# THE HILBERT ZETA FUNCTION IS CONSTRUCTIBLE IN FAMILIES OF CURVES

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VERY PRELIMINARY DRAFT

ABSTRACT. In this note we show that the generating series for the topological Euler characteristic of the Hilbert schemes of points on a curve singularity is constructible in families.

## 1. INTRODUCTION

The goal of this note is to present ideas related to the following expectation:

**Expectation 1.** *Let $(C, p)$ be the germ of a reduced curve singularity over $\mathbb{C}$. The Euler characteristic Hilbert zeta function*

$$Z^{\text{top}}_{C,p}(t) := \sum_{d \geq 0} \chi_{top}(\text{Hilb}^d(C,p))t^d$$

*depends only on the topology of $C$ and combinatorics of an embedded resolution.*

Here $\text{Hilb}^d(C, p)$ is the reduced Hilbert scheme of $d$ points supported on $p \in C$ which can be identified with the parameter space of ideals of codimension $d$ in $\widehat{\mathcal{O}}_{C,p}$.

$$\text{Hilb}^d(C,p) = \{I \subset \widehat{\mathcal{O}}_{C,p} \ : \ \dim_{\mathbb{C}} \widehat{\mathcal{O}}_{C,p}/I = d\}$$

Furthermore $\chi_{top}$ denotes the compactly supported topological Euler characteristic.

By a family of reduced curve singularities $(\mathcal{C} \to B, \sigma)$ we mean a flat family of reduced curves $\mathcal{C} \to B$ as well as a section $\sigma : B \to \mathcal{C}$ such that $\mathcal{C}_b \setminus \sigma(b)$ is nonsingular for all $b \in B$. The main result of this note is the following evidence for Expectation 1.

**Theorem 1.** *Let $(\mathcal{C} \to B, \sigma)$ be a flat family of reduced curve singularities. Then*

$$b \mapsto Z^{\text{top}}_{\mathcal{C}_b, \sigma(b)}(t)$$

*is a constructible function $B \to \mathbb{Z}[\![t]\!]$.*

Theorem 1 implies that the Hilbert zeta function $Z^{\text{top}}_{C,p}(t)$ is a discrete invariant of the singularity $(C, p)$. The main question then is exactly what type of discrete information about the singularity does the Hilbert zeta function encode?

For planar curves Maulik [Mau16], verifying a conjecture of Oblomkov-Shende [OS12], proved $Z^{top}_{C,p}(t)$ is a topological invariant. An answer to the above question for planar curves has recently emerged due to a large body of work connecting $Z^{top}_{C,p}(t)$ to compactified Jacobians, knot invariants, string theory, enumerative geometry of Calabi-Yau threefolds, affine Springer theory, the Hitchin fibration, representation theory of Cherednik algebras, etc (e.g. [Kas15, OS12, Mau16, MS13, MSV15, MY14, DSV13, DHS12, GORS14, OY16, GN15, Ng06]). We hope that Theorem 1 as well as the rationality result of [BRV17] are the first steps in extending parts of this picture to non-planar curves.

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1.1. **Proof of Theorem 1.** In [BRV17], Ranganathan, Vakil and the author showed that $Z^{\text{top}}_{C,p}(t)$ is a rational function of the form $P(t)/(1-t)^s$ where $s$ is the number of branches [BRV17, Theorem 2.3][1]. More precisely, there is an expansion of the following form.

$$
\begin{aligned}
Z^{\text{top}}_{C,p}(t) = &\sum_{a_1=0}^{c_1-1}\cdots\sum_{a_s=0}^{c_s-1}\sum_{d\geq 0}\chi_{top}(\text{Hilb}^{d,a_1,\ldots,a_s}(C,p))t^d \\
&+\sum_{a_1=0}^{c_1-1}\cdots\sum_{a_{s-1}=0}^{c_{s-1}-1}\frac{1}{1-t}\sum_{d\geq 0}\chi_{top}(\text{Hilb}^{d,a_1,\ldots,c_s}(C,p))t^d \\
&+\sum_{a_1=0}^{c_1-1}\cdots\sum_{a_{s-2}=0}^{c_{s-2}-1}\frac{1}{(1-t)^2}\sum_{d\geq 0}\chi_{top}(\text{Hilb}^{d,a_1,\ldots,c_{s-1},c_s}(C,p))t^d \\
&\vdots \\
&+\frac{1}{(1-t)^s}\sum_{d\geq 0}\chi_{top}(\text{Hilb}^{d,c_1,\ldots,c_s}(C,p))t^d
\end{aligned}
$$

