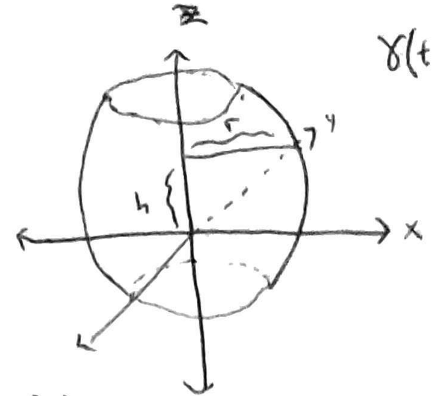
Surfaces of revolution / rotation

$f(t, \varphi) =$

$(r\cos\varphi,\ r\sin\varphi,\ h)$



$\gamma(t) = (r(t), h(t))$

$\cap$

(x, z) plane

regular curve

HW 3

$$f_t = (\dot r\cos\varphi,\ \dot r\sin\varphi,\ \dot h)$$

$$f_\varphi = (-r\sin\varphi,\ r\cos\varphi,\ 0)$$

$$\mathrm{I}: \begin{bmatrix} \dot r^2 + \dot h^2 & 0 \\ 0 & r^2 \end{bmatrix}$$

$$\nu = \frac{f_t \times f_\varphi}{\| f_t \times f_\varphi \|} = \frac{1}{\sqrt{\dot r^2 + \dot h^2}} (-\dot h\cos\varphi,\ -\dot h\sin\varphi,\ \dot r)$$

$$\frac{1}{\left(r^2(\dot r^2 + \dot h^2)\right)^{1/2}} (-r\dot h\cos\varphi,\ -r\dot h\sin\varphi,\ r\dot r)$$

$$f_{tt} = (\ddot r\cos\varphi,\ \ddot r\sin\varphi,\ \ddot h) \qquad f_{t\varphi} = (-\dot r\sin\varphi,\ \dot r\cos\varphi,\ 0)$$

$$f_{\varphi\varphi} = (-r\cos\varphi,\ -r\sin\varphi,\ 0)$$

$$\mathrm{II}: \langle f_{\alpha\beta}, \nu \rangle = \begin{bmatrix} -\ddot r\dot h + \dot r\ddot h & 0 \\ 0 & r\dot h \end{bmatrix} \frac{1}{\sqrt{\dot r^2 + \dot h^2}}$$

$$L = (h_{ij})(g^{ij}) = \operatorname{diag}\left(\frac{1}{(\dot r^2 + \dot h^2)^{3/2}} (-\ddot r\dot h + \dot r\ddot h),\ \frac{1}{(\dot r^2 + \dot h^2)^{1/2}} \frac{\dot h}{r} \right)$$

$$\kappa_1 = \frac{1}{(\dot r^2+\dot h^2)^{3/2}}(-\ddot r\dot h+\dot r\ddot h) \qquad \kappa_2 = \frac{1}{(\dot r^2+\dot h^2)^{1/2}}\frac{\dot h}{r}$$ 26

Rmk note that both I & II are diagonal. This means that

1) the coordinate curves are orthogonal

2) the coordinate curves are lines of curvature

When we take an arclength parametrization $s=t$ for γ, then $\|\gamma'\|^2 = r'^2 + h'^2 = 1$ so

$$\mathrm{I}: \begin{bmatrix} 1 & 0 \\ 0 & r^2 \end{bmatrix} \qquad \mathrm{II}: \begin{bmatrix} -r''h'+r'h'' & 0 \\ 0 & rh' \end{bmatrix}$$

$$\kappa_1 = -r''h'+r'h'' \qquad \kappa_2 = \frac{h'}{r}$$

$e_1 \leftarrow$ principal curvature directions $\rightarrow e_2$

κ_1 = normal curvature in the "vertical" direction, i.e. along the curve γ

Claim κ_1 = curvature of γ as a plane curve

Pf $e_1 = (r', h')$ $\quad e_1' = (r'', h'')$

$e_2 = (-h', r')$

$$\kappa = \langle e_1', e_2\rangle \text{ by Frenet eqn}$$
$$= -r''h'+r'h'' = \kappa_1$$

Upshot the constant φ curves have geodesic curvature $\kappa_g = 0$, so they're geodesics

$r'^2 + h'^2 = 1$

$r'r'' + h'h'' = 0 \qquad r'' = \dfrac{-h'h''}{r'}$

$$K_1 = -r''h' + r'h'' = \frac{h'h''}{r'}h' + r'h'' = \frac{h''}{r'}\left(h'^2 + r'^2\right) = \frac{h''}{r'} = \frac{-r''}{h'}$$

$$\boxed{K = K_1 K_2 = \frac{-r''}{h'}\frac{h'}{r} = \frac{-r''}{r}}$$

Rmk $h' = \pm\sqrt{1-(r')^2}$

so everything is a function of r & derivatives

$$\boxed{H = \frac{1}{2}(K_1+K_2) = \frac{1}{2}\left(\frac{h''}{r'} + \frac{h'}{r}\right) = \frac{1}{2}\left(\frac{rh'' + r'h'}{rr'}\right) = \frac{(rh')'}{(r^2)'}}$$

