The Gauss Map & Curvature

Def 1) let $f : U \to \mathbb{R}^3$ be an ~~[illegible]~~ injective regular surface element. The Gauss map

$$\nu_f : U \longrightarrow S^2 \subseteq \mathbb{R}^3 \quad \text{is given by}$$

$$\nu_f(u_1,u_2) = \frac{\frac{\partial f}{\partial u_1} \times \frac{\partial f}{\partial u_2}}{\left\| \frac{\partial f}{\partial u_1} \times \frac{\partial f}{\partial u_2} \right\|} \in S^2$$

(i.e. (u,v) goes to the unit normal at $f(u,v)$)

2) if $S \subseteq \mathbb{R}^3$ is an orientable regular surface, with orien N the Gauss map is the map

$$N : S \longrightarrow S^2, \quad N(p) \in S^2 \text{ a unit normal}$$

3) If $\{ f_\alpha : U_\alpha \to S \subseteq \mathbb{R}^3 \}$ are a family of charts w/ positive Jacobian determinants inducing the orientation N, then

$$N(p) = \nu_{f_\alpha}(u_1,u_2) \quad \text{where } f_\alpha(u_1,u_2) = p$$

for any (f_α, U_α) with $p \in f_\alpha(U_\alpha)$

$$\begin{tikzcd}
U_\alpha \arrow[r, "f_\alpha"] \arrow[drr, "\nu_{f_\alpha}"'] & V_\alpha \ni p \arrow[dr, "N|_{V_\alpha}"] & \subseteq S \arrow[d, "N"] \\
 & & S^2
\end{tikzcd}$$

4) The <u>Weingarten map</u> or <u>shape operator</u> is minus the derivative $-dN_p$ of the Gauss Map

Rem: Here we assume f_α is at least C^2, e.g. & S is C^2 or C^∞

$$-dN_p : T_pS \longrightarrow T_{N(p)}S^2 \subseteq \mathbb{R}^3$$
$$T_{N(p)}S^2 = T_pS \quad \leftarrow \text{Claim}$$

<u>Claim</u> the image of dN_p is equal to T_pS.

: Choose a chart $f : U \to S^2$ around p, then

$$N(p) = \nu_f(u_1,u_2) \quad \& \quad \boxed{dN_p = d\nu_{f,p} \circ df_p^{-1}} \leftarrow \text{useful formula}$$

$$dN_p\left(\frac{\partial f}{\partial u_i}\right) = d\nu_{f,p}\, e_i = \frac{\partial \nu_{f,p}}{\partial u_i}$$

$$0 = \frac{\partial}{\partial u_i}\langle \nu, \nu\rangle = 2\left\langle \frac{\partial \nu}{\partial u_i}, \nu\right\rangle \quad \text{so} \quad dN_p\left(\frac{\partial f}{\partial u_i}\right) \in \nu(p)^\perp = T_pS$$

Kühnel denotes dN_p by

$$L : T_uf \longrightarrow T_uf$$
$$T_{f(u)}S \xrightarrow{-dN_{f(u)}} T_{f(u)}S$$

<u>Lemma</u> $L : T_uf \to T_uf$ is independent of parametrization f (up to sign!) & L is self adjoint with respect to the first fundamental form.

<u>Pf</u> Independence is clear from the description of dN_p, but we can also check it directly: if $\tilde f = f\circ\alpha$ then $\tilde\nu = \pm\nu\circ\alpha$

(sign depends on sign det $J\alpha$)

$$\tilde L = -d\tilde\nu_p \circ d\tilde f_p^{-1} = \mp\, d\tilde\nu_p \circ d\alpha_p \circ d\alpha_p^{-1} \circ df^{-1} = \pm L$$

