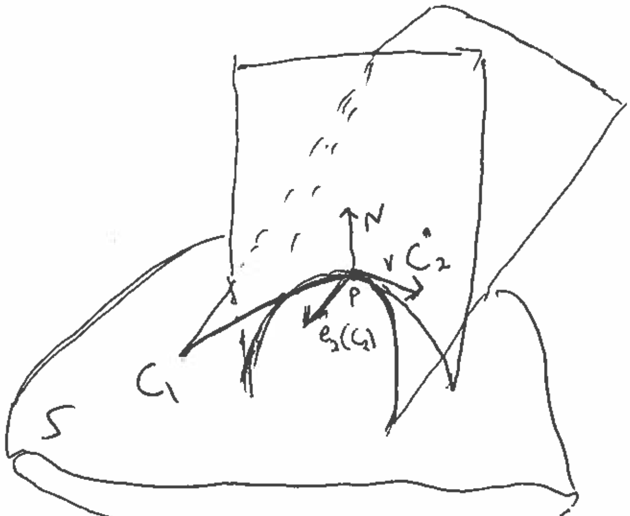
the tangent component $K_t = \| c''_{/\!/} \|$ is the <u>geodesic curvature</u>

Note $K^2 \geq K_n^2$ with equality at $p \iff e_2 \parallel N$ at p

$\iff K_t = 0$ at p $\iff$ the osculating plane of c at p contains $N(p)$

<u>Rmk</u> Since $K_n = \mathrm{II}(X, X)$ where $X = c'$, K_n depends <u>only</u> on the tangent vector of a curve, not on the curve itself!

(Known as Meusnier's theorem)



C_1 & C_2 have the same tangent vector so they have the same normal curvature

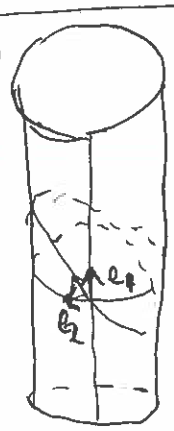
C_1 has nonzero geodesic curvature but

$C_2 = S \cap \mathrm{Span}(N, v) \leftarrow$ "normal section"

has zero geodesic curvature & $K = K_n$

In particular C_2 is the unique curve through p with tangent v & geodesic curvature 0

<u>Ex</u> 1)



$$\mathrm{II}(X,X) = \langle x_1 e_1 + x_2 e_2, L(x_1 e_1 + x_2 e_2) \rangle = x_2^2$$

$L = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ if X is a unit vector,

then K_n varies between 1 if $X = \pm e_2$ & 0 if $X = \pm e_1$

Linear algebra : 1) The eigenvalues of L are real: (8)

Proof: let v be an eigenvector with eigenvalue λ, then $\langle Av, Av\rangle = \langle v, A^2 v\rangle = \langle v, \lambda^2 v\rangle = \lambda^2\langle v, v\rangle = \lambda^2 \|v\|^2$

$$\Rightarrow \lambda^2 = \frac{\langle Av, Av\rangle}{\|v\|^2} \in \mathbb{R}_{\geq 0}$$

$$\Rightarrow \lambda \in \mathbb{R}$$

2) if $\lambda_1 \neq \lambda_2$, are the eigenvalues, then eigen vectors e_1 & e_2 are orthogonal

$$\lambda_1 \langle e_1, e_2\rangle = \langle Le_1, e_2\rangle = \langle e_1, Le_2\rangle = \langle e_1, e_2\rangle \lambda_2$$

$\Rightarrow \langle e_1, e_2\rangle = 0$ since $\lambda_1 \neq \lambda_2$

Def-Prop (Olinde-Rodriguez) let $X \in T_pS$ be a unit tangent vector ($I(X,X) = 1$), then the following are equivalent

1) X is a stationary point for $II(X,X)$ under the constraint $I(X,X) = 1$

2) X is an eigenvector of L

In this case, X is a <u>principal curvature direction</u> & λ (the eigenvalue) is a principal curvature.

Note in this case $II(X,X) = I(X, LX) = \lambda I(X,X) = \lambda$ so λ = normal curvature in X direction

---

recall stationary = derivative is zero. This is a <u>Lagrange multiplier</u> problem. Note that the Gradient of $II(X,X)$ is LX & gradient of $I(X,X) = X$

So X is stationary $\iff \exists$ Lagrange multiplier
with constraint $I(X,X)=1$ λ s.t.