Ex 1) $K = c$ constant $\iff r'' + cr = 0$

2) $H = 0 \iff rh' = \pm r\sqrt{1-r'^2} = c_1$ constant

$r^2(1-r'^2) = c_2 = c_1^2$ = constant

3) $K_1 = K_2 \iff \dfrac{h''}{r'} = \dfrac{h'}{r} \iff \dfrac{h''}{h'} = \dfrac{r'}{r}$

$\dfrac{h''}{h'} = \dfrac{d}{ds}\log h'$

$\dfrac{r'}{r} = \dfrac{d}{ds}\log r = \dfrac{d}{ds}\log h'$

$\iff \log r + C_1 = \log h' \qquad h' = c_2 r$

$\pm\sqrt{1-r'^2} = c_2 r \qquad 1-(r')^2 = c_2^2 r^2$

$$\boxed{1 = (r')^2 + c_2^2 r^2}$$

Surfaces of revolution with constant curvature K:

$$r'' + Kr = 0 \qquad K \text{ constant}$$

$$r(s) = \begin{cases} a\cos(\sqrt{K}\, s) + b\sin(\sqrt{K}\, s) & K > 0 \quad \text{elliptic} \\ as + b \quad (|a| \leq 1 & K = 0 \quad \text{parabolic} \\ a\cosh(\sqrt{-K}\, s) + b\sinh(\sqrt{-K}\, s) & K < 0 \quad \text{hyperbolic} \end{cases}$$

Note that $(h')^2 + (r')^2 = 1$ by arc length parameterization

so in case $K=0$, need $|a| \leq 1$ to ensure

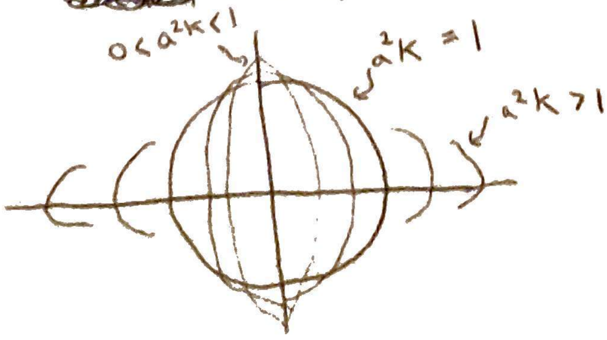
that $h' = \sqrt{1-(r')^2}$ is well defined.

Elliptic

Suppose wlog $b = 0$, $r(s) = a\cos(\sqrt{K}s)$

$$h'(s) = \pm\sqrt{1 - a^2K\sin^2(\sqrt{K}s)} \qquad 0 \leq r'(s)^2 = a^2K\sin^2(\sqrt{K}s) \leq 1$$

$$h(s) = \int_0^t \underbrace{\sqrt{1 - a^2K\sin^2(\sqrt{K}s)}\, ds}$$ so called elliptic integrals

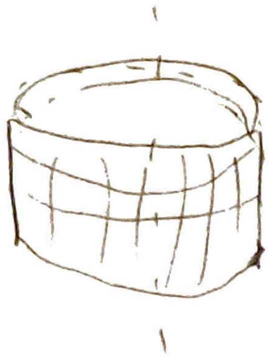


parabolic $K=0$

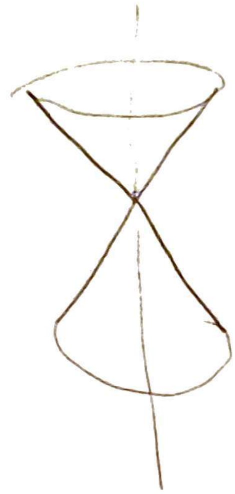
$r(s) = as+b$ $\quad r' = a$ $\quad h' = \sqrt{1-a^2} = c$

