r+1}} & W_{r+1}(A) \\ {\scriptstyle \lambda_{r+1}\varphi^{-1}}\uparrow & & \uparrow{\scriptstyle V} \\ \mathbb{A}_{\mathrm{inf}}(A) & \xrightarrow{\theta_r} & W_r(A) \end{array}$$

where λ_{r+1} is an element satisfying $\theta_{r+1}(\lambda_{r+1}) = V(1) \in W_{r+1}(A)$.

(b) There are commutative diagrams

$$\begin{array}{ccc} \mathbb{A}_{\mathrm{inf}}(A) & \xrightarrow{\tilde{\theta}_{r+1}} & W_{r+1}(A) \\ {\scriptstyle \varphi}\downarrow & & \downarrow{\scriptstyle R} \\ \mathbb{A}_{\mathrm{inf}}(A) & \xrightarrow{\tilde{\theta}_r} & W_r(A) \end{array} \quad \begin{array}{ccc} \mathbb{A}_{\mathrm{inf}}(A) & \xrightarrow{\tilde{\theta}_{r+1}} & W_{r+1}(A) \\ \| & & \downarrow{\scriptstyle F} \\ \mathbb{A}_{\mathrm{inf}}(A) & \xrightarrow{\tilde{\theta}_r} & W_r(A) \end{array} \quad \begin{array}{ccc} \mathbb{A}_{\mathrm{inf}}(A) & \xrightarrow{\tilde{\theta}_{r+1}} & W_{r+1}(A) \\ {\scriptstyle \tilde{\lambda}_{r+1}}\uparrow & & \uparrow{\scriptstyle V} \\ \mathbb{A}_{\mathrm{inf}}(A) & \xrightarrow{\tilde{\theta}_r} & W_r(A) \end{array}$$

where $\tilde{\lambda}_{r+1} = \varphi^{r+1}(\lambda_{r+1})$ is an element satisfying $\tilde{\theta}_{r+1}(\tilde{\lambda}_{r+1}) = V(1) \in W_{r+1}(A)$.

Proof. Parts (a) and (b) are equivalent by composing with φ so we prove (a) only. It suffices to check on Teichmuller representatives. The first and second diagrams follow directly from Lemma 3.6 and the properties of R, F and φ.

For the third diagram, note that $V(1) = p[1]^{1/p} = p = VF$ so

$$\theta_{r+1}(\lambda_{r+1}\varphi^{-1}[x]) = V(1)[x^{(0)}]^{1/p} = VF[x^{(0)}]^{1/p} = V[x^{(0)}] = V\theta_r([x]).$$

□

Corollary 3.10. *We can define* $\lim_{r\to\infty}\theta_r =: \theta_\infty : \mathbb{A}_{\mathrm{inf}}(A) \to W(A)$ *which sits in a commutative diagram*

$$\begin{array}{ccc} \mathbb{A}_{\mathrm{inf}}(A) & \xrightarrow{\theta_\infty} & W(A) \\ \| & & \downarrow \\ W(A^\flat) & \longrightarrow & W(A/pA) \end{array}$$

Proof. We may take the limit by the first diagram of Lemma 3.9(a) and the commutative diagram follows by Corollary 3.7. □

Finally we discuss the composition of θ_r with the ghost map $w : W_r(A) \to A^r$.

Lemma 3.11.

$$w \circ \theta_r = (\theta, \theta\varphi, \theta\varphi^2, \ldots, \theta\varphi^{r-1})$$

Proof. We can compute for any $x \in A^\flat$, we have

$$w(\theta_r(x)) = w([x^{(0)}]) = w(x^{(0)}, 0, \ldots) = (x^{(0)}, x^{(0)p}, x^{(0)p^2}, \ldots) = (\theta x, \theta\varphi x, \theta\varphi^2 x, \ldots).$$

□

4 Perfectoid rings

We would like to interpret the diagrams

$$\begin{array}{ccc} \mathbb{A}_{\mathrm{inf}}(A) & \xrightarrow{\theta_r} & W_r(A) \\ {\scriptstyle R}\downarrow & & \downarrow \\ W_r(A^\flat) & \longrightarrow & W_r(A/pA) \end{array}$$

from Corollary 3.7 as a diagram of pro-infinitessimal thickenings.

