

THE MOTIVIC HILBERT ZETA FUNCTION OF A PLANAR n -FOLD THICKENING OF A SMOOTH CURVE

DORI BEJLERI

1. INTRODUCTION

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In [2], it is shown that the motivic Hilbert zeta function of a reduced curve singularity is a rational function with denominator $(1 - t)^r$ where r is the number of branches. It is natural to ask if this above rationality holds for non-reduced curves. For *generically reduced* curves one expects the answer to be yes:

Conjecture 1. *Let $(C, 0)$ be the germ of a generically reduced curve singularity. Then $Z_{0 \subset C}^{\text{Hilb}}(t)$ is a rational function with denominator $(1 - t)^r$ where r is the number of branches.*

For generically nonreduced curves, the picture seems more complicated. In this note, we compute the example of a “uniformly thickened” planar curve.

Theorem 1.1 (Section 4). *Let C be a smooth curve and let X be a uniformly n -fold thickening with two dimensional tangent space so that $X_{\text{red}} = C$. Then*

$$Z_X^{\text{Hilb}}(t) = \prod_{m=1}^n Z_C(\mathbb{L}^{m-1}t^m)$$

where $Z_C(t)$ is the motivic zeta function of C .

Remark 1.2. *We remark in particular that $Z_X^{\text{Hilb}}(t)$ is in particular a rational function with explicitly computable denominator.*

The key computation in the proof of Theorem 1.1 is the local case of the germ of the non-reduced branch $y^n = 0$ in $(\mathbb{A}^2, 0)$.

Proposition 1.3. *Let $C_n = \text{Speck}[x, y]/(y^n)$. Then the Hilbert zeta function of C_n is given by*

$$Z_{0 \subset C_n}^{\text{Hilb}}(t) := \sum_d [\text{Hilb}^d(C_n, 0)]t^d = \prod_{m=1}^n \left(\frac{1}{1 - \mathbb{L}^{m-1}t^m} \right).$$

Remark 1.4. *Note that*

$$\lim_{n \rightarrow \infty} Z_{0 \subset C_n}^{\text{Hilb}}(t) = \prod_{m=1}^{\infty} \left(\frac{1}{1 - \mathbb{L}^{m-1}t^m} \right) = Z_{(0 \subset \mathbb{A}^2)}^{\text{Hilb}}(t)$$

as expected (see Proposition 2.1).

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2. HILBERT SCHEME OF POINTS ON THE PLANE

In this section, we give some background on $\text{Hilb}^d(\mathbb{A}^2)$ following [7, 4].

The action of $(\mathbb{C}^*)^2$ on $k[x, y]$ by $(t_1, t_2) \cdot (x, y) = (t_1 x, t_2 y)$ induces an action on $\text{Hilb}^d(\mathbb{A}^2)$. The fixed points of the torus action are indexed by partitions $\lambda \vdash d$.

We denote the monomial ideal by I_λ and define

$$B_\lambda := \{x^i y^j \mid (i, j) \in \lambda\}$$

and

$$Z_\lambda := \text{Speck}[x, y]/I_\lambda.$$

The subset

$$U_\lambda := \{[Z] \in \text{Hilb}^n(\mathbb{A}^2) \mid B_\lambda \text{ spans } \mathcal{O}_Z\}$$

is a maximal torus invariant open affine neighborhood of $[Z_\lambda]$. Coordinate functions on U_λ are given by $c_{i,j}^{r,s}$ satisfying

$$x^r y^s = \sum_{\lambda} c_{i,j}^{r,s} x^i y^j \pmod{I}$$

for $[I] \in U_\lambda$.

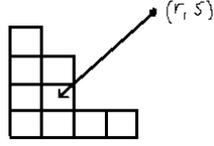


FIGURE 1. The function $c_{i,j}^{r,s}$ depicted as an arrow from box (r, s) to box (i, j) .