$h = cs+d$ $\quad$ line segments

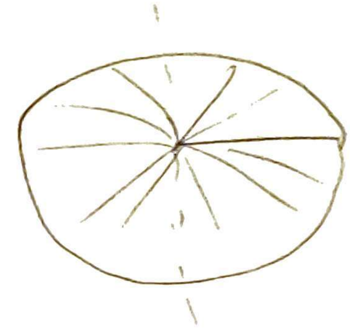
$a=0$



$0 < |a| < 1$



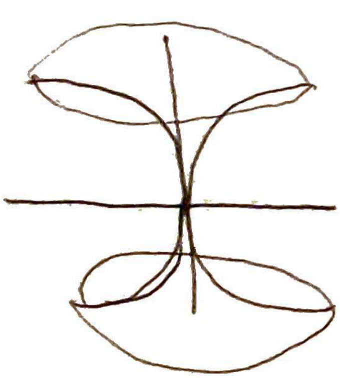
$|a|=1$

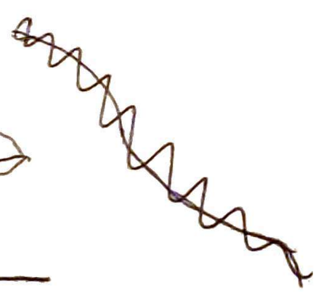


hyperbolic $K<0$

similar story,

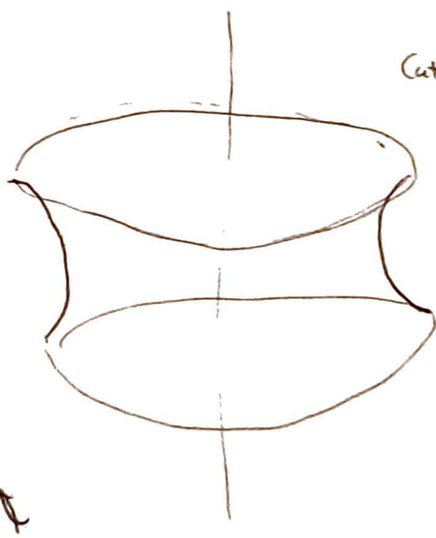
$b^2 > a^2$





catenoid

$b^2 < a^2$



$a=b$

$K=-1$

hyperbolic pseudosphere

Ruled surfaces

has a parametrization of the form

$$F(u,v) = c(u) + v\,X(u)$$

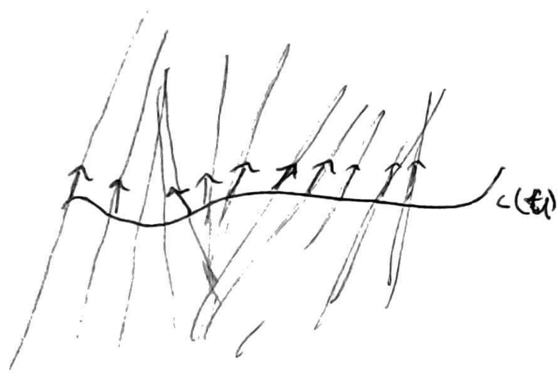
c is a (not-nec regular) curve

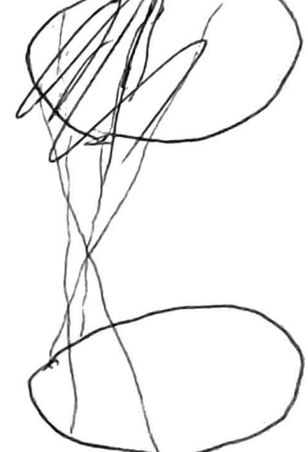
$X(u)$ is a nonvanishing vector field on c

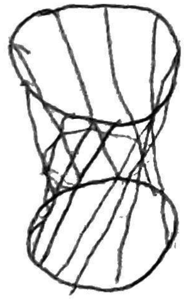
So S is swept out by a family of lines moving along $c(u)$

the directrix

↑ rulings







let $F(t,s) = c(t) + sX(t)$ with $\|X\| = 1$ $X' \neq 0$

Lemma $\exists$ a parametrization

$$F_*(u,v) = c_*(u) + v\,X_*(u)$$

with $\|X_*\| = \|X_*'\| = 1$ $\quad \langle c_*', X_*' \rangle = 0$

Pf X is a regular curve so it has an arc length parametrization $X_*(u) = X(t)$

write $c_*(u) = c(u) + r(u)\,X_*(u)$

Lemma

$$\langle c_*', X_*' \rangle = \langle c' + v(u) X_*' + v' X_*, X_*' \rangle$$

$$= \langle c', X_*' \rangle + v(u) \underbrace{\langle \cancel{X_*'}, X_*' \rangle}_{1} + \underbrace{\langle v' \cancel{X_*}, X_*' \rangle}_{0}$$

$$= \langle c', X_*' \rangle + v(u) = 0 \qquad \text{b/c } X_* \text{ is spherical}$$

$$\Longleftrightarrow \quad v(u) = -\langle c', X_*' \rangle.$$ This determines c_*. ∎

Thm 1) a ruled surface in the above standard parameters is determined by

$$F = \langle c', X \rangle, \quad \lambda = \det(c', X, X')$$

$$J = \det(X, X', X'')$$

up to euclidian motions

2)

$$I = \begin{pmatrix} \langle c', c' \rangle + v^2 & \langle c', X \rangle \\ \langle c', X \rangle & 1 \end{pmatrix} = \begin{pmatrix} F^2 + \lambda^2 + v^2 & F \\ F & 1 \end{pmatrix}$$

Pf 1) J = geodesic curvature of X as a spherical curve. Uniquely determines X up to initial value.

Use $X, X', X \times X'$ orthonormal frame

$$c' = \langle c', X \rangle X + \langle c', X' \rangle X' + \langle c', X \times X' \rangle X \times X'$$

$$= FX + \lambda X \times X'$$

Now solve this ODE to get c