Dupin indicatrix tangents of constant normal curvature in T_pS

e.g. $\{ \mathrm{II}(x,x) = \pm 1 \} \subseteq T_p S$

$X = e_1 r\cos\theta + e_2 r\sin\theta$ e_1, e_2 principal curvature directions

$\mathrm{II}(x,x) = r^2\cos^2\theta\, K_1 + r^2\sin^2\theta\, K_2$ $X = (x_1, x_2)$

$x_1^2 K_1 + x_2^2 K_2$ in the e_1, e_2 basis

$\boxed{x_1^2 K_1 + x_2^2 K_2 = \pm 1}$

1) elliptic point: gives an ellipse which degenerates to a circle when $K_1 = K_2$. No asymptotic directions

2) hyperbolic point: $x_1^2 K_1 + x_2^2 K_2 = \pm 1$ ~~one~~ two hyperbolas with common asymptotes. the asymptotes are the asymptotic directions!

3) parabolic points: we get parallel lines & the asymptotic direction is the parallel one.

Graph of a function

Recall every regular surface is locally the graph of a function $h: \mathbb{R}^2 \to \mathbb{R}$ via a parametrization $\mathfrak{f}(u_1, u_2) = (u_1, u_2, h(u_1, u_2))$. Suppose wlog that

$h(0,0) = 0$ & $\nabla h(0,0) = 0$

on HW Compute that $\mathrm{II}|_{(0,0)} = \nabla^2 h = \left(\frac{\partial^2 h}{\partial u_i \partial u_j}\right)$

Ex (Monkey Saddle)

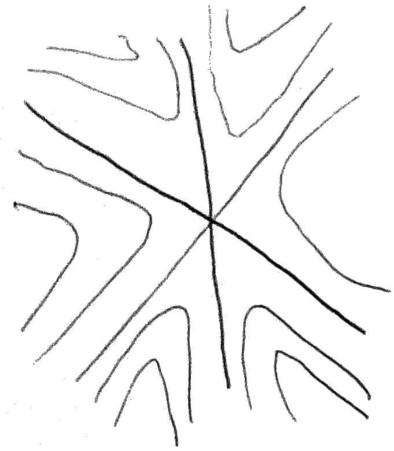
ordinary saddle point $h(x,y) = x^2 - y^2$ so $(0,0)$ hyperbolic point

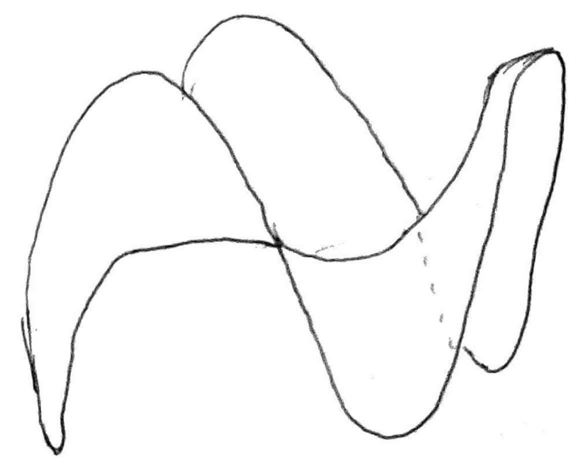
$$\nabla^2 h = \begin{bmatrix} 2 & 0 \\ 0 & -2 \end{bmatrix}$$

monkey saddle $h(x,y) = x^3 - 3xy^2$

$$\nabla^2 h = \begin{bmatrix} 6x & -6y \\ -6y & -6x \end{bmatrix} \qquad \nabla^2 h\big|_{(0,0)} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

so $(0,0)$ is a level point even though it "looks" hyperbolic





$p \in S$

Rmk At an umbilic point with principal curvature $\kappa = \kappa_1 = \kappa_2 \neq 0$, we have an osculating sphere with radius $1/|\kappa|$ & center $p + \frac{1}{\kappa} N$.

This is the osculating sphere of all the normal sections of p. Note these are regular space curves with the same curvature.

If $\kappa = 0$, the osculating sphere degenerates to the tangent plane

Theorem Suppose S is a connected C^∞ surface with every point being an umbilic point. Then S is contained in a sphere $(K \neq 0)$ or a plane $(K = 0)$.

Pf $p \in S$ is umbilic means $L : T_pS \to T_pS$ has a 2-d eigenspace. That is, $LX = KX$ for all $X \in T_pS$. Let $f : U \to S$ be a connected chart at p with Gauss map $\nu = \frac{f_{u_1} \times f_{u_2}}{\|f_{u_1} \times f_{u_2}\|}$. Then

$$L\left(\frac{\partial f}{\partial u_i}\right) \underset{\text{by def}}{=} -\frac{\partial \nu}{\partial u_i} \underset{\text{by assumption}}{=} K \frac{\partial f}{\partial u_i}$$

$$-\frac{\partial^2 \nu}{\partial u_1 \partial u_2} = \frac{\partial K}{\partial u_1}\frac{\partial f}{\partial u_2} + K\frac{\partial^2 f}{\partial u_1 \partial u_2}$$

$$= \frac{\partial K}{\partial u_2}\frac{\partial f}{\partial u_1} + K\frac{\partial^2 f}{\partial u_1 \partial u_2}$$

$$K_{u_1} f_{u_2} = K_{u_2} f_{u_1}$$ f_{u_1} & f_{u_2} form a basis for T_pS

$\Longrightarrow K_{u_1} = K_{u_2} = 0 \Longrightarrow K$ constant in U.

