# MODULI OF VARIETIES OF GENERAL TYPE AND WALL-CROSSING

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There notes from a lecture series given at Moduli Spaces of Algebraic Surfaces in Ann Arbor, May 2025. They are a partially based on lectures from the 2022 AGNES Summer School on higher dimensional moduli. The goal is to introduce the KSBA compactification of the moduli space of varieties of general type, and more generally pairs $(X, \Delta)$ of log general type. Our focus will be on low dimensional examples and the wall-crossing phenomena that govern how the moduli spaces depend on the coefficients of $(X, \Delta)$.

These notes will be expanded into a forthcoming book project on moduli of higher dimensional varieties written with Kristin DeVleming.

We work over $\mathbb{C}$.

## 1. Introduction

The first fundamental goal of moduli theory of varieties is to answer the classification question. The prototypical example is the moduli space of curves of fixed genus. For each genus $g \geq 2$, there is a $3g-3$ dimensional moduli space $\mathscr{M}_g$ and the classification problem for smooth projective curves of genus $g$ becomes the problem of studying the geometry of $\mathscr{M}_g$.

The second fundamental goal of moduli theory is to produce compactifications. The prototypical example here is again the moduli of curves and its compactification $\overline{\mathscr{M}}_g$ by Deligne-Mumford stable curves. The compactification problem can be thought of as the question of classifying degenerate or singular objects, in this case singular curves. We will discuss $\overline{\mathscr{M}}_g$ in depth in Section 2.

### 1.1. **Classification and moduli theory of surfaces.**

To motivate the higher dimensional theory, we will begin with a brief summary of the Kodaira-Enriques classification of smooth projective surfaces and the features of the moduli theory in each case. The primary invariant is the Kodaira dimension.

**Definition 1.1.** Let $X$ be a smooth projective variety. The *Kodaira dimension* $\kappa(X)$ is defined as follows. If $h^0(X, mK_X) = 0$ for all $m > 0$, then we declare $\kappa(X) = -\infty$. Otherwise,

$$h^0(X, mK_X) = am^{\kappa(X)} + \text{lower order terms} \quad a \neq 0$$

is a polynomial in $m$ and we let $\kappa(X)$ be its leading coefficient.

Note that the Kodaira invariant takes values in $\{-\infty, 0, \dots, \dim X\}$. We also recall that a surface is *minimal* if it contains no $(-1)$ curves. Beyond the case of rational surfaces, we generally deal with minimal surfaces.

#### 1.1.1. $\kappa = -\infty$.

(1) **Rational surfaces**, i.e. those surfaces birational to $\mathbb{P}^2$, are characterized by the vanishing of the invariants $q(X) \colon = h^1(X, \mathscr{O}_X) = 0$ and $P_2 \colon = h^0(X, 2K_X) = 0$. From the point of view of moduli theory, the most interesting families are the del Pezzo surfaces which have

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$-K_X$ ample. The secondary invariant is the degree $d\colon = K_X^2 \in \{1,\dots,9\}$. There are 10 families, blowups of $\mathbb{P}^2$ at $m = 0,\dots,8$ generic points which have degree $9-m$, and $\mathbb{P}^1 \times \mathbb{P}^1$ of degree 8. The families have expected dimension $10-2d$.

(2) **Minimal ruled surfaces** which are of the form $\mathbb{P}(\mathscr{E}) \to C$ for $C$ a smooth projective curve and $\mathscr{E}$ a rank 2 vector bundle on $C$. The secondary invariant is the irregularity $q(X)\colon = h^1(X,\mathscr{O}_X) = h^1(C,\mathscr{O}_C)$. In each genus there are two families depending on the parity of $\deg \mathscr{E}$ of expected dimension $6g-6$.

In these cases the moduli spaces are Artin stacks which can have negative dimension due to the existence of positive dimensional stabilizers and one has to deal with subtle issues of stability to produce a reasonable moduli theory. This is the topic of K-stability and K-moduli which we don't discuss here.

1.1.2. $\kappa = 0$.

(1) **Abelian surfaces** are those smooth projective surfaces $X$ whose complex points $X(\mathbb{C})$ are biholomorphic to complex tori $\mathbb{C}^2/\Lambda$ where $\Lambda \subset \mathbb{C}^2$ is a rank 4 integral lattice. There is a 4-dimensional family of complex tori most of which are not algebraic. The algebraic ones are the ones admitting a polarization or ample line bundle $L$. The theory of abelian varieties produces a discrete invariant $\underline{d}$ of $L$ called the polarization type. There are countably many polarization types and for each one, there is a 3 dimensional moduli space $\mathscr{A}_{2,\underline{d}}$ of polarized abelian surfaces of type $\underline{d}$.
(2) **K3 surfaces** are those smooth compact surfaces with $q(X) = 0$ and trivial canonical sheaf $\omega_X \cong \mathscr{O}_X$. There is a connected 20-dimensional family of complex K3 surfaces, most of which are not algebraic. The algebraic ones are again the ones admitting a polarization $L$. The secondary invariant is the degree $L^2 = 2d$ which is a positive even integer. For each degree there is a smooth 19-dimensional moduli space $\mathscr{F}_{2d}$ of polarized K3 surfaces of degree $2d$.
(3) **Enriques surfaces** are smooth projective surfaces with $q(X) = h^0(X,\omega_X) = 0$. There is a 10 dimensional family of Enriques surfaces and all of them can be obtained as a quotient of a K3 surface by a fixed point free involution.
(4) **Bielliptic surfaces** are surfaces of the form $E \times E'/G$ where $E, E'$ are elliptic curves, and $G$ is a finite group which acts by translations on $E'$ and by automorphisms fixing a point on $E$. There are 7 families of such surfaces corresponding to the possible such finite group actions. Two families are 2-dimensional and the rest are 1-dimensional.

Here the moduli spaces are of a Hodge theoretic nature and one usually needs a polarization to form a reasonable algebraic moduli space.

1.2. $\kappa = 1$. In Kodaira dimension 1, all surfaces are elliptic. That is, they have a canonical fibration $f : X \to C$ with genus 1 fibers over a smooth curve $C$. The secondary invariants are the genus $g(C)$ and the Euler characteristic $n = \chi(\mathscr{O}_X)$. Such fibrations are of Kodaira dimension 1 if and only if either $g \geq 2$, or $g = 1$ and $n \geq 1$, or $g = 0$ and $n \geq 3$. As in the Kodaira dimension 0 case, to have a well-behaved moduli theory we need to pick a polarization. The simplest case is when $f$ admits a section $S$. Then $K_X + S$ is a polarization and there is a moduli space $\mathscr{W}_{g,n}$ for such tuples $(f : X \to C, S)$ with $g = g(C)$ and $n = \chi(\mathscr{O}_X)$. The expected dimension is $10n - 2 + 2g$.

1.3. $\kappa = 2$. These are the minimal surfaces of general type. The secondary invariants are the volume $v = K_X^2 > 0$ and Euler characteristic $n = \chi(\mathscr{O}_X)$. A (quasi)-polarization is furnished by the canonical $K_X$ and there is a moduli space $\mathscr{M}_{v,n}$ of expected dimension $10n - 2v$. In general, $\mathscr{M}_{v,n}$ need not be connected. Moreover, for each $v$ there are finitely many $n$ so one can also consider $\mathscr{M}_v = \bigsqcup_n \mathscr{M}_{v,n}$. The geography problem for surfaces of general type is to classify for which $v, n$ is

$\mathscr{M}_{v,n}$ nonempty, how many irreducible components does it have, and what is the general surface parametrized by each irreducible component.

1.4. **Compactifications.** The first goal of these notes is to introduce the *KSB compactification* $\mathscr{M}_v \subset \overline{\mathscr{M}}_v^{KSB}$ of the moduli of minimal surfaces of general type by the moduli of *stable surfaces*. This is the higher dimensional analogue of the Deligne-Mumford compactification $\mathscr{M}_g \subset \overline{\mathscr{M}}_g$.