Here $\text{Hilb}^{d,a_1,\ldots,a_s}(C,p) \subset \text{Hilb}^d(C,p)$ are certain subvarieties indexed by $a_i \in \mathbb{N}$ and $c_i$ are the conductors of each branch (see Section 3).

Furthermore, there are uniform bounds

$$-\delta \leq d - \sum_{i=1}^{s} a_i \leq c - \delta.$$

where $\delta$ and $c = \sum c_i$ are the $\delta$-invariant and total conductor (Section 3). In particular, $d \leq 2c-\delta$ in any of the terms $\text{Hilb}^{d,a_1,\ldots,a_s}(C,p)$ appearing in the expansion above. Multiplying the expression by $(1-t)^s$ we see that the degree of $P(t)$ is bounded above by $2c-\delta+s$. From the expression $Z^{\text{top}}_{C,p}(t) = P(t)/(1-t)^s$, we can then determine $P(t)$ from the first $2c-\delta+s$ coefficients of $Z^{\text{top}}_{C,p}(t)$.

e Now let $(f : \mathcal{C} \to B, \sigma)$ be a family of reduced curve singularity over a base $B$ of finite type. We may suppose without loss of generality that $B$ is normal. Then by Corollary 3.5 and Proposition 3.6 there is a finite stratification of the base over which $\delta, s$ and $c$ are constant so we may suppose $f$ is a $(\delta, s, c)$-constant family. By Proposition 2.4 there exists a projective morphism $\pi_d : \text{Hilb}^d(\mathcal{C}/B) \to B$ whose fiber over $b \in B$ is $\text{Hilb}^d(\mathcal{C}_b, \sigma(b))$. For each $d$ there exists a finite stratification of $B$ so that over each stratum, the fibers of $\pi_d$ have the same topological Euler characteristic. We may take the refinement of all these stratifications for $1 \leq d \leq 2c-\delta+s$. This produces a stratification such that $\chi_{top}(\text{Hilb}^d(\mathcal{C}_b, \sigma(b))$ is constant on strata for all $1 \leq d \leq 2c-\delta+s$. As these coefficients suffice to determine the full zeta function $Z^{\text{top}}_{\mathcal{C}_b,\sigma(b)}(t)$, we are done.

**Remark 1.1.** Note in fact that the proof of Theorem 1 applies verbatim with $\chi_{top}$ replaced by any invariant $\chi$ satisfying the following two properties: (1) $\chi$ factors through the Grothendieck ring of varieties, (2) $\chi$ is constructible in families of varieties. In a future version of this paper we will generalize the result to many other invariants.

[1] In fact we show the same is true in the Grothendieck ring of varieties $K_0(\text{Var})$

## 2. HILBERT SCHEMES OF POINTS ON CURVES

In this section we will define the Hilbert zeta function $Z_{C,p}(t)$ and construct relative Hilbert schemes for families of germs of reduced curve singularities $(f : \mathcal{C} \to B, \sigma)$. We will always assume that $B$ is noetherian and that there is an embedding of germs

$$(\mathcal{C}, \sigma) \subset (\mathbb{C}^N \times B, 0 \times B)$$

so that $(f : \mathcal{C} \to B, \sigma)$ is the germ of a family of reduced affine curves.