Indeed for $r = 1$ and $A = \mathcal{O}_C$ for C a perfectoid field of characteristic 0, the diagram

$$\begin{array}{ccc} \mathbb{A}_{\mathrm{inf}}(\mathcal{O}_C) & \xrightarrow{\theta} & \mathcal{O}_C \\ \downarrow & & \downarrow \\ \mathcal{O}_{C^\flat} & \longrightarrow & \mathcal{O}_C/p \end{array}$$

is familiar in p-adic hodge theory and the map θ behaves as a sort of 1-parameter deformation of $\mathcal{O}_C$. We'll study this case more carefully in the next section.

In this section we'll explore the properties required by the ring for θ_r to behave as such which leads directly to the definition of perfectoid rings.

4.1 Surjectivity and the kernel of θ

Throughout this section, let S be a π-adically complete ring such that $\pi^p|p$.[1]

Lemma 4.1. *The following are equivalent:*

(i) every element of $S/\pi pS$ is a p^{th} power;

(ii) every element of S/pS is a p^{th} power;

(iii) every element of $S/\pi^p pS$ is a p^{th} power;

(iv) $F : W_{r+1}(S) \to W_r(S)$ is surjective for all $r \geq 1$;

(v) $\theta_r : \mathbb{A}_{\mathrm{inf}}(S) \to W_r(S)$ is surjective for all $r \geq 1$;

(vi) $\theta : \mathbb{A}_{\mathrm{inf}}(S) \to S$ is surjective.

Proof. $(i) \implies (ii) \implies (iii)$ is clear since $\pi^p|p|\pi p$.

For $(iii) \implies (i)$, let $y \in S$. We can write $y = x_0^p \mod p$ by assumption so $y = x_0^p + \pi^p y_1$. By induction

$$y = \sum x_i^p \pi^{pi} = \left(\sum x_i \pi^i\right)^p \mod \pi p.$$

For $(iv) \implies (ii)$ we have that $F : W_2(S) \to W_1(S) \cong S$ is given explicitly by $F(a_0, a_1) = a_0^p + a_1 p = a_0^p \mod p$. Since F is surjective, every element of S/pS is a p^{th} power.

$(ii) \implies (iv)$ is a result of Davis-Kedlaya [DK14].

$(iv) \implies (v)$ follows by the definition of θ since F surjective implies $\varprojlim_F W_r(S) \to W_r(S)$ is surjective.

$(v) \implies (vi)$ is clear.

$(vi) \implies (ii)$ follows from the computation $\theta([x]) = [x^{(0)}] = [x^{(1)}]^p \mod p$ so $x^{(0)}$ is a p^{th} power in S/pS. □

Corollary 4.2. *Under the equivalent conditions of Lemma 4.1, there exist units $u, v \in S^\times$ such that $u\pi$ and vp admit a compatible system of p-power roots.*

Proof. Applying Lemma 3.1 to S and $S/\pi pS$ gives an isomorphism

$$\varprojlim_{x \mapsto x^p} S \cong \varprojlim_{\varphi} S/\pi pS.$$

By assumption we can take a compatible system of p-power roots for $\pi \mod \pi p$ which corresponds to a compatible system $x = (x^{(0)}, x^{(1)}, \ldots)$ on the left such that $x^{(0)} = \pi \mod \pi p$. Writing

$$x^{(0)} = \pi + \pi p y = \pi(1 + py)$$

[1]Note this is stronger than the conditions we had on A above.

we see that $x^{(0)}$ differs from π by the unit $1 + py$. The same argument works with π replaced by p.

$\square$

Now we move on to studying the kernel of θ.

Definition 4.3. An element $\xi \in \ker\theta$ is distinguished if ξ_1 is a unit in $S^\flat$ where $\xi = (\xi_0, \xi_1, \dots) \in W(S^\flat)$ is its Witt vector expansion.

Proposition 4.4. *Let S as above and suppose that $\varphi : S/\pi S \to S/\pi^p S$ is surjective (so that S satisfies all the equivalent properties of Lemma 4.1).*

(i) if $\ker\theta$ is principal, then we have

  (a) $S/\pi S \to S/\pi^p S$ is an isomorphism;

  (b) any generator of $\ker\theta$ is a non-zero divisor;

  (c) $\xi \in \ker\theta$ is a generator if and only if ξ is distinguished;

  (d) if $\theta_r(\xi) = V(1)$ for some r then $\xi \in \ker\theta$ and ξ is distinguished.