We represent these as arrows starting at box (r, s) and ending at box $(i, j) \in \lambda$. Note that if $(r, s) \in \lambda$, then

$$c_{i,j}^{r,s} \equiv \begin{cases} 1 & (r, s) = (i, j) \\ 0 & \text{else} \end{cases}$$

Therefore, the nonconstant functions correspond to arrows that start at $(r, s) \in \mathbb{N}^2 \setminus \lambda$ and end in λ .

For each box $(i, j) \in \lambda$, there are two distinguished arrows $d_{i,j}$ and $u_{i,j}$ pointing southeast and northwest respectively as depicted:

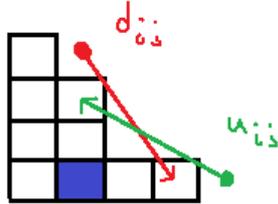


FIGURE 2. The distinguished arrows $d_{i,j}$ and $u_{i,j}$ associated to box (i, j) in blue.

The torus acts on \mathcal{O}_{U_λ} by

$$(t_1, t_2) \cdot c_{i,j}^{r,s} = t_1^{r-i} t_2^{s-j} c_{i,j}^{r,s}$$

The cotangent space T_λ^* to the monomial subscheme $[Z_\lambda]$ in U_λ has basis given by the set of distinguished arrows $d_{i,j}$ and $u_{i,j}$ as (i, j) runs through each box in λ .

Let $\sigma : \mathbb{C}^* \rightarrow (\mathbb{C}^*)^2$ be a 1-parameter subgroup $\sigma(t) = (t^p, t^q)$ for $q \gg p > 0$. This induces a Bialynicki-Birula decomposition of $\text{Hilb}^d(\mathbb{P}^2)$ into affine cells. With these weights, a cell is either contained in $\text{Hilb}^n(\mathbb{A}^2, 0)$ or is disjoint from it. Thus we get a Bialynicki-Birula decomposition of $\text{Hilb}^d(\mathbb{A}^2, 0)$ into affine cells $D_\lambda \cong \mathbb{A}^{b(\lambda)}$ indexed by partitions.

The cell $D_\lambda \subset \mathcal{U}_\lambda$ is the vanishing locus of all *positive weight* coordinate functions $c_{i,j}^{r,s}$. From the choice of weights, we see that $c_{i,j}^{r,s}$ is positive weight for σ if and only if $s > j$ (weakly south pointing arrows) or $s = j$ and $r > i$ (strictly west pointing arrows). In particular, the cotangent space to $[Z_\lambda]$ in D_λ is spanned by the set of $u_{i,j}$ that are not horizontal. Therefore

$$b(\lambda) = \dim D_\lambda = \#\{(i, j) \in \lambda \mid u_{i,j} \text{ is not horizontal}\}$$

Let $|\lambda|$ denote the number of boxes, $h(\lambda)$ the height (longest column) of the diagram and $l(\lambda)$ the length (longest row) of the diagram. Then a combinatorial argument shows that

$$b(\lambda) = |\lambda| - l(\lambda)$$

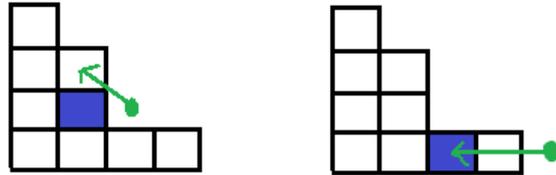


FIGURE 3. The arrow $u_{i,j}$ is not horizontal if and only if the box (i, j) is not top most in its column. Such boxes are clearly in bijection with boxes not in the first row.

Now we can compute $Z_{(0 \subset \mathbb{A}^2)}^{\text{Hilb}}(t)$.