2.1. **Hilbert zeta functions.** For $X$ a quasiprojective variety, the Hilbert scheme $\mathrm{Hilb}^d(X)$ is the moduli space for flat families of length $d$ subschemes of $X$. Using the identification between length $d$ subschemes $Z \subset X$ and ideal sheaves $J$ with $\mathrm{length}(\mathcal{O}_X/J) = d$, we will represent the closed points of $\mathrm{Hilb}^d(X)$ by the corresponding ideals.

There is a well defined Hilbert-Chow morphism (see, for example, [FGI+05, Chapter 7])

$$h : \mathrm{Hilb}^d(X) \to \mathrm{Sym}^d(X)$$

sending a subscheme to its support:

$$[J] \mapsto \sum_{p \in \mathrm{Supp}(\mathcal{O}_X/J)} \mathrm{length}(\mathcal{O}_{X,p}/J_p)[p].$$

When $X$ is a smooth curve, $h$ is an isomorphism.

Let $Y \subset X$ be a closed $k$-subvariety. Then $\mathrm{Sym}^d(Y) \subset \mathrm{Sym}^d(X)$ is a closed subvariety and we define $\mathrm{Hilb}^d(X, Y)$ the *Hilbert scheme with support in* $Y$ as the scheme theoretic preimage $h^{-1}(\mathrm{Sym}^d(Y))$ by the Hilbert-Chow morphism $h : \mathrm{Hilb}^d(X) \to \mathrm{Sym}^d(X)$. Set theoretically, $\mathrm{Hilb}^d(X, Y) \subset \mathrm{Hilb}^d(X)$ consists of length $d$ subschemes $Z \subset X$ with support $\mathrm{supp}(\mathcal{O}_Z)$ contained in $Y$.

We define the *motivic Hilbert zeta function with support in* $Y$ as:

$$Z_{Y \subset X}(t) := \sum_{d \geq 0} [\mathrm{Hilb}^d(X, Y)] t^d \in 1 + tK_0(\mathrm{Var})[\![t]\!]$$

Furthermore, for any ring homomorphism $\mu : K_0(\mathrm{Var}) \to A$ we define the *$\mu$-Hilbert zeta function with support in* $Y$ by

$$Z_{Y \subset X}(t, \mu) := \sum_{d \geq 0} \mu(\mathrm{Hilb}^d(X, Y)) t^d \in 1 + tA[\![t]\!].$$

The Hilbert zeta function respects the following scissor relation for $Y \subset X$ closed with open complement $U$ [BRV17, Lemma 2.5].

$$Z_X(t) = Z_U(t) \cdot Z_{Y \subset X}(t)$$

Our main object of study is $Z_{C,p}(t)$ for $p \in C$ a singular point on a reduced curve and its specialization $Z^{top}_{C,p} = Z_{C,p}(t, \chi_{top})$. Note that $\mathrm{Hilb}^d(C, p)$ depends only on the completed local ring $\widehat{\mathcal{O}}_{C,p}$. In fact there is a natural identification of the punctual Hilbert scheme

$$\mathrm{Hilb}^d(C, p) = \{I \mid I \subset \widehat{\mathcal{O}}_{C,p},\ \dim_{\mathbb{C}}(\widehat{\mathcal{O}}_{C,p}/I) = d\}$$

as a parameter space for colength $d$ ideals in $R$.

2.2. **Relative Hilbert schemes of points.** Let $(f : \mathcal{C} \to B, \sigma)$ be a family of reduced curve singularities and $\mathfrak{m} \subset \mathcal{O}_{\mathcal{C}}$ the ideal of the section $S := \sigma(B) \subset \mathcal{C}$. The proof of the following closely follows [Ber12, Lemma 2.22]

**Lemma 2.1.** *Let $Z \subset \mathcal{C}$ be a subscheme flat and proper over $B$ of degree $d$. Then $\mathfrak{m}^d \subset I(Z)$.*