(ii) Conversely, if π is a non-zero divisor and $\varphi : S/\pi S \to S/\pi^p S$ is an isomorphism, then $\ker\theta$ is principal.

Proof sketch. By Corollary 4.2 we can suppose that π has a compatible system of p-power roots and let $\varpi \in S^\flat$ be the corresponding element under $\varprojlim_{x\mapsto x^p} S \cong S^\flat$. Using surjectivity of θ and that $\pi^p | p$, we may write

$$p + \pi^p\theta(x) = 0$$

for some x. Define $\xi := p + [\varpi]^p x$ so that $\theta(\xi) = 0$. We want to use the diagram

$$\begin{array}{ccc} \mathbb{A}_{\mathrm{inf}}(S)/\xi & \xrightarrow{\theta} & S \\ \big\downarrow{\scriptstyle \bmod [\varpi]^p} & & \big\downarrow \\ \mathbb{A}_{\mathrm{inf}}(S)/(\xi, [\varpi]^p) & \longrightarrow & S/\pi^p S \end{array} \tag{1}$$

Here the bottom map is $\mathbb{A}_{\mathrm{inf}}(S)/(\xi, [\varpi]^p) = \mathbb{A}_{\mathrm{inf}}(S)/(p, [\varpi]^p) = S^\flat/\varpi^p S^\flat \to S/\pi^p S$.

For (i) suppose $\ker(\theta)$ is generated by some ξ'. One shows that ξ is also a generator by writing $\xi = a\xi'$ and computing in Witt vector components that ξ'_1 and a_0 must be units. This implies that a is a unit and that any $\xi'' \in \ker\theta$ with component ξ''_1 must also be a generator.

To see that ξ (and therefore any other generator) is a non-zero divisor one again expands out $\xi b = 0$ into Witt vector components and uses π-adic separatedness to force $b = 0$.

For (a), since ξ generates $\ker\theta$, the top morphism in diagram 1 is an isomorphism and so the bottom map $S^\flat/\varpi^p S^\flat \to S/\pi^p S$ is an isomorphism. On the other hand $S^\flat$ is perfect so we have

$$\begin{array}{ccc} S^\flat/\varpi^p S^\flat & \xrightarrow{\sim} & S/\pi^p S \\ \big\uparrow{\scriptstyle \varphi} & & \big\uparrow{\scriptstyle \varphi} \\ S^\flat/\varpi S^\flat & \xrightarrow{\sim} & S/\pi S \end{array}$$

where the left, top and bottom are isomorphisms so the right is an isomorphism.

Finally suppose ξ is any element with $\theta_r(\xi) = V(1) = (0, 1, 0, \ldots)$. Then $\theta(\xi) = 0$ by compatibility of θ_r with restriction. On the other hand, $R(\xi) = (\xi_0, \xi_1, \ldots, \xi_{r-1}) = \theta_r(\xi) \mod p = (0, 1, 0, \ldots, 0)$ by Corollary 3.7 so $\xi_1 = 1 \mod p$ lifts to a unit in $S^\flat$.

For part (ii), suppose conversely that $S/\pi S \to S/\pi^p S$ is an isomorphism and π is a non-zero divisor and let ξ, ϖ as in diagram 1.

Using surjectivity, one checks surjectivity $S/\pi^{1/p^n} S \to S/\pi^{1/p^{n-1}} S$ and concludes that ϖ generates the kernel of $S^\flat \to S/\pi S$ by expanding any element into components under the isomorphism of Lemma 3.1 and checking that π^{1/p^n} divides the corresponding component of any element in the kernel. It follows that $S^\flat/\varpi S^\flat \to S/\pi S$ is an isomorphism.