Proposition 2.1.

$$Z_{(0 \subset \mathbb{A}^2)}^{\text{Hilb}}(t) = \prod_{m=1}^{\infty} \left(\frac{1}{1 - \mathbb{L}^{m-1} t^m} \right)$$

Proof. Since $\text{Hilb}^d(\mathbb{A}^2, 0)$ is stratified by affine spaces, it suffices to compute the Betti number generating function. We see from above that

$$b_{2i}(\text{Hilb}^d(\mathbb{A}^2, 0)) = \#\{\lambda \vdash d \mid b(\lambda) = i\}.$$

Let

$$P(q, t) := \sum_{\lambda} q^{l(\lambda)} t^{|\lambda|}.$$

Then the generating function for the Betti numbers (up to a factor of 2) is

$$P(1/q, qt) = \sum_{\lambda} q^{|\lambda| - l(\lambda)} t^{|\lambda|}.$$

Since $l(\lambda)$ is the number of parts (columns) of the partition, $P(q, t)$ is just the generating function for the number of parts of a partition. This is

$$P(q, t) = \prod_{m \geq 1} \left(\frac{1}{1 - qt^m} \right)$$

and we get the result by substituting $q \mapsto 1/\mathbb{L}$ and $t \mapsto \mathbb{L}t$. □

3. PROOF OF PROPOSITION 1.3

Let $C_n = \text{Speck}[x, y]/(y^n)$. An ideal I defines a subscheme of C_n if and only if $y^n \in I$. It follows that $\text{Hilb}^d(C_n)$ is locally out of $\text{Hilb}^d(\mathbb{A}^2)$ by the vanishing of the functions $c_{i,j}^{0;n}$ for all $(i, j) \in \lambda$ on the open set U_λ .

In particular, a monomial ideal I_λ defines a subscheme of C_n if and only if the height $h(\lambda)$ is bounded by n . That is, λ fits inside a horizontal height n strip. Equivalently, each part (column) of the partition is at most size n .

Therefore,

$$\text{Hilb}^d(C_n) \subset \left(\bigcup_{h(\lambda) \leq n} U_\lambda \right) \setminus \left(\bigcup_{h(\lambda) > n} U_\lambda \right).$$

The affine cell D_λ is defined by the vanishing of positive weight arrows. If $h(\lambda) \leq n$, then $c_{i,j}^{0;n}$ has weight $q(n-j) - pi > 0$ since $j < n$. Therefore, $c_{i,j}^{0;n}$ is identically zero on D_λ . That is:

Lemma 3.1. *If $h(\lambda) \leq n$, $D_\lambda \subset \text{Hilb}^d(C_n, 0)$ and otherwise $D_\lambda \cap \text{Hilb}^d(C_n, 0) = \emptyset$. That is, $\text{Hilb}^d(C_n, 0)$ admits an affine stratification by the cells D_λ for $h(\lambda) \leq n$.*

Proposition 3.2. *The Hilbert zeta function of C_n is given by*

$$Z_{0 \subset C_n}^{\text{Hilb}}(t) = \prod_{m=1}^n \left(\frac{1}{1 - \mathbb{L}^{m-1} t^m} \right).$$

Proof. As before, it suffices to compute the Betti number generating function (up to a factor of 2) and $\dim D_\lambda = |\lambda| - l(\lambda)$. Letting

$$P(q, t) = \sum_{\lambda, h(\lambda) \leq n} q^{l(\lambda)} t^{|\lambda|},$$

the generating function is given by $P(1/q, qt)$. This is the generating function for partitions with parts bounded by n and statistic given by number of parts. As above this is given by

$$P(q, t) = \prod_{m=1}^n \left(\frac{1}{1 - qt^m} \right)$$

and we obtain the Hilbert zeta function by $q \mapsto 1/\mathbb{L}$ and $t \mapsto \mathbb{L}t$. □

Remark 3.3. *For a general monomial curve $C \subset \mathbb{A}^2$, one can run the same argument as to write the Hilbert zeta function as a sum over partitions of explicit powers of \mathbb{L} . However, these powers become more complicated to compute since Lemma 3.1 no longer holds. In this case $D_\lambda \cap \text{Hilb}^d(C, 0)$ is an explicit affine subspace of D_λ given by the vanishing of certain coordinate functions $c_{i,j}^{r,s}$ depending on the monomials generating the ideal of C .*

4. LOCALLY PLANAR UNIFORMLY THICKENED CURVES

A ribbon is a uniform double structure on a smooth curve [1]. More generally, we define a *uniformly n -fold thickened curve* to be a non-reduced curve X with smooth $X^{\text{red}} = C$ and such that the completed local ring at every point of X is isomorphic to the germ of $(C_n, 0)$.