*Proof.* We may suppose without loss of generality $B$ is affine. Since our family embeds into $(\mathbb{C}^n \times B, 0 \times B)$ as the germ of some subvariety, $\mathfrak{m}$ is finitely generated. Consider $\bar{\mathfrak{m}} := \mathfrak{m}/I \subset \mathcal{O}_{\mathcal{C}}/I = \mathcal{O}_Z$ where $I = I(Z)$ is the ideal sheaf of $Z$. Note that $I \subset \mathfrak{m}$ and $\sqrt{I} = \sqrt{\mathfrak{m}}$ as $Z$ is proper over $B$ so it is necessarily supported on the section. Therefore every element of $\bar{\mathfrak{m}}$ is nilpotent and $\bar{\mathfrak{m}}$ is finitely generated so $\bar{\mathfrak{m}}^n = 0$ for large $n$.

On the other hand, $\bar{\mathfrak{m}}$ is the ideal of $\sigma(B)$ inside $Z$ so it is contained in every maximal ideal of $\mathcal{O}_Z$. Therefore by Nakayama's lemma [AM69, Proposition 2.6] $\bar{\mathfrak{m}}^k = \bar{\mathfrak{m}}^{k+1}$ implies that $\bar{\mathfrak{m}} = 0$. In particular, $\bar{\mathfrak{m}}^n = 0$ if and only if $n \geq n_0 = \min\{k : \bar{\mathfrak{m}}^k = \bar{\mathfrak{m}}^{k+1}\}$. It follows that $\bar{\mathfrak{m}}^j/\bar{\mathfrak{m}}^{j-1} \neq 0$ for any $j < n_0$ and so for any $k \leq n_0$

$$\mathcal{O}_Z/\bar{\mathfrak{m}}^k$$

has rank at least $k$ above some point $b \in B$. Since $Z$ is finite of degree $d$ we must have $k \leq d$. Therefore $n_0 \leq d$ and $\bar{\mathfrak{m}}^d = 0$.

□

Let $S_d = \mathrm{Spec}_B(\mathcal{O}_C/\mathfrak{m}^d)$ be the $d^{th}$ formal neighborhood of the section in $\mathcal{C}$.

**Lemma 2.2.** *$S_d$ is finite over $B$.*

*Proof.* $S_d \to B$ is quasi-finite and the induced morphism $(S_d)_{red} \to B_{red}$ is an isomorphism by existence of a section so $S_d \to B$ is proper. □

In particular, $S_d \to B$ is projective with relatively ample line bundle $\mathcal{O}_{S_d}$. By Lemma 2.1, every flat and proper subscheme $Z \subset \mathcal{C}$ of degree $d$ over $B$ is a subscheme of $S_n$ for $n \geq d$.

**Definition 2.3.** We define the relative Hilbert scheme $\mathrm{Hilb}^d(\mathcal{C}/B, \sigma)$ of length $d$ subschemes supported on a family of curve singularities to be the Hilbert scheme $\mathrm{Hilb}^d(S_d/B)$.

**Proposition 2.4.** *$\mathrm{Hilb}^d(\mathcal{C}/B, \sigma)$ is a projective $B$-scheme and for each $b \in B$, we have an identification*

$$\mathrm{Hilb}^d(\mathcal{C}/B, \sigma) \times_B k(b) = \mathrm{Hilb}^d(\mathcal{C}_b, \sigma(b)).$$

*Proof.* Since $S_d \to B$ is a projective morphism and $B$ is Noetherian, then $\mathrm{Hilb}^d(S_d/B)$ exists and is projective over $B$ by a theorem of Grothendieck (e.g. [FGI+05, Theorem 5.14]). Furthermore, the formation of $\mathrm{Hilb}^d(S_d/B)$ is compatible with basechange [FGI+05, (5), page 114] so that

$$\mathrm{Hilb}^d(S_d/B) \times_B k(b) = \mathrm{Hilb}^d(\mathrm{Spec}(\mathcal{O}_{C_b}/\mathfrak{m}_b^d)).$$

By Lemma 2.1, every subscheme of $\mathcal{C}_b$ of length $d$ supported on $\sigma(b)$ is a subscheme of $\mathrm{Spec}(\mathcal{O}_{C_b}/\mathfrak{m}_b^d)$ and so we may identify the right hand side with $\mathrm{Hilb}^d(\mathcal{C}_b, \sigma(b))$. □

**Remark 2.5.** Note that $\mathrm{Hilb}^d(\mathcal{C}/B, \sigma)$ does not represent the functor for flat families of flat and proper subschemes of $\mathcal{C}$ of degree $d$ over $B$. However, this is ok for our applications as the invariants we are interested in are insensitive to the scheme structure.