Recall, an operator is self-adjoint for an inner product if $\langle LX, Y\rangle = \langle X, LY\rangle$, note that

$$L\frac{\partial f}{\partial u_i} = -\frac{\partial \nu}{\partial u_i}$$

so

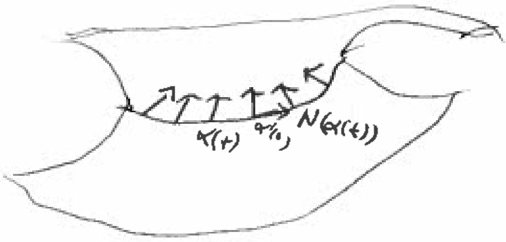
$$I\left(L\frac{\partial f}{\partial u_i}, \frac{\partial f}{\partial u_j}\right) = \left\langle L\frac{\partial f}{\partial u_i}, \frac{\partial f}{\partial u_j}\right\rangle = \left\langle -\frac{\partial \nu}{\partial u_i}, \frac{\partial f}{\partial u_j}\right\rangle$$

$$= \cancel{\frac{\partial}{\partial u_i}\underbrace{\left\langle -\nu, \frac{\partial f}{\partial u_j}\right\rangle}_{=0}}^{\,0} + \left\langle \nu, \frac{\partial^2 f}{\partial u_i \partial u_j}\right\rangle$$

$$= \left\langle \nu, \frac{\partial^2 f}{\partial u_j \partial u_i}\right\rangle = \left\langle L\frac{\partial f}{u_j}, \frac{\partial f}{u_i}\right\rangle$$

Recall vectors $X(p)$ in T_pS can be represented as derivatives of curves $\alpha : (-\varepsilon, \varepsilon) \to S$, $\alpha(0) = p$, $\alpha'(0) = X$

Then $L(X) = -dN_p(X) = -\frac{d}{dt} N \circ \alpha \Big|_{t=0}$ measures rate of change of N along α:



Ex 1) if $S \subseteq \mathbb{R}^2$ is a plane, then $S = n^\perp$ for a constant normal vector n

So $N : S \to S^2$ is constant & $dN \equiv 0$

$\boxed{II = III \equiv 0}$

2) Consider $S^2 = \{x^2 + y^2 + z^2 = 1\}$, $N(x,y,z) = \pm(x,y,z)$ are Gauss maps since (x,y,z) is normal to S^2 at (x,y,z) : $xx' + yy' + z'z$ $\quad \alpha(t) = (x(t), y(t), \ldots)$ in S^2

$\langle (x,y,z), (x', y', z')\rangle = 0$

Choose $-$ sign. Then $L = -dN = \mathrm{Id}$

$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$\boxed{I = II = III}$

Ex Compute L using charts.

The Cylinder $\{x^2+y^2=1\} \subseteq \mathbb{R}^3$

any unit normal is given

by $N(x,y,z) = \pm(x,y,0)$

choose - sign

$$dN_p(X) = -N'(t)\Big|_{t=0} = \left(x'(t)\big|_{t=0},\ y'(t)\big|_{t=0},\ 0\right)$$

$\alpha(0)=p$

$\alpha'(0)=X$, now for any $p \in S$, $T_pS = \text{Span}\langle u, v\rangle$

$v \parallel$ z-axis & $u \perp$ z-axis

then $L(u)=0$ $L(v)=1$ $L = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$

In coordinates: $f(h,\theta) = (\cos\theta, \sin\theta, h)$

$$N = \frac{-f_\theta \times f_h}{\|f_\theta \times f_h\|}$$

$$= \frac{-1}{\cos^2+\sin^2}(\cos\theta, \sin\theta, 0)$$

$$= (-\cos\theta, -\sin\theta, 0)$$

$f_\theta = (-\sin\theta, \cos\theta, 0)$ $f_h = (0,0,1)$

$$L\left(\frac{\partial F}{\partial \theta}\right) = -dN_{f(h,\theta)}\left(\frac{\partial F}{\partial \theta}\right) = \frac{\partial N}{\partial \theta} = (-\sin\theta, \cos\theta, 0) = \frac{\partial F}{\partial \theta}$$

$$L\left(\frac{\partial F}{\partial h}\right) = -dN_{f(h,\theta)}\frac{\partial F}{\partial h} = \frac{\partial N}{\partial h} = 0$$

so $L = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ in the basis $\left\langle \frac{\partial F}{\partial h}, \frac{\partial F}{\partial \theta} \right\rangle$ as claimed.

$$\boxed{L^2 = L \text{ so } \mathrm{III} = \mathrm{II} \quad \mathrm{II}(f_\theta, f_\theta) = 1 \quad \mathrm{II}(f_\theta, f_h) = 0 \quad \mathrm{II}(f_h, f_h) = 0}$$