Example 4.1. *Let $C \subset S$ be a smooth curve inside a smooth surface with ideal I . Then the curve X with ideal I^n is a uniformly n -fold thickened curve.*

Theorem 4.2. *Let X be a uniformly n -fold thickened curve with reduced subvariety C . Then*

$$Z_X^{\text{Hilb}}(t) = \prod_{m=1}^n Z_C(\mathbb{L}^{m-1}t^m).$$

In particular, it is a rational function.

Proof. There is a Hilbert-Chow morphism $h : \text{Hilb}^d(X) \rightarrow \text{Sym}^d(C)$ sending a subscheme of X to its support. We can stratify $\text{Sym}^d(C)$ by partitions $d = \sum i d_i$ where the d points have collided into d_i points of multiplicity i . Then over each stratum h is a Zariski locally trivial fibration with fiber

$$\prod \text{Hilb}^i(C_n, 0)^{d_i}.$$

From the explicit form of the power structure on the Grothendieck ring of varieties (see also [6]), we see that

$$Z_X^{\text{Hilb}}(t) = (Z_{0_C C_n}^{\text{Hilb}})^{[C]} = \prod_{m=1}^n \left(\frac{1}{1 - \mathbb{L}^{m-1}t^m} \right)^{[C]}.$$

By [5, Statement 2],

$$\left(\frac{1}{1 - \mathbb{L}^{m-1}t^m} \right)^{[C]} = \left(\frac{1}{1 - t} \right)^{[C]} \Big|_{t \mapsto \mathbb{L}^{m-1}t^m} = Z_C(\mathbb{L}^{m-1}t^m),$$

thus completing the proof. \square

One expects that sometimes the moduli space of sheaves on a ribbon, or more generally a uniformly n -fold thickened curve X , should be related to the moduli space of rank n vector bundles on the underlying smooth curve (see, for example, [3]).

Question 1. *Is the expression above for $Z_X^{\text{Hilb}}(t)$ related to motivic invariants of the moduli space of rank n vector bundles on the smooth curve X^{red} ?*

REFERENCES

- [1] D. BAYER AND D. EISENBUD, *Ribbons and their canonical embeddings*, Trans. Amer. Math. Soc., 347 (1995), pp. 719–756. [4](#)
- [2] D. BEJLERI, D. RANGANATHAN, AND R. VAKIL, *Motivic Hilbert zeta functions of curves are rational*, J. Inst. Math. Jussieu, 19 (2020), pp. 947–964. [1](#)
- [3] D. CHEN AND J. L. KASS, *Moduli of generalized line bundles on a ribbon*, J. Pure Appl. Algebra, 220 (2016), pp. 822–844. [4](#)
- [4] G. ELLINGSRUD AND S. A. STRØMME, *On the homology of the Hilbert scheme of points in the plane*, Invent. Math., 87 (1987), pp. 343–352. [2](#)
- [5] S. M. GUSEIN-ZADE, I. LUENGO, AND A. MELLE-HERNÁNDEZ, *A power structure over the Grothendieck ring of varieties*, Math. Res. Lett., 11 (2004), pp. 49–57. [4](#)
- [6] ———, *Power structure over the Grothendieck ring of varieties and generating series of Hilbert schemes of points*, Michigan Math. J., 54 (2006), pp. 353–359. [4](#)
- [7] M. HAIMAN, *t, q-Catalan numbers and the Hilbert scheme*, Discrete Math., 193 (1998), pp. 201–224. Selected papers in honor of Adriano Garsia (Taormina, 1994). [2](#)

DORI BEJLERI, DEPARTMENT OF MATHEMATICS, HARVARD UNIVERSITY, CAMBRIDGE, MA 02138 USA
 Email address: bejleri@math.harvard.edu