## 3. SINGULAR CURVES AND THEIR DEFORMATIONS

In this section we will recall some facts about reduced curve singularities and their equisingular deformations including semicontinuity of $\delta$ and $s$. Furthermore, we show that the conductor $c$ is also constructible, insuring the existence of a $(\delta, s, c)$-constant stratification for any family of reduced curve singularities.

Let $(C,p) \subset (\mathbb{C}^N, 0)$ be the germ of a reduced curve singularity with $s$ branches $C_i$ and let $\mathcal{O}_C = \widehat{\mathcal{O}}_{C,p}$ denote the corresponding completed local ring. Let $n : \tilde{C} \to C$ be the normalization. By picking uniformizers for each branch, we identify $\mathcal{O}_{\widetilde{C}}$ with the ring $\prod_{i=1}^s \mathbb{C}[\![x_i]\!]$. The normalization induces a finite extension

$$\mathcal{O}_C \hookrightarrow \mathcal{O}_{\widetilde{C}} \cong \prod_{i=1}^{s} \mathbb{C}[\![x_i]\!]$$

of rings which factors through the inclusions $\mathcal{O}_{C_i} \subset \mathbb{C}[\![x_i]\!]$ corresponding to the $i^{th}$ branch $n_i : \widetilde{C}_i \to C_i \subset C$ of the normalization.

(1) Let

$$\delta := \dim_{\mathbb{C}}(n_* \mathcal{O}_{\widetilde{C}}/\mathcal{O}_C)$$

be the $\delta$-*invariant* of $C$. Similarly, we denote by $\delta_i$ the $\delta$-invariant $\dim_{\mathbb{C}} \mathbb{C}[\![x_i]\!]/\mathcal{O}_{C_i}$ of the $i^{th}$ branch.

(2) Let

$$\mathfrak{c} := \operatorname{Ann}_{\mathcal{O}_C}(n_* \mathcal{O}_{\widetilde{C}}/\mathcal{O}_C)$$

be the *conductor ideal*. This an ideal of both $\mathcal{O}_{\widetilde{C}}$ and $\mathcal{O}_C$. In particular $\mathfrak{c}$ is generated by monomials, say $x_i^{c_i}$, as an ideal of $\prod_{i=1}^s k[\![x_i]\!]$. It's clear from the definition that $c_i$ is the smallest positive integer such that for all $n \geq c_i$, $x_i^n \in \mathcal{O}_C$. We will refer to $c_i$ as the conductor of the $i^{th}$ branch, denote by

$$c := \dim_{\mathbb{C}}(\mathcal{O}_{\widetilde{C}}/\mathfrak{c}) = \sum_{i=1}^{s} c_i$$

the conductor of $C$, and by $\underline{c} = (c_1, \ldots, c_s)$ the conductor branch-length vector. More generally, for any finite homomorphism of rings $\varphi : R \to S$ the *conductor* of $\varphi$ is defined as

$$\mathfrak{c}(\varphi) := \operatorname{Ann}_{\varphi(R)}(S/\varphi(R)).$$

Then it is clear that

$$c_i = \dim_{\mathbb{C}}(\mathcal{O}_{\widetilde{C}_i}/\mathfrak{c}(n_i)).$$

(3) The Milnor number $\mu(C)$ is defined as $\dim_{\mathbb{C}}(\omega_C/d\mathcal{O}_C)$ where $d : \mathcal{O}_C \to \omega_C$ is the differential composed with the canonical map $\Omega^1_C \to n_*\Omega^1_{\tilde{C}} \cong n_*\omega_{\tilde{C}} \to \omega_C$ to the dualizing sheaf of $C$. The Milnor number satisfies

$$\mu(C) = 2\delta(C) - s + 1$$

(see [BG80]).