More generally, we can consider pairs $(X, \Delta)$ where

$$\Delta = \sum_{i=1}^{n} a_i D_i$$

is a $\mathbb{Q}$-divisor with $a_i \in (0,1] \cap \mathbb{Q}$. This added flexibility allows us to handle for example the cases of Kodaira dimension $\kappa(X) < 2$ by adding a choice of divisor $\Delta$ into our moduli problem. In this case there is a moduli space $\mathscr{M}_{v,\vec{a}}$ depending on numerical invariants $\vec{a} = (a_1, \ldots, a_n)$ and the *volume* $v = \mathrm{vol}(X, \Delta)$ which we will describe below. Then $\mathscr{M}_{v,\vec{a}}$ admits a compactification

$$\mathscr{M}_{v,\vec{a}} \subset \overline{\mathscr{M}}_{v,\vec{a}}^{KSBA}$$

by the *KSBA moduli space* parameterizing *stable pairs*.

The second goal of these notes is to describe the *wall-crossing phenomena* that governs how these compactifications depend on the coefficient vector $\vec{a}$.

## 2. Moduli of stable curves

We will begin by reviewing the story for the moduli space of curves to get a feel for the features we hope to extract in higher dimensions.

Let $\mathscr{M}_{g,n}$ be the moduli space of smooth $n$-pointed genus $g$ curves $(C, p_1, \ldots, p_n)$ where the $p_i$ are distinct. When $2g - 2 + n > 0$, $\mathscr{M}_{g,n}$ is a $3g - 3$ dimensional separated Deligne-Mumford stack with quasi-projective coarse moduli space $M_{g,n}$ parametrizing isomorphism classes of pointed curves.

**Question 2.1.** How do we compactify $\mathscr{M}_{g,n}$? How are different compactifications related?

In a 1-parameter family of pointed curves, the points $p_i$ can collide. Thus our compactification procedure has to include a choice of what happens when points collide. Moreover, we know from experience that we have to consider non-normal curves to obtain a compact moduli space[2] so we also have to decide which non-normal curves to allow.

**Definition 2.2** (Deligne-Mumford-Knudsen, Hassett)**.** Fix a rational *weight vector* $\vec{a} = (a_1, \ldots, a_n)$ with $0 < a_i \leq 1$. An $\vec{a}$-weighted stable curve of genus $g$ is a tuple $(C, p_1, \ldots, p_n)$ such that

(a) (singularity) $C$ is a curve with arithmetic genus $g$ and at worst nodal singularities and $p_i$ are smooth points,

(b) (singularity) a subset of the marked points are allowed to coincide only if the sum of the weights is at most 1: for all $p \in C$,

$$\sum_{p_i = p} a_i \leq 1$$

(c) (stability) the line bundle $\omega_C^M(M \sum a_i p_i)$ is ample for $M$ large and divisible enough.

Points $(a) + (b)$ are conditions on the singularities of the pair $(C, \sum a_i p_i)$ which pick out a large enough class of pairs so that limits of arbitrary families of pointed curves exist. The stability condition $(c)$ cuts down this class just enough so that these limits are unique.

**Definition 2.3.** We say a weight vector $\vec{a}$ is *admissible* if $2g - 2 + \sum a_i > 0$ and we let $\mathscr{P}_{g,n}^a$ denote the space of admissible weight vectors.

[2]Why? Hint: monodromy

Note that $\mathscr{P}^a_{g,n}$ is (essentially) the rational points of rational polytope and by abuse of notation we will call it the ample polytope. We have the following theorem of Deligne–Mumford, Knudsen and Hassett.

**Theorem 2.4** ([DM69, Knu83a, Knu83b, Has03])**.** *For any $\vec{a} \in \mathscr{P}^a_{g,n}$, there exists a smooth and proper Deligne-Mumford stack $\overline{\mathscr{M}}_{g,\vec{a}}$ with projective coarse moduli space $\overline{M}_{g,\vec{a}}$ parametrizing $\vec{a}$-weighted stable genus $g$ curves. Moreover, $\mathscr{M}_{g,n} \subset \overline{\mathscr{M}}_{g,\vec{a}}$ is open and dense.*

*Proof sketch.* The proof is naturally divided into steps. These are the same steps we would like to understand in higher dimensions but they each become much harder.

(1) **Boundedness.** The key proposition is the following.

**Proposition 2.5.** *Let $C$ be a connected nodal curve, $D \subset C^{sm}$ an effective Cartier divisor and $L = \omega_C^k(D)$ is ample. Then $L^N$ is very ample $H^1(C, L^N) = 0$ for any $N \geq 3$.*

Applying the proposition to the line bundle $\omega^M(M\sum a_i p_i)$ where $M$ is a fixed integer such that $Ma_i$ is an integer for each $i$, we conclude that $L^3$ embeds $(C, p_1, \ldots, p_n)$ into a fixed projective space $\mathbb{P}^{N}$[3] as a degree $d := M(2g - 2 + \sum a_i)$ curve. Thus we can consider the incidence variety $H \subset (\mathbb{P}^N)^n \times \mathrm{Hilb}_{1,d}$ where $\mathrm{Hilb}_{1,d}$ is the Hilbert scheme of degree $d$ curves in $\mathbb{P}^N$ and $V = \{(p_1, \ldots, p_n), C) \mid p_i \in C\}$. By general results on Hilbert schemes, $V$ is a finite type quasi-projective variety with a universal family $(\sigma_1, \ldots, \sigma_n \in \mathscr{C}) \to H$ that $(C, p_1, \ldots, p_n)$ appears as a fiber of this family for any $\vec{a}$-weighted stable curve of genus $g$. Moreover, every automorphism of $(C, p_1, \ldots, p_n)$ is induced by a projective transformation of $\mathbb{P}^N$ (Why?).

(2) **Algebraicity.** Next we need to show that stability is an algebraic condition on $V$.

**Lemma 2.6.** *Let $(\sigma_1, \ldots, \sigma_n \in \mathscr{C} \subset \mathbb{P}^N_V) \to V$ be any flat family of curves with sections $\sigma_i$. Then the locus*

$$U = \{v \in V \mid (\mathscr{C}, \sigma_i)_v \text{ is an } \vec{a}\text{-weighted stable curve}\} \subset V$$

*is a locally closed subvariety.*

*Proof.* Each of the following properties define locally closed subvarieties (exercise): $\mathscr{C}_v$ is nodal, $\sigma_i$ is contained in the smooth locus of $\mathscr{C} \to V$, $L^3 \cong \mathscr{O}_{\mathscr{C}}(1)$. Their intersection gives a locally closed subvariety where all the properties are satisfied so replace $V$ with the subvariety where each of these properties hold. Let $Z_I$ be the intersection of $\sigma_i$ for $i \in I$ where $I \subset \{1, \ldots, n\}$ is a subset with total weight $> 1$. The images of $Z_I$ are closed subvarieties $V_I$ and now $U = V \setminus \cup V_I$ over all such $I$ does the job.[4] □

Thus we have produced a finite type $U$ with an action of $PGL(N+1)$ carrying an equivariant family of embedded $\vec{a}$-weighted stable curves of genus $g$ such that every curve appears in this family and two curves in this family are isomorphic if and only if they related by the $PGL(N+1)$ action. The induced map $U \to \overline{\mathscr{M}}_{g,\vec{a}}$ is a smooth cover which exhibits that $\overline{\mathscr{M}}_{g,\vec{a}}$ is an algebraic stack.

(3) **Properness.** The valuative criterion for properness for stacks reduces properness to the following statement whose proof makes use of two fundamental tools: semi-stable reduction and the minimal model program.

[3] Why? What is $N$?

[4] In general we have to put the correct scheme structure on $U$. For curves this isn't an issue since our moduli space ends up being smooth but in higher dimension we have to be more careful.

**Proposition 2.7.** *Let $U = D \setminus 0$ be a punctured DVR and let $(C, \sigma_1, \ldots, \sigma_n) \to U$ be a family of $\vec{a}$-weighted stable curves. Then there exists an extension of DVRs $D' \to D$ and a unique family $(C', \sigma_i') \to D'$ of $\vec{a}$-weighted stable curves extending the pullback*

$$(C, \sigma_i) \times_D D'.$$

*Proof.* Let us assume first that $C \to U$ is a family of smooth curves. Then the Semi-stable Reduction Theorem tells us that there exists $D' \to D$ and $C' \to D'$ extending $C \times_D D'$ such that $C'$ is a smooth surface and the central fiber $C_0'$ is a reduced nodal curve. Replacing $D$ with $D'$ we may assume that $C \to U$ admits such an extension, call it $\bar{C} \to D$. Taking the closure of the sections $\bar{\sigma}_i$ and blowing up further, we may assume that $\sigma_i$ extend to sections which are either disjoint or identically equal.

