Curves in $\mathbb{R}^n$

Def

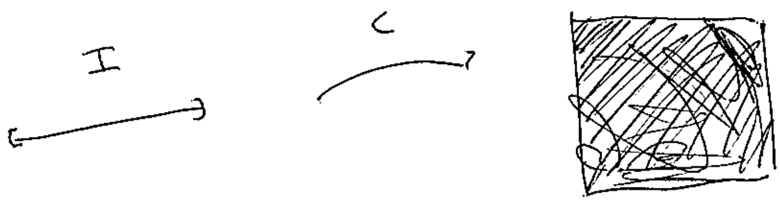
A parametrized curve is a continuously differentiable (sometimes denoted C^1) map

$$c : \underset{\mathbb{R}}{\underset{\cap}{I}} \longrightarrow \mathbb{R}^n \qquad I = [a, b]$$

> **Rmk**
> C^n = "n times continuously differentiable"
> $n = 0, 1, \ldots, \infty$

Rmk Why not continuous? There exist continuous, for example space filling curves! That is, continuous

$$c : [0,1] \to \mathbb{R}^2 \quad \text{s.t.} \quad \mathrm{Im}(c) = [0,1] \times [0,1]$$

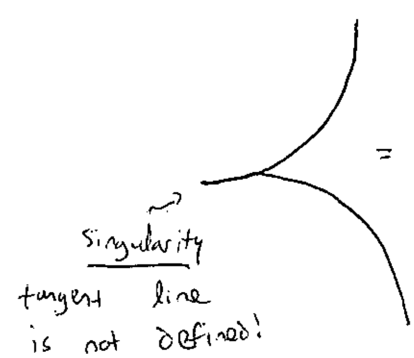


Ex: a) $c : [0, 2\pi] \to \mathbb{R}^2 \qquad c(t) = (\cos(t), \sin(t))$

$\mathrm{Im}(c)$ or trace of c =

the unit circle

b) $c(t) = (t^2, t^3) : \mathbb{R} \to \mathbb{R}^2$



$= c(\mathbb{R})$

$$\left.\frac{dc}{dt}\right|_{t=0} = \left.(2t, 3t^2)\right|_{t=0} = (0, 0)$$

Def A regular parametrized (C^1) curve is a C^1-curve $c: \underset{\mathbb{R}}{\overset{\cap}{I}} \to \mathbb{R}^n$ s.t.

$$\frac{dc}{dt} \neq 0 \quad \text{for all} \quad t \in I.$$

Aside on multivariable Calculus

$f: \underset{\mathbb{R}^n}{\overset{\cap}{U}} \to \mathbb{R}^m$ is C^1 $\qquad x \in U$

$y_0 = f(x_0)$

$$df_{x_0} : T_{x_0}\mathbb{R}^n \longrightarrow T_{y_0}\mathbb{R}^m$$

$$df_{x_0}(v) = Jf_{x_0} v + y$$

$$\underset{\text{Jacobian}}{Jf_x}\Big|_{x_0} = \left[\frac{\partial f_i}{\partial x_j} \right] \Bigg|_{x = x_0} \qquad \begin{aligned} f &= (f_1, \ldots, f_m) \\ x &= (x_1, \ldots, x_n) \end{aligned}$$

$$f(x_0 + v) = f(x_0) + Jf_{x_0} v + o(\|v\|)$$

$\leftarrow$ higher order term such that $\lim_{v \to 0} \frac{o(\|v\|)}{\|v\|} = 0$

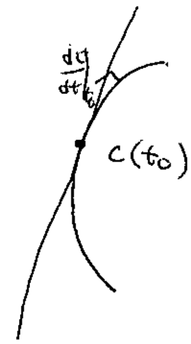
If $m \leq n$ & Jf_{x_0} is full rank, we say f is an *immersion* at x_0.

So a par. curve c is regular if its an immersion everywhere on I.

$dc_{t_0} : \mathbb{R}^1 \longrightarrow \mathbb{R}^n$ $\qquad$ $dc_{t_0}(r) = r \left.\frac{dc}{dt}\right|_{t_0} + c(t_0)$

$I \ni t_0$

$dc_{t_0} : \mathbb{R} \longrightarrow$



So dc_{t_0} embeds $T_{t_0} I = \mathbb{R}$ as the tangent line of $Im(c)$ at $c(t_0)$.

<u>Slogan</u> regular par. curves have a well defined tangent line ~~along~~ to $c(I)$ at $c(t)$ for each $t \in I$

<u>Def</u> a) regular C^1-curves c_1 & c_2 are equivalent if there is $\varphi : I_2 \to I_1$ C^1 & bijective <u>$\varphi' > 0$</u>

$$\begin{array}{ccc} c_1 : I_1 & \longrightarrow & \mathbb{R}^n \\ \quad \uparrow \varphi & \nearrow & \\ c_2 : I_2 & & \end{array}$$

such that $c_2 = c_1 \circ \varphi$

b) a regular curve is an equivalence class of regular parametrized curves.