Using diagram 1, write $x \in \ker(\theta)$ as $x = \xi y_0 + [\varpi] x_1$ since by commutativity x must become 0 in the composition $\mathbb{A}_{\text{inf}}(S)/\xi \to S^\flat/\varpi^p S^\flat \to S^\flat/\varpi S^\flat = S/\pi S$. Then

$$0 = \theta(x) = \theta(\xi y_0) + \theta([\varphi] x_1) = \pi\theta(x_1)$$

so $\theta(x_1) = 0$ since π is not a zero divisor. Then $x_1 = \xi y_1 + [\varpi] x_2$ and so on so by induction x is in the ideal generated by ξ.

□

4.2 Perfectoid rings

Now we are equipped to defined perfectoid rings.

Definition 4.5. A ring S is perfectoid if

- S is π-adically complete for some $\pi \in S$ such that $\pi^p | p$;
- $\varphi : S/pS \to S/pS$ is surjective (equivalently θ is surjective);
- $\ker(\theta)$ is a principal ideal.

The idea is that perfectoid rings are precisely the rings so that $\theta : \mathbb{A}_{\text{inf}}(S) \to S$ is a 1-parameter pro-infinitessimal deformation. We can think of it as deforming S in the ξ direction.

Remark 4.6. (Perfectoid rings in characteristic p) Suppose S is a characteristic p ring. Then S is perfectoid if and only if it is perfect. Indeed if S is perfect then it's 0-adically complete and the Frobenius is an isomorphism so $S^\flat \cong S$. Thus $\theta : \mathbb{A}_{\text{inf}}(S) = W(S) \to S$ corresponds to Witt vector restriction and has kernel generated by p.

On the other hand, if S is perfectoid then $p \in \ker(\theta)$ since p is zero in S but $p = V(1)$ so it's distinguished by Lemma 4.4(i)(d). Thus p generates $\ker(\theta)$ and so

$$S^\flat = W(S^\flat)/p \cong S$$

is perfect.

Lemma 4.7. *Suppose S is perfectoid and ξ a generator of $\ker\theta$. Then the non-zero divisor*

$$\xi_r := \xi\varphi^{-1}(\xi)\ldots\varphi^{-(r-1)}(\xi)$$

is a generator for $\ker(\theta_r)$*. Similarly, the non-zero divisor*

$$\tilde{\xi}_r = \varphi^r(\xi)\varphi^{r-1}(\xi)\dots\varphi(\xi)$$

is a generator for $\ker(\tilde{\theta}_r)$*.*

Proof. The two statements are equivalent by applying φ^r so we prove the first. By 4.4(i)(d), we may suppose that $\theta_{r+1}(\xi) = V(1)$ after multiplying by a unit.

Suppose ξ_r is a non-zero divisor generating $\ker(\theta_r)$. Consider the commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathbb{A}_{\mathrm{inf}}(S) & \xrightarrow{\xi\varphi^{-1}} & \mathbb{A}_{\mathrm{inf}}(S) & \xrightarrow{\theta} & S & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle\theta_r} & & \downarrow{\scriptstyle\theta_{r+1}} & & \| & & \\ 0 & \longrightarrow & W_r(S) & \xrightarrow{V} & W_{r+1}(S) & \xrightarrow{R} & S & \longrightarrow & 0 \end{array}$$

The top row is exact since ξ is a non-zero divisor and the bottom row is exact by surjectivity of θ_r. Commutativity is by Lemma 3.9. A diagram chase implies that since ξ_r generates $\ker(\theta_r)$, then $\xi\varphi^{-1}(\xi_r) = \xi_{r+1}$ generates $\ker(\theta_{r+1})$. Indeed if $\theta_{r+1}(x) = 0$ then $\theta(x) = 0$ so $x = \xi\varphi^{-1}(y)$ but $y \in \ker(\theta_r)$ by commutativity of the first square. $\square$

4.3 Perfectoid rings with enough roots of unity

The kernel of θ_r has particularly nice generators in the case where S is a perfectoid ring with many roots of unity. More specifically, suppose S is perfectoid and contains a compatible system $1, \zeta_p, \zeta_{p^2}, \dots$ of primitive p-power roots of unity.[2] This includes as the most impotant example the ring of integers $\mathcal{O}_C$ of a perfectoid field C. We will study this example more closely in the next section.