We have filled in the family such that $(\bar{C}_0, \bar{p}_i)$ has the right singularities, but

$$\omega_{\bar{C}_0}(\sum a_i \bar{p}_i) = \omega_{\bar{C}}(\sum a_i \sigma_i)|_{\bar{C}_0}$$

may not be ample (where $\bar{p}_i = (\bar{\sigma}_i)|_{\bar{C}_0}$). To make the log canonical divisor ample, we have to contract curves $R \subset \bar{C}_0 \subset \bar{C}$ where $\deg_R(K_C + \sum a_i \sigma_i) \leq 0$. This is accomplished by the *minimal model program.* The output of this contraction (the so-called relative log canonical model) yields our desired family and is unique. This can be summarized in the following diagram.

$$\begin{array}{ccccccccc}
C & \longleftarrow & C' & \hookrightarrow & \bar{C} & \xrightarrow{\rho} & \bar{C}' & \xrightarrow{\phi} & \bar{C}'' \\
\downarrow & & \downarrow & & \downarrow{\scriptstyle \pi} & & \downarrow & & \downarrow \\
U & \longleftarrow & U' & \hookrightarrow & D' & = & D' & = & D'
\end{array}$$

Here $\mu$ is the contraction of $(-1)$-curves which exists and produces a smooth $\bar{C}'$ by Castelnuovo's Contraction Theorem. Then $K_{\bar{C}'}$ is nef but not ample and $\rho$ is the contraction of $(-2)$-curves which results in a family of nodal curves $\bar{C}'' \to D'$ with ample canonical line bundle and at worst A-type singularities on the total space. Uniqueness follows from identifying $\bar{C}''$ with $\mathrm{Proj}_{D'} \oplus_{m \geq 0} \pi_* \mathscr{O}_{\bar{C}}(mK_{\bar{C}})$.

For the general case, we normalize $C \to U$ to get several families $(C_i, \Sigma_{i,j}) \to U$ such that $C$ is obtained by gluing together the $C_i$ along combinations of $\Sigma_{i,j}$. Now we run the above argument for each $i$ where we take $\Sigma_{i,j}$ to be marked with coefficient 1. Finally we glue the resulting extensions along the closures of the sections $\Sigma_{i,j}$ which are still disjoint (and disjoint from any of other marked sections $\sigma_i$) as they have weight 1. □

(4) **Deformation Theory.** We can regard the data of an $\vec{a}$-weighted stable curve as a map $s : S = \{1, \ldots, n\} \to C$ where $S$ is a finite set (viewed as a discrete scheme). Since the condition that $(C, p_1, \ldots, p_n)$ is stable is open for a family of pointed nodal curves, the deformation theory of $\overline{\mathscr{M}}_{g,\vec{a}}$ is the same as the deformation theory of the map $S \to C$.

**Proposition 2.8.** *There exists an extension*

$$0 \to \Omega_C \to \Omega_{s:S \to C} \to \bigoplus_{i=1}^{n} \mathscr{O}_{p_i} \to 0$$

*such that $T^i(-) := \mathrm{Ext}^i_C(\Omega_{s:S\to C}, -)$ for $i = 0, 1, 2$ forms a deformation-obstruction theory for the functor of flat deformations of the map $s : S \to C$.*

If you're not familiar with deformation theory, the key points are that $\mathrm{Ext}^0(\Omega_{s:S\to C}, \mathscr{O}_C)$ is the space of infinitessimal automorphisms of $s$, $\mathrm{Ext}^1(\Omega_{s:S\to C}, \mathscr{O}_C)$ is the tangent space to $[s : S \to C]$ in the moduli space of maps, and $\mathrm{Ext}^2(\Omega_{s:S\to C}, \mathscr{O}_C)$ is the obstruction space to extending a deformation $s_n$ of $s$ over $\operatorname{Spec} k[x]/(x^{n+1})$ to a deformation $s_{n+1}$ over $\operatorname{Spec} k[x]/(x^{n+2})$.

**Exercise 2.9.** If $(C, p_i)$ is $\vec{a}$-weighted stable and $s : S \to C$ as above, then

$$\hom_C(\Omega_{s:S\to C}, \mathscr{O}_C) = 0,\ \operatorname{ext}^1_C(\Omega_{s:S\to C}, \mathscr{O}_C) = 3g-3+n,\ \operatorname{ext}^2(\Omega_{s:S\to C}, \mathscr{O}_C) = 0.$$

**Corollary 2.10.** *$\overline{\mathscr{M}}_{g,\vec{a}}$ is a smooth Deligne-Mumford stack of dimension $3g-3+n$.*

(5) **Projectivity.** By the Keel-Mori theorem, $\overline{\mathscr{M}}_{g,\vec{a}}$ admits a proper coarse moduli space $M_{g,\vec{a}}$. To prove it is projective, we need to find an ample line bundle on $M_{g,\vec{a}}$. A general fact about stacks tells us that for every line bundle $\mathscr{L}$ on a proper Deligne-Mumford stack $\mathscr{M}$, there exists an $N$ such that $\mathscr{L}^N$ descends to a line bundle $L$ on the coarse moduli space $M$ and $L$ is ample if and only if the pullback of $\mathscr{L}$ to any curve $C \to \mathscr{M}$ has positive degree.

Kollár developed a general method for producing such line bundles on a moduli space using the universal family. The idea is to use the sheaves

$$V_k := \pi_*\omega_\pi^k\left(k\sum a_i\sigma_i\right) Q_{d,j} := \sigma_j^*\omega_\pi^d\left(d\sum a_i\sigma_i\right)$$

where $\pi : (\mathscr{C}, \sigma_i) \to \overline{\mathscr{M}}_{g,\vec{a}}$ is the universal family. Kollár's ampleness lemma gives a general method for show that some tensor combination of determinants of these vector bundles is ample.

(6) **Interior.** The condition that $C$ is smooth is open so we have an open substack $\overline{\mathscr{M}}^{sm}_{g,\vec{a}}$ containing $\mathscr{M}_{g,n}$ where $C$ is smooth. Moreover, $\mathscr{M}_{g,n} \subset \overline{\mathscr{M}}^{sm}_{g,\vec{a}}$ is open since this corresponds to the locus where $s : S \to C$ is an isomorphism onto its image and being an isomorphism is an open condition. Finally, to show that $\overline{\mathscr{M}}^{sm}_{g,\vec{a}}$ is dense, we can either use deformation theory to show that any $\vec{a}$-stable curve can be deformed to one where $C$ is smooth (Hint: use the local to global spectral sequence), or we can show that $\overline{\mathscr{M}}_{g,\vec{a}}$ is connected by picking a particularly nice stable curve that every curve degenerates to (Hint: pick a curve where every component is genus 0 and then induct on the genus $g$).

□

With the moduli space at hand, it is natural to ask the following. Let us denote the universal family $\mathscr{C}_{g,\vec{a}} \to \overline{\mathscr{M}}_{g,\vec{a}}$ with sections $\{\sigma_i\}_{i=1}^n$.

**Question 2.11.** How does the moduli space and universal family vary as we vary the weight $\vec{a} \in \mathscr{P}^a_{g,n}$?

The idea is that if we start with an $\vec{a}$-weighted stable curve $(C, p_i)$ and pick another weight $\vec{b} \le \vec{a}$[5] in $\mathscr{P}^a_{g,n}$, we can consider the line segment $\vec{v}(t) := t\vec{a} + (1-t)\vec{b}$ as $t \in [0,1]$. If $(C, p_i)$ is already $\vec{b}$-stable then it is $\vec{v}(t)$ stable for all $t \in [0,1]$ and nothing happens. Otherwise, we can start decreasing $t$ until we hit a point $0 < t_1 < 1$ where

$$\omega_C\left(\sum v_i(t)p_i\right)$$

has degree 0 on some component. This means that $(C, p_i)$ os not $v_i(t_1)$-stable, and we need to contract those components to make it stable. Since $C$ only has finitely many components, this process occurs at finitely many times $0 < t_m < t_{m-1} < \ldots < t_1 < 1$ and the output is an $\vec{b}$-stable curve. The key point is that this can be done on the level of the universal family too!