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Def 2^{nd} Fundamental form

$$\mathrm{II}(X,Y) = \mathrm{I}(LX, Y) \overset{\text{by self adjoint}}{=} \mathrm{I}(X, LY)$$

3^{rd} Fundamental form

$$\mathrm{III}(X,Y) = \mathrm{I}(LX, LY) = \mathrm{I}(L^2X, Y)$$

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1) Note that II & III are symmetric bilinear forms by self adjointness of L

2) Fact: $\mathrm{III} - tr(L)\mathrm{II} + det(L)\mathrm{I} \equiv 0$

Indeed, by Cayley-Hamilton, $L^2 - tr(L)L + det(L) \equiv 0$

& by bilinearity of I, we conclude

In coordinates: fix a chart $f : U \to S$ with gauss map $\nu : U \to S^2$

$$\mathrm{I} : \quad (g_{ij}) = \left\langle \frac{\partial f}{\partial u_i}, \frac{\partial f}{\partial u_j} \right\rangle$$

$$\mathrm{II} : \quad (h_{ij}) = \left\langle \nu, \frac{\partial^2 f}{\partial u_i \partial u_j} \right\rangle = -\left\langle \frac{\partial \nu}{\partial u_i}, \frac{\partial f}{\partial u_j} \right\rangle$$

$$\mathrm{III} : \quad (e_{ij}) = \left\langle \frac{\partial \nu}{\partial u_i}, \frac{\partial \nu}{\partial u_j} \right\rangle \qquad \text{b/c} \quad L\left(\frac{\partial f}{\partial u_j}\right) = \frac{\partial \nu}{\partial u_j}$$

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matrix of L :

$$L\left(\frac{\partial f}{\partial u_i}\right) = \frac{\partial \nu}{\partial u_i} = \sum_j h_i^j \frac{\partial f}{\partial u_j} \implies h_{ik} = \sum_j h_i^j g_{jk}$$

$$\left\langle L\left(\frac{\partial f}{\partial u_i}\right), \frac{\partial f}{\partial u_k} \right\rangle = h_{ik}$$

equivalently, if (g^{ij}) denotes the inverse matrix, then

$$h_i^{\ j} = \sum_k h_{ik} g^{kj} \qquad (g^{ij}) = \frac{1}{\det(g_{ij})}\begin{bmatrix} g_{22} & -g_{12} \\ -g_{21} & g_{11} \end{bmatrix}$$

$$(h_{ij}) = \begin{bmatrix} L & M \\ M & N \end{bmatrix} = \begin{bmatrix} e & f \\ f & g \end{bmatrix} \qquad = \frac{1}{EG-F^2}\begin{bmatrix} G & -F \\ -F & E \end{bmatrix}$$

Rmk

$$\begin{tikzcd} U \arrow[r, "f"] \arrow[rr, bend right, "\nu"'] & S \arrow[r, "N"] & S^2 \end{tikzcd}$$

we can think of ν as a parametrization of S^2 & $\mathrm{III} = \mathrm{I}_\nu$, the 1^{st} fundamental form of ν.

curves on a surface

Fix orientation N on S

let $c : (-\varepsilon, \varepsilon) \to S \subseteq \mathbb{R}^3$ $\quad c(0) = p, \quad c'(0) = X \in T_pS$

$K = \|c''\|$ write $c'' = c''_{\parallel} + c''_{\perp}$

tangent to S $\quad$ normal to S

That is, $c''_{\perp} = \langle c'', N\rangle N$

$\langle c'', N\rangle = K\langle e_2, N\rangle$ is the normal curvature K_n

In coordinates,

$f : U \to S$

$\nu : U \to S^2$

$$c''_{\perp} = \left\langle \frac{d^2c}{ds^2}, \nu\right\rangle \nu = -\left\langle c', \frac{\partial \nu}{\partial s}\right\rangle \nu = \langle X, LX\rangle \nu$$

$$\frac{d}{ds}\langle c', \nu\rangle = 0 \qquad \boxed{\frac{\partial \nu}{\partial s} = -LX} \qquad \mathrm{II}(X,X)\nu$$