Definition 4.8. Let S and ζ_{p^r} be as above. Define elements

$$\epsilon := (1, \zeta_p, \zeta_{p^2}, \dots) \in S^\flat, \quad \mu := [\epsilon] - 1 \in W(S^\flat) = \mathbb{A}_{\mathrm{inf}}(S).$$

and

$$\xi := 1 + [\epsilon^{1/p}] + [\epsilon^{2/p}] + \dots + [\epsilon^{(p-1)/p}] = \frac{\mu}{\varphi^{-1}(\mu)} \in W(S^\flat) = \mathbb{A}_{\mathrm{inf}}(S).$$

Lemma 4.9. ξ *is a generator of* $\ker(\theta)$ *satisfying* $\theta_r(\xi) = V(1)$ *for all* $r > 1$*.*

Proof. Note that

$$\theta(\xi) = 1 + \zeta_p + \zeta_p^2 + \dots + \zeta_p^{p-1} = 0$$

by Lemma 3.6 and the definition of ζ_p. By functoriality of Witt vectors and the map θ, it suffices to prove the statement for $\mathbb{Z}_p^{cycl} := (\mathbb{Z}_p[\zeta_{p^\infty}])^\wedge_p$ [3] as the choice of $\zeta_{p^r} \in S$ determines a unique map $\mathbb{Z}_p^{cycl} \to S$.

In particular, we may assume that S is p-torsion free. In this case the ghost map $w : W_r(A) \to A^r$ is injective and so it suffices to compute

$$w(\theta_r(\xi)) = w(V(1)).$$

[2]If S is not an integral domain, a primitive root of unity is defined to be a root of the corresponding cyclotomic polynomial.

[3]This denotes the p-adic completion

By Lemma 3.11 We can compute for any $x \in A^\flat$, we have

$$w(\theta_r(\xi)) = (\theta\xi, \theta\varphi\xi, \theta\varphi^2\xi, \ldots).$$

Since $\theta(\xi) = 0$ and $w(V(1)) = w(0, 1, 0, \ldots) = (0, p, p, \ldots)$, it suffices to show that

$$\theta\varphi^i(\xi) = p$$

for all $i \geq 1$. In this case

$$\begin{aligned}\theta\varphi^i(\xi) &= \theta(1 + [\epsilon^{p^{i-1}}] + [\epsilon^{p^{i-1}}]^2 + \ldots + [\epsilon^{p^{i-1}}]^{p-1}) \\ &= 1 + (\zeta_p^{p^{i-1}}) + (\zeta_p^{p^{i-1}})^2 + \ldots + (\zeta_p^{p^{i-1}})^{p-1} \\ &= p\end{aligned}$$

□

We obtain the following by Lemma 4.7 and a computation.

Corollary 4.10. *The kernel of θ_r is generated by*

$$\xi_r = \xi\varphi^{-1}(\xi)\ldots\varphi^{-(r-1)}(\xi) = \sum_{i=1}^{p^r-1}[\epsilon^{1/p^r}]^i$$

and the kernel of $\tilde{\theta}_r$ is generated by

$$\tilde{\xi}_r = \varphi^r(\xi_r) = \sum_{i=0}^{p^r-1}[\epsilon]^i.$$

Proposition 4.11. *Let S be a perfectoid ring which is flat over $\mathbb{Z}_p$ and contains a compatible sequence of primitive p^{th} roots of unity. Let $\epsilon, \xi_r, \tilde{\xi}_r$ and μ as above. Then for any r,*

(i) μ is a non-zero divisor;

(ii) $\tilde{\theta}_r(\mu) = [\zeta_{p^r}] - 1 \in W_r(S)$ is a non-zero divisor;

(iii) $\mu = \xi_r\varphi^{-r}(\mu)$ and $\varphi^r(\mu) = \tilde{\xi}_r\mu$;

(iv) μ divides $\tilde{\xi}_r - p^r$.

Proof. $\tilde{\theta}_r(\mu) = [\zeta_{p^r}] - 1$ by Lemma 3.6. Since S is flat over $\mathbb{Z}_p$, it is torsion free so $w : W_r(S) \to S^r$ is injective. Thus it suffices to check that that

$$w(\tilde{\theta}_r(\mu)) = (\zeta_{p^r} - 1, \zeta_{p^{r-1}} - 1, \ldots, \zeta_p - 1)$$

is not zero divisor (where we computed the expression using Lemma 3.11 and $\theta_r = \tilde{\theta}_r \circ \varphi^r$. Now $\zeta_{p^r} - 1$ divides p and p is not a zero divisor since S is flat over $\mathbb{Z}_p$ so $\zeta_{p^r} - 1$ is not a zero divisor.