**Theorem 2.12** ([Has03])**.** *There exists a finite rational polyhedral decomposition of $\mathscr{P}^a_{g,n}$ such that the following hold.*

[5]The ordering is the one where $b_i \le a_i$ for all $i$

*(1) If $\vec{a}, \vec{a}'$ are in the same chamber, then there are natural isomorphism*

$$\begin{array}{ccc} \mathscr{C}_{g,\vec{a}} & \xrightarrow{\cong} & \mathscr{C}_{g,\vec{a}'} \\ \downarrow & & \downarrow \\ \overline{\mathscr{M}}_{g,\vec{a}} & \xrightarrow{\cong} & \overline{\mathscr{M}}_{g,\vec{a}'}. \end{array}$$

*(2) If $\vec{b} \leq \vec{a}$, then there is a natural reduction morphisms $\rho_{\vec{b},\vec{a}} : \overline{\mathscr{M}}_{g,\vec{a}} \to \overline{\mathscr{M}}_{g,\vec{b}}$ as well as a compatible contraction $\mathscr{C}_{g,\vec{a}} \to \mathscr{C}_{g,\vec{b}}$ of universal families which fiberwise is the one described in the paragraph above.*

*(3) The reduction morphisms for different weights $\vec{c} \leq \vec{b} \leq \vec{a}$ are compatible:*

$$\rho_{\vec{c},\vec{b}} \circ \rho_{\vec{b},\vec{a}} = \rho_{\vec{c},\vec{a}}.$$

*Proof sketch.* The walls are where the collection of possible stable curves changes. The discussion above suggests that this occurs exactly when the degree of $\omega_C(\sum a_i p_i)$ becomes zero on some component of some weighted stable curve. Such a component must be rational and containing exactly one node of $C$ (Why?). Let $(C, p_i)$ be an $\vec{a}$-weighted stable curve and let $E \subset C$ be such a rational component which contains the marked points $p_j$ for $j \in J \subset \{1, \ldots, n\}$. Then

$$\deg \omega_C \left(\sum a_i p_i\right)|_E = \sum_{j \in J} a_j - 1.$$

It follows that the walls are given by

$$\sum_{j \in J} a_j = 1$$

where $J$ ranges over all subsets of $\{1, \ldots, n\}$[6]. The chambers are by definition the complements of the walls. In particular, they are rational polyhedral and there are only finitely many of them.

It's not hard to see that for $\vec{a}, \vec{a}'$ in the same chamber, a pointed curve $(C, p_i)$ is $\vec{a}$-stable if and only if it is $\vec{a}'$-stable. Thus the universal family over $\overline{\mathscr{M}}_{g,\vec{a}}$ may be viewed as an $\vec{a}'$ stable family inducing a morphism $\overline{\mathscr{M}}_{g,\vec{a}} \to \overline{\mathscr{M}}_{g,\vec{a}'}$ and vice versa. We conclude these spaces are canonically isomorphic.

For $\vec{c} \leq \vec{a}$, let $\vec{v}(t)$ for $t \in [0,1]$ as before. By the first part, the line segment $\vec{v}(t)$ as $t$ varies intersects finitely many walls, say at times $0 < t_m < t_{m-1} < \ldots < t_1 < 0$. Moreover, at time $t_1$, we have that

$$\omega_\pi \left(\sum v_i(t)\sigma_i\right)$$

is not $\pi$-ample but has degree 0 on some components of the universal family. Here $\pi : \mathscr{C}_{g,\vec{a}} \to \overline{\mathscr{M}}_{g,\vec{a}}$ is the universal family. The key input is the following proposition.

**Proposition 2.13.** *Let $C$ be a nodal curve, $D \subset C$ an effective divisor contained in the smooth locus of $C$, and $L = \omega_C^k(D)$ has degree $\geq 0$ on each component of $C$. Then $L^m$ is basepoint free and $h^1(L^m) = 0$ for all $m \geq 3$. Moreover, the morphism $\phi_{L^m} : C \to C'$ induced by $L^m$ contracts the components $E \subset C$ on which $\deg L|_E = 0$ to a point and $\phi_{L^m}$ is birational on every other component of $C$. Finally, $C'$ has at worst nodal singularities.*

This proposition is exactly what we saw in action in the example with

$$L = \omega_C^k \left(k \sum v_i(t_1) p_i\right)$$

[6] There are some boundary cases for $J$. What are they?

where $k$ clears the denominators of $v_i(t_1)$. Now we can take the universal version

$$\mathscr{L} = \omega_\pi^k \left(k \sum v_i(t_1)\sigma_i\right)$$

and let

$$\mathscr{C}_1 = \mathrm{Proj}_{\overline{\mathscr{M}}_{g,\vec{a}}} \bigoplus_{m\geq 0} \pi_* \mathscr{L}^m$$

By the proposition and the theorem on Cohomology and Base Change, up to replacing $\mathscr{L}$ with a power, these pushforwards are locally free sheaves whose formation commutes with basechange and the graded algebra is finitely generated so the relative Proj makes sense. Moreover, there is a morphism $\mathscr{C}_{g,\vec{a}} \to \mathscr{C}_1$ which fiberwise is just the basepoint free contraction $C \to C'$ from the proposition. Thus, $\mathscr{C}_1$ with the images of $\sigma_i$ is a $\vec{v}(t_1)$-weighted stable family over $\mathscr{M}_{g,\vec{a}}$ and thus is pulled back from the universal family $\mathscr{C}_{g,\vec{v}(t_1)}$ via a morphism

$$\overline{\mathscr{M}}_{g,\vec{a}} \to \overline{\mathscr{M}}_{g,\vec{v}(t_1)}.$$

This gives us a reduction morphism $\rho_{\vec{v}(t_1),\vec{a}}$ and now we repeat the argument and induct on the number of walls $t_i$ to get the required reduction morphism.

By construction each of these morphisms extend the identity on $\mathscr{M}_{g,n}$ and so their compositions agree in the obvious way since the moduli space is separated. □

## 3. Canonical models and singularities of the minimal model program

Our goal now is to understand how much of this picture can be generalized to higher dimensions. To a first approximation, we can phrase it as follows. Here pairs $(C, \sum a_i p_i)$ of curves with weighted marked points are replaced by *normal crossings pairs*, that is, pairs

$$(X, \Delta = \sum a_i D_i)$$

of a variety equipped with a weighted linear combination of Weil divisors $D_i \subset X$ such that $X$ is smooth and $\Delta$ locally looks like a union of smooth hypersurfaces intersecting tranversely, i.e. the pair $(X, \Delta)$ is *normal crossings*. Let $r$ denote the least common denominator of the coefficients $a_i$. The admissibility condition $2g - 2 + \sum a_i > 0$ becomes the following condition in higher dimensions.

**Definition 3.1.** (1) We say a $\mathbb{Q}$-divisor $D$ on $X$ is big if

$$\lim_{m\to\infty} \frac{h^0(X, mrD)}{m^{\dim X}} > 0$$

where $r$ is the greatest common denominator of the coefficients of $D$.

(2) We say the pair $(X, \Delta)$ is of log general type if the $\mathbb{Q}$-divisor $K_X + \Delta$ is big.

**Goal 3.2.** *Construct compactifications of the moduli space of log general type pairs* $(X, \Delta)$ *and understand how they depend on the coefficients of* $\Delta$.

The first question is what plays the role of $\mathscr{M}_{g,n}$, the so-called interior of the moduli space? Ideally one would like to say the moduli space of normal crossings pairs $(X, \Delta)$. Unfortunately, these do not form a reasonable moduli space due to the existence of different birational models in higher dimensions. This problem already begins for surfaces of general type, i.e. $\dim X = 2$ and $\Delta = 0$.

**Example 3.3.** Let $X \to \mathbb{D}$ be a degenerating family of minimal surfaces of general type such that $X_t$ has no $(-2)$-curves for $t \neq 0$ but $X_0$ contains a $(-2)$-curve $E \subset X_0$. There exists a commutative diagram

$$\begin{array}{ccc} X & \overset{\mu}{\dashrightarrow} & X' \\ & \searrow \quad \swarrow & \\ & \mathbb{D} & \end{array}$$

where $\mu$ is a birational surgery called a flop. This is an example of the *Atiyah flop*. It replaces the $(-2)$-curve $E$ with a different $(-2)$-curve $E'$. The resulting family $X' \to \mathbb{D}$ is not isomorphic to $X \to \mathbb{D}$ but they agree away from $X_0$. Thus, we have two different families that extend the same family over the punctured disc $\mathbb{D} \setminus \{0\}$ which shows that the moduli functor of higher dimensional varieties cannot be separated in general.