Denote by $v_i : \mathcal{O}_{\widetilde{C}} \to \mathbb{N}$ the composition of the projection onto $k[\![x_i]\!]$ with the valuation on $k[\![x_i]\!]$. This gives the order of vanishing of a function along the $i^{th}$ branch of the normalization.

3.1. **Equisingular families.** Let $(f : \mathcal{C} \to B, \sigma)$ be a flat family of germs of reduced curve singularities. Recall we will always assume that $B$ is Noetherian and that there is an embedding of germs

$$(\mathcal{C}, \sigma) \subset (\mathbb{C}^N \times B, 0 \times B)$$

so that $(f : \mathcal{C} \to B, \sigma)$ is the germ of a family of reduced affine curves.

**Definition 3.1.** A morphism $\nu : \mathcal{C}' \to \mathcal{C}$ is a *simultaneous normalization* of $f$ if for any $b \in B$, $\nu_b : \mathcal{C}'_b \to \mathcal{C}_b$ is the normalization. We say that $f$ is *equinormalizable* if the normalization $\widetilde{\mathcal{C}} \to \mathcal{C}$ of the total space is a simultaneous normalization of $f$.

**Theorem 3.2** (Tessier [Tei77], Reynaud, Chiang-Hsieh–Lipman [CHL06])**.** *Let $(f : \mathcal{C} \to B, \sigma)$ be a flat family of reduced curve singularities over a normal base $B$. Then $f$ is equinormalizable if and only if $\delta(C_b, \sigma(b))$ is constant for $b \in B$.*

**Definition 3.3.** Suppose $B$ is connected, smooth and 1-dimensional with a basepoint $0 \in B$. We say that the family $(f : \mathcal{C} \to B, \sigma)$ is *equisingular*[2] if there is a homeomorphism

$$(\mathcal{C}, \sigma(B)) \cong_{top} (B \times \mathcal{C}_0, B \times \sigma(0))$$

compatible with the maps to $B$.

**Theorem 3.4** (Buchweitz–Greuel [BG80, Theorems 5.2.2 and 6.1.7])**.** *Let $(f : \mathcal{C} \to B, \sigma)$ be a flat family of reduced curve singularities.*

*(a) The function $\mu(\mathcal{C}_b, \sigma(b))$ for $b \in B$ is upper semicontinuous.*
*(b) Suppose $B$ is a smooth, connected and 1-dimensional base. Then the following are equivalent:*
  *(i) $(f : \mathcal{C} \to B, \sigma)$ is equisingular;*
  *(ii) the Milnor number $\mu(\mathcal{C}_b, \sigma(b))$ is constant for $b \in B$;*
  *(iii) $\delta(\mathcal{C}_b, \sigma(b))$ and the number of branches $s(\mathcal{C}_b, \sigma(b))$ are constant.*

**Corollary 3.5.** *There exists a stratification $B = \bigsqcup B_i$ such that the pullback $f_i : \mathcal{C}_i \to B_i$ is a $\mu$-constant family for each $i$. Furthermore, $f_i$ is $(\delta, s)$-constant and if $B_i$ is normal then $f_i$ is equinormalizable.*

We call such families $(\delta, s)$-*constant* or *equisingular* families. If $(f : \mathcal{C} \to B, \sigma)$ is an equisingular family, then the normalization $\widetilde{f} : \widetilde{\mathcal{C}} \to B$ is a family of $s$ germs of smooth curves with degree $s$ multisection. That is, $\widetilde{\mathcal{C}}_b \cong \bigsqcup_{i=1}^s \widehat{\mathbb{A}}^1$ where $\widehat{\mathbb{A}}^1 = \mathrm{Spec}(\mathbb{C}[\![x]\!])$.