$c_1(t) = (\cos(t), \sin(t))$ $\qquad$ $c_2 = (\cos(2t), \sin(2t))$

$\varphi : [0, 1/2] \longrightarrow [0, 1]$ $\qquad$ $\varphi(t) = 2t$

Arc length

$c : [a,b] \longrightarrow \mathbb{R}^n$ regular C^1-curve

$$L(c) = \int_a^b \left\| \frac{dc}{dt} \right\| dt$$

<u>Def</u> $c : I \to \mathbb{R}^n$ is a parametrization by arc-length if $\left\| \frac{dc}{dt} \right\| = 1$

$$\left(\Leftrightarrow \int_a^s \left\| \frac{dc}{dt} \right\| dt = s - a \quad \text{for all } s \in I \right)$$

<u>Lemma</u>
Any regular curve can be parametrized by arc-length.

<u>Proof</u> $c : [a,b] \longrightarrow \mathbb{R}^n$, $L = L(c)$, $I = [0, L]$

define $$s(t) := \int_a^t \left\| \frac{dc}{dt}(\tau) \right\| d\tau \in I$$

$\psi : [a,b] \longrightarrow I \qquad \psi(t) = s(t)$

$\frac{d\psi}{dt} = \left\| \frac{dc}{dt} \right\| \neq 0$ by regularity so

ψ is invertible w/ inverse φ, $\frac{d\varphi}{dt} = 1 / \left\| \frac{dc}{dt} \right\| > 0$

define $c_1 = c \circ \varphi : [0, L] \longrightarrow \mathbb{R}^n$ $c_1 \sim c$

$$\frac{dc_1}{dt} = \frac{1}{\left\|\frac{dc}{dt}\right\|} \frac{dc}{dt}$$

so c_1 is an arc-length parametrization.

Rmk c_1 is unique up to translation

$s \mapsto s + s_0$.

Notation

$c(t)$ regular par. curve

$c(s)$ par. by arc length

$$\dot{c}(t) = \frac{ds}{dt} c' = \|\dot{c}\| c'$$

$\dot{c}(t) = \frac{dc}{dt}$ tangent vector

$c'(s) = \frac{dc}{ds}$ unit tangent vector

Ex 1) $ax + by = 0$

$c(t) = (at, bt)$ reg

$c_1(t) = (at^2, bt^2)$ not reg

$\dot{c}(t) = (a, b)$

$\|\dot{c}(t)\| = \sqrt{a^2 + b^2}$ constant so we define

$$c_2(s) = \left(\frac{as}{\sqrt{a^2+b^2}}, \frac{bs}{\sqrt{a^2+b^2}} \right)$$

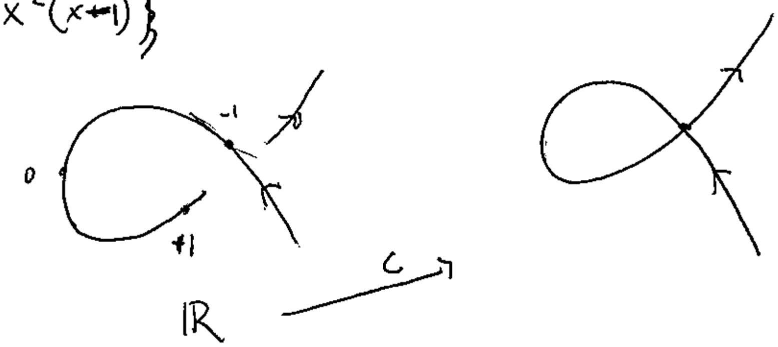
$s = \left(\sqrt{a^2+b^2}\right) t$

then $\left\| \frac{dc_2}{ds} \right\| = 1$ so c_2 is an par. by arc-length

2) $c(t) = (t^2 - 1, t^3 - t)$ regular par. curve

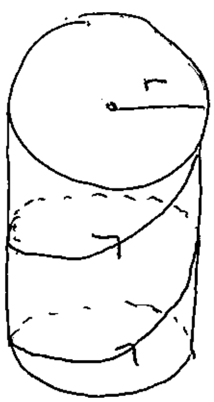
but $c(1) = c(-1) = (0,0)$ so c is <u>not</u> injective

$\mathrm{Im}(c) = \{ y^2 = x^2(x+1) \}$



c) the <u>helix</u>

$$C(t) = \left(r\cos(\alpha t),\ r\sin(\alpha t),\ bt \right)$$



$$\dot{c}(t) = \left(-r\alpha \sin(\alpha t),\ r\alpha\cos(\alpha t),\ b \right)$$

$$\|\dot{c}(t)\| = \sqrt{r^2\alpha^2 + b^2} \qquad \text{constant}$$

$$s = t\sqrt{r^2\alpha^2 + b^2}$$

$$c\left(\frac{s}{\sqrt{r^2\alpha^2+b^2}} \right) =: c_1(s)$$ is an arc-length parameterization

d) $f(t) := \begin{cases} 0 & \text{if } t \le 0 \\ t e^{-1/t} & \text{if } t > 0 \end{cases}$ <u>fact</u> f is C^∞ (∞ - differentiable) & $f^{(n)}(0) = 0$ for all n

$c(t) = \left(f(t), f(-t) \right)$ is a C^∞ - par curve

$\mathbb{R} \xrightarrow{c} \mathbb{R}^2$ but <u>not</u> regular

$\mathrm{Im}(c) =$