The solution is given to us by the minimal model program. The log canonical models give us a unique birational representative for our moduli space to parametrize.

**Log canonical models.** Let $(X, \Delta)$ be a projective normal crossings pair of log general type.[7] Recall this means that $K_X + \Delta$ is big (the higher dimensional analogue of the admissibility condition $2g - 2 + \sum a_i > 0$). Let us also assume that the coefficients of $\Delta$ are in $(0, 1)$ so that $(X, \Delta)$ has klt singularities. The minimal model program posits the existence of the following diagram of birational maps.

(1)
$$\begin{array}{ccc} & & (X, \Delta) \\ & \overset{\rho}{\swarrow} & \vdots \; \varphi \\ (X^{min}, \Delta^{min} = \rho_*\Delta) & & \downarrow \\ & \overset{\phi}{\searrow} & \\ & & (X^{lc}, \Delta^{lc} = \varphi_*\Delta) \end{array}$$

Here $\rho$ is a an explicit sequence of birational transformations (extremal contractions and flips) of the minimal model progra (mmp), while

$$\phi = \phi_{m(K_X^{min} + \Delta^{min})} \quad \varphi = \phi_{m(K_X + \Delta)}$$

are the respective log canonical linear series. $(X^{min}, \Delta^{min})$ is a not-necessarily unique minimal model and

$$(X^{lc}, \Delta^{lc}) = \mathrm{LCM}(X, \Delta)$$

is the unique log canonical model.

**Claim 3.4.** *A minimal model is characterized by the property that it has klt singularities and $K_{X^{min}} + \Delta^{min}$ is nef*[8]*. The log canonical model is unique. It is uniquely characterized by either of the following properties:*

- *$(X^{lc}, \Delta^{lc})$ has log canonical singuarlties and $K_{X^{lc}} + \Delta^{lc}$ is ample, or*
- *the log canonical ring $R(K_X + \Delta)$ is finitely generated and*

$$X^{lc} = \operatorname{Proj} R(K_X + \Delta) = \operatorname{Proj} \bigoplus_{m \geq 0} H^0(X, m(K_X + \Delta)).$$

*Remark* 3.5. We call $R(K_X + \Delta) \colon = \oplus_{m \geq 0} H^0(X, m(K_X + \Delta))$ the *canonical ring* of $(X, \Delta)$.

One can takes these as the working definitions of klt and log canonical (lc) singularities though we will review the actual definitions below.

[7]For these lectures we will always assume $\Delta$ is a $\mathbb{Q}$-divisor.

[8]A $\mathbb{Q}$-Cartier divisor is nef if it has non-negative degree on every curve

**Definition 3.6.** Let $(X,\Delta)$ be a normal variety with Weil divisor $\Delta = \sum a_i D_i$. Suppose $a_i \in (0,1] \cap \mathbb{Q}$. Let $\mu : Y \to X$ be a log resolution so that $\mu_*^{-1}\Delta + \sum E_i$ is normal crossings where $E_i$ are the divisorial components of the exceptional locus viewed as reduced Weil divisors and $\mu_*^{-1}$ denotes strict transform. Then we can write

$$K_Y + \mu_*^{-1}\Delta + \sum E_i = \mu^*(K_X + \Delta) + \sum a_i E_i.$$

(1) We say that $(X,\Delta)$ is log canonical (lc) singularities if for all log resolutions $\mu$, $a_i \geq 0$ for all $i > 0$.
(2) We say that $(X,\Delta)$ has klt singularities if $\lfloor \Delta \rfloor = 0$ and for all log resolutions $\mu$, $a_i > 0$ for all $i$. Here $\lfloor \Delta \rfloor$ denotes the round down of the divisor.

The number $a_i =: a_i(X,\Delta,E_i)$ is called the *log discrepancy* of $(X,\Delta)$ with respect to $E_i$.

*Remark* 3.7. It suffices to check the conditions in the above definition for one log resolution (exercise).

*Remark* 3.8. Log canonical singularities are the largest class of normal singularities for which the log canonical ring is preserved when taking a resolution:

$$R(K_X + \Delta) = R(K_Y + \mu_*^{-1}\Delta + \sum E_i).$$

This follows from the well-known negativity lemma. For klt singularities, we have the slightly stronger equality

$$R(K_X + \Delta) = R(K_Y + \mu_*^{-1}\Delta + \sum (1-\epsilon_i)E_i)$$

for $0 < \epsilon_i \ll 0$.

*Remark* 3.9. The key feature of klt singularities is that it is an open condition on the coefficients of $\Delta$. If $(X,\Delta)$ has klt singularities and $\Delta'$ is a $\mathbb{Q}$-Cartier divisor, then $(X, \Delta+\epsilon\Delta')$ has klt singularities for $0 < \epsilon \ll 1$. Moreover, if the log canonical model of $(X,\Delta)$ happens to also be a minimal model, then the log canonical model is also invariant under perturbation: $\mathrm{LCM}(X,\Delta) = \mathrm{LCM}(X,\Delta+\epsilon\Delta')$ for $\Delta'$ as above and $0 < \epsilon \ll 1$.

Existence of minimal and log canonical models is provided by the seminal work of Birkar–Cascini–Hacon–Mckernan.

**Theorem 3.10** ([BCHM10])**.** *If $(X,\Delta)$ is a projective log smooth pair where $\Delta$ is a $\mathbb{Q}$-boundary, then $R(K_X+\Delta)$ is finitely generated and such a diagram exists.*

There is also a relative version for a projective morphism $\pi : (X,\Delta) \to B$ such that $K_X + \Delta$ is $\pi$-big. In this case, we have a *relative log canonical model*

$$\mathrm{LCM}(X/B,\Delta) = \mathrm{Proj} \bigoplus_{m \geq 0} \pi_* \mathscr{O}_X(m(K_X+\Delta)).$$

This is again a unique birational model characterized by the property of having log canonical singularities and $\pi$-ample log canonical divisor.

We already saw this construction earlier.

**Example 3.11.** (a) The unique stable limit of a family of curves $(C,\sigma_i) \to U \subset D$ with semi-stable extension $(\bar{C},\bar{\sigma}_i) \to D$ is the relative log canonical model

$$\mathrm{LCM}\left(\bar{C}/D, \sum a_i \bar{\sigma}_i + \bar{C}_0\right).$$

(b) The family of $\vec{b}$-weighted stable curves which induces the wall crossing morphism

$$\overline{\mathscr{M}}_{g,\vec{a}} \to \overline{\mathscr{M}}_{g,\vec{b}}$$

is the relative log canonical model

$$\mathrm{LCM}\left(\mathscr{C}_{g,\vec{a}}/\overline{\mathscr{M}}_{g,\vec{a}}, \sum b_i\sigma_i\right)$$

*Remark* 3.12. Log canonical singularities and klt singularities respectively can also be characterized as the singularities that appear on a (relative) log canonical model or as the output of a (relative) mmp.

The upshot of this discussion is that if we want a reasonable (at the very least bounded and separated) moduli space of varieties of log general type, we should be considering the moduli space of log canonical models.

We have the following more general definitions.

**Definition 3.13.** A log pair $(X, \Delta)$ is a *log canonical model* if $(X, \Delta)$ has log canonical singularities, and $K_X + \Delta$ is an ample $\mathbb{Q}$-Cartier divisor.

**Definition 3.14.** If $(X, \Delta)$ is a normal projective log pair, we say that $\Delta$ is a $\mathbb{Q}$-boundary if $\Delta$ is an effective $\mathbb{Q}$-divisor with coefficients $\leq 1$.

**Definition 3.15.** If $(X, \Delta)$ is a normal projective log pair with boundary $\Delta$ such that $K_X + \Delta$ is $\mathbb{Q}$-Cartier and big, the *log canonical model* of $(X, \Delta)$

$$\mathrm{LCM}(X, \Delta)$$

is defined to be $\mathrm{LCM}(Y, \Delta_Y + E)$ where $\mu : Y \to X$ is a log resolution of $(X, \Delta)$, $\Delta_Y = \mu_*^{-1}\Delta$ is the strict transform, and $E$ is the reduced $\mu$-exceptional divisor. This is independent of the choice of resolution (exercise).

With a view toward wall-crossing, we pose the following question.

**Question 3.16.** How does $\mathrm{LCM}(X, \Delta)$ change as we vary the coefficients of $\Delta$?

We will discuss this question more in depth in the next two lectures but for now lets make an observation and give a few examples.