i) $K \equiv 0$, then $LX = 0$ for all X so the normal vector N is <u>constant</u> thus

$$\frac{\partial}{\partial u_i}\langle f(u,v), N\rangle = \left\langle \frac{\partial f}{\partial u_i}, N\right\rangle = 0$$ so $\boxed{\langle f(u,v), N\rangle = \text{constant}}$

ii) $\underline{K \neq 0}$ Consider the center

$m(u,v) := F(u,v) + \frac{1}{K}\nu(u,v)$ of the osculating sphere

$$\frac{\partial m}{\partial u_i} = \frac{\partial F}{\partial u_i} + \frac{1}{K}\frac{\partial \nu}{\partial u_i} = \frac{\partial F}{\partial u_i} + \frac{1}{K}\left(-K\frac{\partial F}{\partial u_i}\right) = 0$$

thus, m is <u>constant</u>! Then $\|F(u,v) - m(u,v)\|^2 = \frac{1}{K^2}$

So $F(u,v) \in$ osculating sphere for all $(u,v) \in \mathcal{U}$.

Now use connectedness of S:

$$f_i : U_i \to V_i \subseteq S$$

f_i are a <u>finite</u> sequence of charts connecting P_1 & P_2

K is constant on each V_i & independent of choice of chart on $V_i \cap V_j$

Thus, K constant all along the path P_1 to P_2 & each nbhd is contained in the <u>same</u> sphere or plane

∎

Prop if $p \in S$ is elliptic, then there exist a neighborhood $p \in V \subseteq S$ s.t. each point of V is on the same side of T_pS

" " hyperbolic, then for each nbhd $p \in V \subseteq S$, there exist points of V on each side of T_pS.

Pf assume $f: U \to S$ a chart

$f(0,0) = p$

$$f(u,v) = f(0,0) + f_u(0,0)u + f_v(0,0)v + \frac{1}{2}\left(f_{uu}(0,0)u^2 + f_{uv}(0,0)uv + f_{vv}(0,0)v^2\right) + o(u^2+v^2)$$

taylor expansion

$$d = \langle f - p, N_p \rangle = \langle f(u,v) - f(0,0), \nu(0,0) \rangle$$

signed distance to T_pS

$$= \frac{1}{2} \mathrm{II}_p(\underbrace{f_u(0,0)u + f_v(0,0)v}_{w}) + o(u^2+v^2)$$

~~[illegible]~~ if p elliptic then II_p has the same sign for all (u,v) so d has the same sign for u,v small

if p hyperbolic, II_p takes on different signs so d does as well for any small u,v $\blacksquare$

$f: U \to S$ for a chart with $f(0,0) = p$

$\alpha : (-\varepsilon, \varepsilon) \to S \qquad \alpha(t) = f(u_1(t), u_2(t))$

line of curvature $\iff$ $L\alpha' = \lambda \alpha'$ for some $\lambda(t)$

$$L(f_{u_i}) = \sum h_i^j f_{u_j} \qquad (h_i^j) = \begin{bmatrix} L & M \\ M & N \end{bmatrix}\begin{bmatrix} E & F \\ F & G \end{bmatrix}^{-1}$$

$$= \frac{1}{EG - F^2}\begin{bmatrix} L & M \\ M & N \end{bmatrix}\begin{bmatrix} G & -F \\ -F & E \end{bmatrix}$$

$$\Rightarrow L = \begin{bmatrix} \dfrac{LG - MF}{EG - F^2} & \dfrac{ME - LF}{EG - F^2} \\[2ex] \dfrac{MG - NF}{EG - F^2} & \dfrac{NE - FM}{EG - F^2} \end{bmatrix}$$

in this basis. $\alpha' = u_1' f_{u_1} + u_2' f_{u_2}$

$L\alpha' = \lambda \alpha' \rightsquigarrow$

$$(ME - LF)(u_1')^2 + (NE - LG) u_1' u_2' + (NF - MG)(u_2')^2 = 0$$

$$\det \begin{bmatrix} u_2' & -u_1' u_2' & u_1' \\ E & F & G \\ L & M & N \end{bmatrix} = 0$$

DE for Asymptotic curves

$$\mathrm{II}(\alpha'(t)) = 0 \quad \rightsquigarrow \quad L(u_1')^2 + 2M u_1' u_2' + N(u_2')^2 = 0$$

Upshot For this chart the ~~lines~~ coordinate lines are lines of curvature

$\iff F = M = 0$ (non-umbilic point)

" " " " are asymptotic

$\iff E = N = 0$ (hyperbolic point)