$\iff X$ eigenvector with eigenvalue λ.

$$\operatorname{grad}_{II} X = \lambda \operatorname{grad}_{I} X$$

$$LX = \lambda X$$

<u>Rmk</u> We may pick orthonormal eigenbasis e_1, e_2 for T_pS with eigenvalues λ_1, λ_2
then any unit vector $X = \cos\theta\, e_1 + \sin\theta\, e_2$ so

$$K_n = II(X,X) = I(LX,X) = I(\lambda_1\cos\theta\, e_1 + \lambda_2 \sin\theta\, e_2, \cos\theta\, e_1 + \sin\theta\, e_2)$$

$$= \boxed{\lambda_1 \cos^2\theta + \lambda_2 \sin^2\theta = K_n}$$

<u>Rmk</u> (Signs) everything we've done here is independent of choices $\boxed{\text{except!}}$ in the choice of orientation. There are two choices of N which differ by sign $\Rightarrow$
L differs by a sign but II does not,
λ_1 & λ_2 differ by $\lambda_1\lambda_2 = \det(L)$ doesn't, etc ...

(Sometimes we'll call λ_i, K_i for curvature)

<u>Def</u> 1) The <u>Gaussian Curvature</u> $K = K_1 K_2 = \det(L)$

2) The <u>mean curvature</u> $H = \frac{1}{2}\operatorname{Tr}(L) = \frac{1}{2}(K_1+K_2)$

<u>Rmk</u> 1) $H^2 - K = \frac{1}{4}(K_1-K_2)^2 \geq 0$ with equality $\iff K_1 = K_2$

recall then since K_i are the max & min of normal curvature then this happens $\iff$ the normal curvature is constant fix a chart $f: U \to S$

$$L = (h_i^j) = (h_{ab})(g^{cd})$$

h_{ab} = coefficients of II

$$h_i^j = \sum_k h_{ik} g^{kj} \implies K = \det(L) = \frac{h_{11}h_{22} - h_{12}^2}{g_{11}g_{22} - g_{12}^2} = \frac{eg - f^2}{EG - F^2} = \frac{LN - M^2}{EG - F^2}$$

Recall: $dA = \sqrt{EG - F^2}\, du_1 du_2$

& ν = parameterization of S^2 via $N \circ f : U \to S^2$

with 1st fundamental form (h_{ij})

Thus, K is measuring the difference or "distortion" between the area form on S & the area form on S^2 via Gauss map.

$$H = \frac{1}{2} Tr(L) = \frac{1}{2}(h_1^1 + h_2^2) = \frac{1}{2} \sum_{i,j} h_{ij} g^{ji}$$

$$= \frac{1}{2\det(g_{ij})} \left(h_{11} g_{22} - 2h_{12} g_{12} + h_{22} g_{11} \right)$$

Def: a point $p \in S$ is called

- elliptic if $K(p) > 0$
- hyperbolic if $K(p) < 0$
- parabolic if $K(p) = 0$ but $H(p) \neq 0$
- umbilic if $\kappa_1(p) = \kappa_2(p)$

strictly umbilic if umbilic & $K \neq 0$

level point if $\kappa_1 = \kappa_2 = 0$

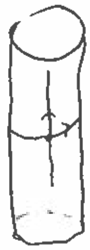
Ex: 1) Sphere of radius 1 $L = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ so $k_1 = k_2 = 1$

& every point is strictly umbilic

2) Cylinder $L = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$

$k_1 = 0 < k_2 = 1$ $K = 0$

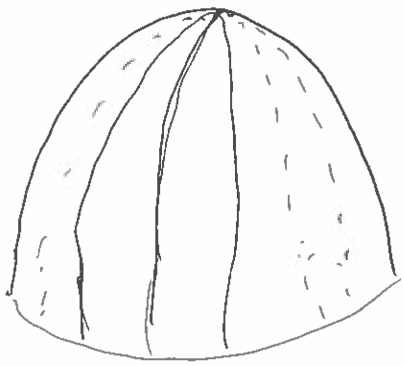
every point is parabolic



3) elliptic point

$a^2x^2 + b^2y^2 + c^2z^2 = 1$

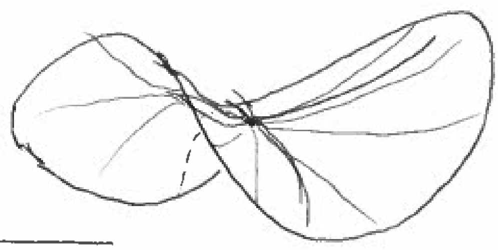
k_1 & k_2 have the same sign



4) hyperbolic point

$x^2 + y^2 - z^2 = 1$

"saddle point"



Def 1) a line of curvature is a regular curve $c: I \to S$ s.t. c' is a principal curvature direction at $c(t)$ for all $t \in I$

2) an asymptotic direction is a direction in which the normal curvature is zero

3) an asymptotic curve is a curve for which $c'(t)$ is an asymptotic direction for all $t \in I$.