*Remark* 3.17. Sometimes it happens that the log canonical model is also a minimal model. That is, in Diagram 1, $\phi : X^{min} \to X^{lc}$ is an isomorphism (Remark 3.9. In this case, $\mathrm{LCM}(X, \Delta)$ is klt and we can perturb the coefficients of $\Delta$ without changing the model. More generally, it could happen that $\mathrm{LCM}(X, \Delta)$ is klt but $\phi$ is a small contraction. Then again we can perturb the coefficients of $\Delta$ and the log canonical model remains the same *up to a flip of the small contraction* $\phi$. In particular, it is unchanged in codimension 1. This is the phenomena that leads to wall-crossing. To a first approximation, the walls occur when the LCM of a member of our family is strictly log canonical or is the base of a flipping contraction, while the open chambers occur when $\mathrm{LCM}(X, \Delta)$ are minimal models for all members in our family.

**Example 3.18.** (a) The walls

$$\sum_{j \in J \subset \{1,\ldots,n\}} a_j = 1$$

for Hassett space are exactly those coefficients for which $\overline{\mathscr{M}}^{sm}_{g,\vec{a}}$ contains strictly log canonical pairs, namely the pointed curve pairs $(C, \sum a_i p_i)$ where $p_j = p_{j'} = p$ for all $j, j' \in J$ so that $p$ appears with coefficient 1 in $\sum a_i p_i$.

(b) Consider the pair $(\mathbb{P}^2, aC + (1-\epsilon)l)$ where $C$ is a cuspidal cubic $y^2z = x^3$ and $l$ is a generic line that meets $C$ transversely away from the cusp. One can compute that this pair is stable if $a \leq \frac{5}{6}$ and in fact even klt if $a < \frac{5}{6}$. For $a > \frac{5}{6}$ the pair is not log canonical and its log canonical model is obtained by taking a log resolution and running the mmp.[9] Thus, $\frac{5}{6}$ is a wall for the moduli space that parametrizes this particular pair. This number is called the *log canonical threshold* of the curve cusp $C \subset \mathbb{P}^2$.

$(X, \Delta)$ be a log canonical model with

$$\Delta = \sum_{i=1}^{n} a_i D_i.$$

When $\dim X = 1$, we needed to fix the genus $g$, number of components $n$ and coefficients $\vec{a} = (a_1, \ldots, a_n)$ to have a reasonable moduli space. In general it makes sense to fix $\vec{a}$ and $n$ but what is the analogue of fixing the genus $g$?

**Question 3.19.** Which numerical invariants are locally constant in a family of log canonical models?

This question belies the fact that, unlike the 1-dimensional case, it is actually quite challenging and subtle to even define an appropriate notion of a family of log canonical models! We will discuss this in more detail later but for now a hint is given in the formula

$$X = \operatorname{Proj} R(K_X + D).$$

One hopes that this description of $X$ remains true fiberwise in a family of pairs. More precisely, let

$$\pi : (X, D) \to B$$

be a suitably nice family of log canonical models. We would hope that that

$$X = \operatorname{Proj}_B R(X/B, K_{X/B} + D) = \operatorname{Proj}_B \bigoplus_{m \geq 0} \pi_* \mathscr{O}_X(m(K_{X/B} + \Delta))$$

In particular, $K_{X/B} + \Delta$ should be a $\mathbb{Q}$-Cartier divisor the degree of the projective embedding of $X_b$ induced by $K_{X_b} + \Delta_b$ should be locally constant as $b$ varies. This motivates the following definition.

**Definition 3.20.** Let $(X, \Delta)$ be a normal projective log pair with $\mathbb{Q}$-boundary such that $K_X + \Delta$ is $\mathbb{Q}$-Cartier and big. The *volume* of $(X, \Delta)$ is defined as the intersection number

$$\operatorname{vol}(X, \Delta) = (K_{X^{lc}} + \Delta^{lc})^{\dim X}$$

where $(X^{lc}, \Delta^{lc}) = \operatorname{LCM}(X, \Delta)$.

Modulo the subtle question of families which we will revisit, we have settled on the following definition for the interior of the moduli space.

**Definition 3.21.** Fix $d$, $v$, and coefficients $\vec{a} = (a_1, \ldots, a_n)$. Then $\mathscr{M}_{d,v,\vec{a}}$ is the moduli functor of families of log canonical models $(X, \Delta = \sum_{i=1}^{n} a_i D_i)$ of dimension $d$ and volume $v$.

In the case of surfaces of general type with no boundary divisor, $\Delta = 0$, the minimal model $X^{min}$ is simply the minimal surface in the classical sense (no $(-1)$-curves) and the canonical contraction $X^{min} \to X^{can}$ contracts any trees of $(-2)$-curves which yields ADE singularities, sometimes also called rational double point singularities. These are a very well behaved class of surface singularities. In particular they are rational and Gorenstein. In this case $\mathscr{M}_{2,v}$ is the moduli space of canonically polarized surfaces with at worst ADE singularities, which we can identify at least set-theoretically[10] the moduli space $\mathscr{M}_v$ of minimal surfaces of general type and volume $v$ from the introduction.

[9]Exercise: compute the log canonical model for $a = 1$.

[10]More precisely, there is a map $\mathscr{M}_v \to \mathscr{M}_{2,v}$ which is a bijection on points. However, $\mathscr{M}_v$ is not separated due to Example 3.3 so we can view this map as the maximal separated quotient of $\mathscr{M}_v$. This is made precise by the theory of Artin-Brieskorn resolutions [Art74].

**Theorem 3.22.** *There is a separated Deligne-Mumford stack $\mathscr{M}_{2,v}$ parametrizing log canonical models $(X, \Delta)$ of dimension 2 with volume $(K_X + \Delta)^2 = v$.*

*Proof sketch.*

- **Boundedness:** The case of canonical models of surfaces of general type ($\Delta = 0$) follows from a theorem of Bombieri [Bom73] that says that the linear series $|5K_X|$ is very ample on the canonical model. The general case of pairs with log canonical singularities is a theorem of Alexeev [Ale94].
- **Algebraicity:** Kollár–Shepherd-Barron [KSB88] and Kollár [Kol23] give the correct notion of families which we will describe in more detail later and algebraicity is proved in general in [Kol23].
- **Separated:** This follows from the minimal model program, and namely, uniqueness of canonical models. Indeed any two different families over a DVR $(X_i, \Delta_i) \to \operatorname{Spec} R$ with the same generic point are birational and thus have the same log canonical model over $R$ but $(X_i, \Delta_i)$ are by assumption isomorphic to this unique log canonical model so in particular they are isomorphic to eachother.

□

## 4. Semi-log canonical singularities and moduli of stable pairs

To obtain a compact moduli space we need to extend our moduli functor to satisfy the valuative criterion of properness. That is, we need a class of varieties containing log canonical models for which *stable reduction* holds for this class: for any family $X \to U$ over the punctured spectrum of a DVR $U = D \setminus 0$ where $(D, 0) = (\operatorname{Spec} R, \mathfrak{m})$, there exists an extension of DVRs $D' \to D$ and an extension $\bar{X}' \to D'$ of $X' = X \times_D D'$ with $\bar{X}'_0$ in the the class.

As we saw in the curves case, we need to deal with non-normal varieties to compactify the moduli space. A complication in this higher dimensional theory is that we really need to consider the properties of divisors such as $K_X$ or $K_X + \Delta$ when $X$ is not normal. In general, divisor theory and the canonical divisor are not well behaved on non-normal varieties so we need to restrict to a certain class of non-normal varieties.

**Definition 4.1.** A reduced equidimensional variety $X$ is said to be *demi-normal* if $X$

- satisfies Serre's condition $S_2$, and
- has at worst nodal singularities at all codimension 1 points.

The second condition says that $X$ "look's like" a nodal curve in codimension 1 and we already know how to handle nodal curves. Serre's condition $S_2$ roughly says that most of the geometry of $X$ is uniquely determined by what is happening in codimension 1, namely, on the nodal locus. Using these two facts we can make sense of $K_X$ and divisor theory.