(ii) follows from (i) since $\tilde{\theta}_r(\mu)$ is a non-zero divisor in each $W_r(S)$ and $\mathbb{A}_{\inf}(S) = \varprojlim_F W_r(S)$.

(iii) is computed by noting that $\xi\varphi^{-1}(\mu) = \mu$.
(iv) follows because $[\epsilon] = 1 \mod \mu$ so

$$\tilde{\xi}_r = \sum_{i=0}^{p^r-1}[\epsilon]^i = p^r \mod \mu.$$

□

5 The case of a perfectoid field

The most important case of the above constructions is when $C = \mathbb{C}_p$ is a complete non-archimedean algebraically closed field of mixed characteristic and $S = \mathcal{O} := \mathcal{O}_C$ is the ring of integers. In this case we denote $A_{\text{inf}} := \mathbb{A}_{\text{inf}}(\mathcal{O})$. More generally everything holds when C is any perfectoid field of mixed characteristic.

As $\mathcal{O}$ is a flat $\mathbb{Z}_p$ algebra with enough roots of unity, the discussion in Section 4.3 holds and we let ϵ, ξ, μ, etc be as in *loc. cit.*

A_{inf} is well known in p-adic hodge theory and its relation to Fontaine's other period rings is crucial to comparison theorems. We recall these other rings here.

Definition 5.1. (a) Let A_{crys} be the p-adic completion of the A_{inf}-subalgebra of $A_{\text{inf}}[\frac{1}{p}]$ generated by all elements of the form $\frac{\xi^m}{m!}$.

(b) Let $B^+_{crys} = A_{crys}[\frac{1}{p}]$ and $B_{crys} = A_{crys}[\frac{1}{\mu}] = B^+_{crys}[\frac{1}{\mu}]$.

(c) Let B^+_{dR} be the ξ-adic completition of B^+_{crys} and $B_{dR} = B^+_{dR}[\frac{1}{\xi}]$ be its fraction field.

Remark 5.2. (a) The ring A_{crys} is the universal p-adically complete divided power thickening of $\mathcal{O}$ over $\mathbb{Z}_p$.

(b) The last equality in (b) uses the computation $\mu^{p-1} = \xi^p \mod p$ so that $\mu^{p-1} \in pA_{crys}$.

(c) B^+_{dR} is a DVR with residue field C.

The ring A_{inf} satisfies the following properties that we won't prove here:

Lemma 5.3. *The kernel of the map*

$$\theta_\infty : A_{\text{inf}} \to W(\mathcal{O})$$

is generated by μ. That is,

$$\bigcap_r \frac{\mu}{\varphi^{-r}(\mu)} A_{\text{inf}} = \mu A_{\text{inf}}.$$

In particular, the ideal (μ) is independent of the choice of roots of unity.

If C is a spherically complete field, then the cokernel of θ_∞ is zero. More generally, the cokernel is $W(\mathfrak{m}^\flat)$-torsion where $\mathfrak{m}^\flat$ is the maximal ideal of $\mathcal{O}^\flat$.

Remark 5.4. Recall that a non-archimedean field is spherically complete if any decreasing sequence of discs has nonempty intersection.

Another useful property is coherence. Recall that a ring is coherent if every finitely generated ideal is finitely presented.

Proposition 5.5. *For each $r \geq 1$, the ring $W_r(\mathcal{O})$ is coherent.*

Remark 5.6. It's not known whether A_{inf} is coherent.

Finally it is instructive to think of A_{inf} analagously to a two dimensional regular local ring.[4] Indeed the pair (p, ξ) where ξ is any generator of $\ker(\theta)$ is a regular sequence, A_{inf} is (p, ξ)-adically complete, and the radical of (p, ξ) is the maximal ideal of A_{inf}.

[4]Note that A_{inf} is *not* two dimensional. In fact it is known that the Krull dimension is at least 3 though we don't know it exactly.

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