**Proposition 3.6.** *Let $(f : \mathcal{C} \to B, \sigma)$ be a $(\delta, s)$-constant family of reduced curve singularities. Then the conductor $c$ is constructible on $B$.*

*Proof.* Since the function $b \to c(\mathcal{C}_b, \sigma(b))$ depends only on the closed points of $B$, we may assume without loss of generality that $B$ is normal. In this case $f$ is equisingular and the normalization $n : \widetilde{\mathcal{C}} \to \mathcal{C}$ is the simultaneous normalization. Consider the sequence

$$0 \to \mathcal{O}_C \to n_*\mathcal{O}_{\widetilde{C}} \to Q \to 0.$$

As $f$ is equinormalizable, we have exactness of

$$0 \to \mathcal{O}_{C_b} \to n_*\mathcal{O}_{\widetilde{C}_b} \to Q_b \to 0$$

so that $\mathrm{length}(Q_b) = \delta$ is constant for all $b \in B$. Thus $Q$ is finite of constant rank over $B$ so it is flat.

[2] There are several notions of equisingular deformations in the literature that are not always equivalent.

**Lemma 3.7.** *Let $(f : \mathcal{C} \to B, \sigma)$ be a family of reduced curve singularities and let $Q$ be a coherent sheaf on $\mathcal{C}$ that is flat and finite over $B$. Then $b \to \mathrm{colength}_{\mathcal{O}_{C_b}}(\mathrm{Ann}_{\mathcal{O}_{C_b}}(Q_b))$ is constructible.*

*Proof.* Let $d$ be the degree of $Q$ over $B$ and for any $k \leq d$ consider $\mathrm{Hilb}^k(\mathcal{C}/B, \sigma)$ with tautological subscheme $Z_k \subset \mathrm{Hilb}^k(\mathcal{C}/B, \sigma) \times_B \mathcal{C}$. Let $Q_H$ the pullback of $Q$ to $\mathrm{Hilb}^k(\mathcal{C}/B, \sigma) \times_B \mathcal{C}$ and $Q_Z$ the pullback of $Z_k$. Then $Q_H$ is flat over $\mathrm{Hilb}^k(\mathcal{C}/B, \sigma)$ of constant degree $d$ over and $Q_Z$, as a quotient of $Q_H$, has degree at most $d$ over $\mathrm{Hilb}^k(\mathcal{C}/B, \sigma)$.

Let $H^k_d \subset \mathrm{Hilb}^k(\mathcal{C}/B, \sigma)$ be the closed subset where $Q_Z$ has degree exactly $d$, or equivalently the locus over which $Q_H \to Q_Z$ is an isomorphism. The image of $H^k_d$ via $\mathrm{Hilb}^k(\mathcal{C}/B, \sigma) \to B$ is constructible in $B$ and is by construction the locus over which $Q$ is supported on a subscheme of length at most $k$. In particular, the image of $H^d_d$ is all of $B$ and the function

$$\varphi : b \to \min\{k \ : \in \mathrm{im}(H^d_k)\}$$

is constructible. On the other hand, since $V(\mathrm{Ann}_{\mathcal{O}_{C_b}}(Q_b)) = \mathrm{Supp}(Q_b)$ is the smallest subscheme on which $Q_b$ is supported, then

$$\varphi(b) = \mathrm{colength}_{\mathcal{O}_{C_b}}(\mathrm{Ann}_{\mathcal{O}_{C_b}}(Q_b)).$$

□

To complete the proof, note that $\delta$ is constant so

$$c(\mathcal{C}_b, \sigma(b)) = \delta + \mathrm{colength}_{\mathcal{O}_{C_b}}(\mathrm{Ann}_{\mathcal{O}_{C_b}}(Q_b))$$

is constructible by the lemma.

□

**Corollary 3.8.** *For any $(\delta, s)$-constant family, we may further stratify so that $c$ is constant and $Z \to B$ is flat. We call such families $(\delta, s, c)$-*constant *families.*

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