*Remark* 4.2. The $S_2$ condition says that local sections of $\mathscr{O}_X$ extend over subsets of codimension 2: $i_*\mathscr{O}_U = \mathscr{O}_X$ whenever $i : U \to X$ is the inclusion of an open set $U = X \setminus Z$ where $Z$ has codimension at least 2. More generally, on an $S_2$ scheme, we have that the sheaf

$$i_*\mathscr{L}$$

where $L$ is a line bundle on any such $U$ is $S_2$ and in particular satisfies the same extension property. In particular, we can talk about the canonical divisor or canonical line bundle on a demi-normal $X$ via the formulas

$$K_X = i_*K_U \quad \omega_X = i_*\omega_U$$

where $U \subset X$ is the open subset where $X$ has nodal singularities. Nodal singularities are Gorenstein so $U$ admits a Cartier canonical divisor and we have the natural formula $\mathscr{O}_X(K_X) = \omega_X$ of Weil divisorial sheaves. More generally, via the same observation, we have a well behaved theory of Weil divisors and divisorial sheaves $\mathscr{O}_X(D)$ on such a scheme as long as we only consider those $D$ for which there exists an open set $U$ with complement of codimension $\geq 2$ such that $D|_U$ is Cartier.

**Exercise 4.3.** Let $X$ be demi-normal with normalization $\nu : Y \to X$. Then there exists a unique *conductor divisor* $D \subset Y$ such that

$$\nu^* K_X = K_Y + D.$$

**Definition 4.4.** Let $(X, \Delta)$ be a pair where $X$ is demi-normal and $\Delta = \sum a_i D_i$ where each $D_i$ is a pure codimension 1 subvariety not contained in the singular locus of $X$. We say $(X, \Delta)$ is *semi-log canonical (slc)* if

(a) $K_X + \Delta$ is $\mathbb{Q}$-Cartier, and
(b) $(Y, D + \mu_*^{-1}\Delta)$ is log canonical where $\mu : Y \to X$ is the normalization and $D$ is the conductor.

*Remark* 4.5. The map $D \to X$ is generically 2-to-1 onto its image so we often call $D$ (or its image) the *double locus*.

We can also define slc in terms of *semi-resolutions* analogously to the definition of log canonical in terms of a resolution. For simplicity we work with surfaces.

**Definition 4.6.** A surface is *semi-smooth* if it is demi-normal and has at worst pinch point singularities of the form $\{xy^2 = z^2\}$.

*Remark* 4.7. Semi-smooth surfaces are exactly the demi-normal surfaces with smooth normalization. The double locus $D \to X$ in this case is smooth and has smooth image $\bar{D} \subset X$ and $D \to \bar{D}$ is a branched cover which is ramified exactly at the pinch points.

**Definition 4.8.** Let $X$ be a demi-normal surface. A *semi-resolution* is a projective morphism $\mu : Y \to X$ from a semi-smooth $Y$ such that $\mu$ is an isomorphism in codimension 1 on $X$ and no components of the double locus of $Y$ are $\mu$-exceptional. If $\Delta$ is a divisor with no component contained in the singular locus, we say $\mu$ is a semi-log resolution if $\mu_*^{-1}\Delta + \sum E_i$ is a normal crossings divisor which meets every component of the double locus of $Y$ transversely. Here $E_i$ are the exceptional divisors of $\mu$ which by assumption are not contained in the double locus.

**Proposition 4.9** (K, complete moduli)**.** *Any demi-normal surface admits a semi-resolution.*

We now can state the alternative characterization of semi-log canonical.

**Proposition 4.10.** *Let $(X, \Delta)$ be a pair where $X$ is a demi-normal surface and $\Delta = \sum a_i D_i$ where each $D_i$ is a pure codimension 1 subvariety not contained in the singular locus of $X$. Then $(X, \Delta)$ is slc if and only if $K_X + \Delta$ is $\mathbb{Q}$-Cartier and for any semi-log resolution $\mu : Y \to X$,*

$$K_Y + \mu_*^{-1}\Delta + \sum E_i = \mu^*(K_X + \Delta) + \sum a_i E_i$$

*for $a_i \geq 0$.*

*Remark* 4.11. The coefficients $a_i$ are well defined as they can be computed at the generic points of $E_i$ which are smooth points of $Y$ by assumption.

As in the case of log canonical singularities, slc singularities are the largest class of singularities for which the canonical ring $R(K_X + \Delta)$ is unchanged by taking a log resolution:

$$R(K_X + \Delta) = R(K_Y + \mu_*^{-1}\Delta + \sum E_i).$$

*Remark* 4.12. Unlike the normal case, $R(K_X + \Delta)$ need not be finitely generated for an slc pair, even if $\Delta = 0$ and even if $X$ is semismooth.

While this definition is natural as the smallest reasonably behaved class of singularities containing nodes and log canonical singularities, it also falls out when trying to mimic the proof of stable reduction for curves (and this is how Kollár and Shepherd-Barron arrived at this definition initially). The key idea is the following theorem (a combination of results due to many people).

**Theorem 4.13** (Inversion of adjunction)**.** *Let $(X, S+\Delta)$ be a normal log pair with boundary such that $S$ is Cartier and the round down $\lfloor \Delta \rfloor = 0$ is zero. Then $(X, S+\Delta)$ is log canonical in a neighborhood of $S$ if and only if $(S, \Delta|_S)$ is semi-log canonical.*

We are now ready to state the main definition which is the higher dimensional version of Definition 2.2. As before the definition involves a singularity part and a stability part.

**Definition 4.14.** A pair $(X, \Delta)$ is a KSBA-*stable pair* if

(a) (singularity) $(X, \Delta)$ has semi-log canonical singularities, and
(b) (stability) $K_X + \Delta$ is an ample $\mathbb{Q}$-Cartier divisor.

**Example 4.15.** In dimension 1, the stable pairs are exactly the weighted stable curves.

Combining the existence of relative log canonical models, semi-stable reduction [KKMSD73], and inversion of adjunction, we obtain properness of the moduli space of stable pairs.

**Theorem 4.16.** *The family of stable pairs is proper. More precisely, given a flat family $(X^0, \Delta^0) \to U^0$ over a punctured spectrum of a DVR $U^0 = U \setminus 0$, $U = \operatorname{Spec} R$, there exist an extension of DVRs $U' \to U$ and a unique family of stable pairs $(X, \Delta) \to U'$ extending*

$$(X^0, \Delta^0) \times_U U'.$$

*Proof sketch.* Assume that $X^0$ is normal. We begin by taking a log resolution $(Y^0, \tilde{\Delta}^0 + E^0)$ of $(X^0, \Delta^0)$. By semi-stable reduction, there exists $U' \to U$ and a semistable family $Y \to U'$ extending $Y^0 \times_U U'$. Here semistable means that $(Y, Y_0)$ is log smooth. Taking the closures of the pullbacks of $\tilde{\Delta}^0$ and $E^0$, and possibly blowing up further we can assume that $(Y, Y_0 + \tilde{\Delta} + E)$ is a normal crossings pair and that the log canonical model of the generic fiber of $(Y, Y_0 + \tilde{\Delta} + E) \to U'$ is $(X^0, \tilde{\Delta}^0) \times_U U'$. Now we take the relative log canonical model

$$\operatorname{LCM}(Y/U', Y_0 + \tilde{\Delta} + E) = (X, X_0 + \Delta) \to U'$$

where $X_0$ is the central fiber. By inversion of adjunction, $(X_0, \Delta|_{X_0})$ has semi-log canonical singularities and by adjunction

$$K_{X_0} + \Delta|_{X_0} = (K_X + X_0 + \Delta)|_{X_0}$$

is ample so $(X_0, \Delta|_{X_0})$ is a stable pair and it is unique by uniqueness of log canonical models. □

**Example 4.17.** Let us consider the example of a line arrangement on $\mathbb{P}^2$ following [Ale15]. Suppose we have a collection of lines in $L_i \subset \mathbb{P}^2_{k[t]}$ over $k[[t]]$ which are generic for $t \neq 0$ but at $t = 0$, $L_1, L_2$ and $L_3$ all meet at a point $p$. Consider the family of pairs $(\mathbb{P}^2, c\sum L_i) \times_{\operatorname{Spec} k[[t]]} k((t))$. To compute the stable limit first we blow up $\mathbb{P}^2_{k[t]}$ at $p$ in the central fiber. Let $X$ denote the blowup and let $L_i'$ be the strict transforms of the family of lines and $E \cong \mathbb{P}^2$ the central fiber. Then $L_i' \cap E = \emptyset$ for $i \neq 1,2,3$ and $L_i \cap E = l_i$ is a line on $E$ for $i = 1,2,3$. The strict transform of the central fiber is $\mathbb{F}_1 = \operatorname{Bl}_p \mathbb{P}^2$ and this is glued to $E$ along the exceptional curve. Now $K_X + c\sum L_i'$ restricts to $E$ to $K_E + l_0 + c(l_1 + l_2 + l_3)$ where $l_0 = E \cap \mathbb{F}_1$. Then $K_E + l_0 + c(l_1+l_2+l_3) \sim_{\mathbb{Q}} (3c-2)l$ which is ample if and only if $c > 2/3$. Thus for $c > 2/3$ the KSBA-stable limit is $\mathbb{F}_1 \cup_{l_0} \mathbb{P}^2$ with a line arrangement of 3 lines on the $\mathbb{P}^2$. On the other hand, if $c < 2/3$, then we must contract $E$ and so the stable limit is simply $\mathbb{P}^2$ with the arrangement of lines where the first three are allowed to be coincident. This tracks with the fact that a planar pair $(\mathbb{A}^2, c(\{x^3 = y^3\}))$ is log canonical if and only if $c \leq 2/3$.

**Kollár's gluing theory.** In order to remove the assumption that $X^0$ is normal, we need to use Kollár's gluing theory to deal with non-normal varieties [Kol13, Section 5]. More precisely, we want to be able to take the normalization $\nu : Y^0 \to X^0$ with double locus $D^0$, compute the stable reduction of $(Y^0, D^0 + \nu^*\Delta^0)$, and then glue the resulting log canonical models to obtain the stable limit of $(X^0, \Delta^0)$.

In order to do this we need to first understand the gluing data. Towards that end let $(X, \Delta)$ be an slc pair with normalization $\nu : Y \to X$ and double locus $D$. Since $X$ is nodal in codimension 1, the map $D \to X$ is generically 2-to-1 onto its image and so $D$ becomes equipped with a rational involution $D \dashrightarrow D$. This extends to an involution $\tau : D^n \to D^n$ on the normalization of $D$.

**Definition-Proposition 4.18.** *There exists a unique divisor* $\mathrm{Diff}_{D^n}(\Delta)$ *such that*

$$K_{D^n} + \mathrm{Diff}_{D^n}(\Delta) = (K_X + \Delta)|_{D^n}.$$

**Theorem 4.19** (Kollár's gluing)**.** *Sending* $(X, \Delta)$ *to* $(Y, D + \nu^*\Delta, \tau : D^n \to D^n)$ *induces a bijection*

$$\{\textit{stable pairs } (X, \Delta)\} \mapsto \left\{ \begin{matrix} (Y, D + \Delta, \tau : D^n \to D^n) \textit{ where} \\ (Y, D + \Delta) \textit{ is log canonical, } D \textit{ is} \\ \textit{a reduced divisor, and } \tau \textit{ is an invo-} \\ \textit{lution which fixes } \mathrm{Diff}_{D^n}(\Delta) \end{matrix} \right\}.$$

With some care, this can be used to prove Theorem 4.16 without the normality assumption.

We are now ready to state the main theorem which guarantees the existence of the so-called KSBA moduli space of stable pairs $\overline{\mathscr{M}}_{d,v,\vec{a}}$. This is the combined work of many people over several decades. Each step of the proof roughly follows the outline from Section 2: boundedness, algebraicity, properness via the minimal model program, and projectivity using Kollár's ampleness lemma.

**Theorem 4.20** ([KSB88, Ale94, BCHM10, HMX14, HX13, KP17, Kol23])**.** *Fix* $d, v$ *and* $\vec{a} = (a_1, \ldots, a_n)$*. There exists a proper Deligne-Mumford stack* $\overline{\mathscr{M}}_{d,v,\vec{a}}$ *with projective coarse moduli space parametrizing stable pairs of dimension* $d$*, volume* $v$ *and coefficients* $\vec{a}$*. Moreover, it contains the* $\mathscr{M}_{d,v,\vec{a}}$ *the moduli space of log canonical models as an open substack.*

*Remark* 4.21. Some warnings:

(1) So far we only know the objects $\overline{\mathscr{M}}_{d,v,\vec{a}}$ parametrizes not its functor of points. The functor of points is quite subtle and its development is detailed in [Kol23]. We will revisit this later.
(2) Unlike the case $d = 1$, $\mathscr{M}_{d,v,\vec{a}}$ is *not* dense in $\overline{\mathscr{M}}_{d,v,\vec{a}}$. In fact $\overline{\mathscr{M}}_{d,v,\vec{a}}$ can have many irreducible or connected components not contained in $\mathscr{M}_{d,v,\vec{a}}$.
(3) Similarly, the singularities of $\overline{\mathscr{M}}_{d,v,\vec{a}}$ can be arbitrarily complicated for $d \geq 2$ ([Vak06]) and a general understanding of its deformation theory is still lacking.

4.1. **Wall-crossing.** Finally, we come back to the question of wall-crossing for these spaces generalizing Theorem 2.12. The first higher dimensional cases of wall-crossing were for the moduli space of hyperplane arrangements [Ale15] and the moduli space of elliptic surfaces [AB21]. Recently, a wall-crossing theorem was proven in full generality [ABIP23, MZ23]

For simplicity, we consider the following situation. Let Let $(X, D_1, \ldots, D_n) \to B$ be a family of smooth normal crossings pairs over a smooth connected base $B$ and let $P$ be a finite, rational polytope of weight vectors $\vec{a} = (a_1, \ldots, a_n)$ such that $a_i < 1$ and $(X, \sum a_i D_i) \to B$ is a family of stable pairs for each $\vec{a} \in P$. Then for each $\vec{a} \in P$, we have a map

$$\varphi_{\vec{a}} : B \to \overline{\mathscr{M}}_{d,v,\vec{a}}.$$

where $\varphi_{\vec{a}}(b)$ is the point classifying the stable pair $(X, \sum a_i D_i)$. We let $\mathscr{N}_{\vec{a}}$ be the normalization of the closure of the image of this map. This is the *KSBA compactification* of the input family of pairs $(X, D_1, \ldots, D_n) \to B$.

The reader should imagine that this is a family of pairs of interest (e.g. $n$-pointed genus $g$ smooth curves or hyperplane arrangements on $\mathbb{P}^n$). Then the theory of stable pairs furnishes a compactification $\mathscr{N}_{\vec{a}}$ for each weight vector $\vec{a} \in P$. Denote by

$$\mathscr{X}_{\vec{a}} \to \mathscr{N}_{\vec{a}}$$

the universal family over $\mathscr{N}_{\vec{a}}$.

**Theorem 4.22** ([ABIP23, MZ23])**.** *Then there exists a finite, rational polyhedral wall-and-chamber decomposition of $P$ such that the following hold.*

*(a) For $\vec{a}, \vec{a}'$ contained in the same chamber, there are canonical isomorphisms*

$$\begin{array}{ccc} \mathscr{X}_{\vec{a}} & \xrightarrow{\cong} & \mathscr{X}_{\vec{a}'} \\ \downarrow & & \downarrow \\ \mathscr{N}_{\vec{a}} & \xrightarrow{\cong} & \mathscr{N}_{\vec{a}'} \end{array}$$

*(b) For $\vec{a}, \vec{b} \in P$ contained in different chambers and satisfying $b_i \leq a_i$ for all $i$, there are canonical birational wall-crossing morphisms*

$$\rho_{\vec{b},\vec{a}} : \mathscr{N}_{\vec{a}} \to \mathscr{N}_{\vec{b}}$$

*such that for any third weight vector $\vec{c}$ with $c_i \leq b_i$, we have $\rho_{\vec{c},\vec{b}} \circ \rho_{\vec{b},\vec{a}} = \rho_{\vec{c},\vec{a}}$. Moreover, the map $\rho_{\vec{b},\vec{a}}$ is induced by a birational map $h^{b,a} : \mathscr{X}_{\vec{a}} \dashrightarrow \rho^*_{\vec{b},\vec{a}} \mathscr{X}_{\vec{b}}$ such that, for a generic $u \in \mathscr{N}_{\vec{a}}$, the fiberwise map $h^{b,a}_u : (\mathscr{X}_{\vec{a}})_u \dashrightarrow (\mathscr{X}_{\vec{b}})_{\rho(u)}$ is the canonical model of $((\mathscr{X}_{\vec{a}})_u, \sum b_i (\mathscr{D}_i)_u)$.*

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