Intrinsic geometry of surfaces

Directional derivative: Y differentiable vector field on $p \in U \subseteq \mathbb{R}^{n+1}$, $X \in T_pU$ a tangent vector at p,

$$D_X Y|_p := DY|_p(X) = \lim_{t\to 0} \frac{Y(p+tX) - Y(p)}{t}$$

Here, we can view $Y : U \longrightarrow \mathbb{R}^{n+1}$

DY = derivative of Y = Jacobian matrix

if $c : (-\varepsilon, \varepsilon) \longrightarrow U$ $\quad c(0) = p \quad c'(0) = X$

then $D_X Y|_p = \frac{d}{dt}(Y \circ c)\Big|_{t=0}$ $\qquad Y \circ c : (-\varepsilon, \varepsilon) \to \mathbb{R}^m$

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when $X = e_i$ the i^{th} coordinate direction, then

~~[illegible]~~ coordinates on $\mathbb{R}^{n+1}$ are $(x^1, \dots, x^{n+1})$

$$D_{e_i} Y|_p = \frac{\partial Y}{\partial x^i}\Big|_p$$

"$e_i = \frac{\partial}{\partial x^i}$"

$X = \sum_i X^i \frac{\partial}{\partial x^i}$ $\qquad X^i$ functions

$$D_X Y|_p = DY|_p\Big(\sum_i X^i \frac{\partial}{\partial x^i}\Big)$$

$$\overset{\text{linearity of } DY|_p}{=} \sum_i X^i \, DY|_p\Big(\frac{\partial}{\partial x^i}\Big) = \sum X^i \frac{\partial Y}{\partial x^i}\Big|_p$$

If $S \subseteq \mathbb{R}^{n+1}$ is a (hyper) surface, Y a vector field on $V \subseteq S$ open.

$X \in T_pS$, $D_X Y|_p$ makes sense

In coordinates: $\mathcal{F} : U \to V \overset{p}{\subseteq} S$ a chart

$\bar{Y} := Y \circ \mathcal{F}$ a vector field on U, $X = \sum x^i \frac{\partial \mathcal{F}}{\partial u_i}$

$$D_X Y\big|_p = \sum_i x^i D_{\frac{\partial \mathcal{F}}{\partial u_i}} \bar{Y}\big|_p \qquad D_{\frac{\partial \mathcal{F}}{\partial u_i}} \bar{Y}\big|_p = \lim_{t\to 0} \frac{1}{t}\left[\bar{Y}(u^1, \dots, u^i + t, \dots, u^n) - \bar{Y}(u^1, \dots, u^n)\right]$$

$\parallel$ derivative in u_i direction

$$\boxed{\text{if } \mathcal{F}(0,\dots,0) = p \quad = \frac{d}{dt}\left(Y \circ \mathcal{F}(0, \dots, t, \dots, 0)\right)\Big|_{t=0}}$$

In particular, $D_{\frac{\partial \mathcal{F}}{\partial u^i}} \frac{\partial \mathcal{F}}{\partial u^j} = \frac{\partial^2 \mathcal{F}}{\partial u^i \partial u^j}$

Problem: $D_X Y$ is not tangent even if X & Y are tangent!

Def: If X, Y are tangent vector fields along $\mathcal{F} : U \to S$, the covariant derivative

$$\nabla_X Y = D_X Y - \langle D_X Y, \nu\rangle \nu \in T_{\mathcal{F}} U$$

$\langle Y, \nu\rangle = 0 \Rightarrow D_X \langle Y, \nu \rangle = 0$

$= \langle D_X Y, \nu\rangle + \boxed{\langle Y, D_X \nu\rangle}$ $\parallel$ $\mathrm{II}(Y, -X)$

So

$$\langle D_X Y, \nu \rangle = \mathrm{II}(X, Y) \qquad \& \qquad \boxed{D_X Y = \nabla_X Y + \mathrm{II}(X, Y)}$$

Rmk D defined for vector fields on $\mathbb{R}^{n+1}$

∇ defined for tangent vector fields to S

(i.e. ∇ is "intrinsic")

Properties of D & ∇, X, X_1, X_2, Y, Y_1, Y_2 (tangent) vector fields

$\varphi, \varphi_1, \varphi_2$ scalar functions

1) $D_{\varphi_1 X_1 + \varphi_2 X_2} Y = \varphi_1 D_{X_1} Y + \varphi_2 D_{X_2} Y$ (linearity)

$\nabla_{\varphi_1 X_1 + \varphi_2 X_2} Y = \varphi_1 \nabla_{X_1} Y + \varphi_2 \nabla_{X_2} Y$

2) $D_X(Y_1 + Y_2) = D_X Y_1 + D_X Y_2$ (additivity)

$\nabla_X(Y_1 + Y_2) = \nabla_X Y_1 + \nabla_X Y_2$

3) $D_X(\varphi Y) = \varphi D_X Y + (D_X \varphi) Y$ (product rule)

$\nabla_X(\varphi Y) = \varphi \nabla_X Y + (\nabla_X \varphi) Y$

Note We define $\nabla_X \varphi := D_X \varphi$ for φ scalar

4) $D_X \langle Y_1, Y_2 \rangle = \langle D_X Y_1, Y_2 \rangle + \langle Y_1, D_X Y_2 \rangle$

$\nabla_X \langle Y_1, Y_2 \rangle = \langle \nabla_X Y_1, Y_2 \rangle + \langle Y_1, \nabla_X Y_2 \rangle$

(compatibility w/ $\langle , \rangle$)

Rmk Commutativity fails: $D_X Y \neq D_Y X$

$\nabla_X Y = \nabla_Y X$

Ex $e_i = \frac{\partial}{\partial x^i}$ on $\mathbb{R}^2$

$D_{e_i} e_j = 0$ so $D_{x^1 e_2} e_1 = x^1 D_{e_2} e_1 = 0$

but $D_{e_1} x^1 e_2 = e_2 + x^1 \cancel{D_{e_1} e_2}$

$= e_2 \neq 0$

Def (Lie bracket)

$[X, Y] = D_X Y - D_Y X$ in general

$= \nabla_X Y - \nabla_Y X$ if X, Y are tangent

Lemma if f is a chart,

1) $\left[\frac{\partial f}{\partial u^i}, \frac{\partial F}{\partial u^j}\right] = 0$ 2) $[X, Y] = \sum_{i,j} \left(\xi^i \frac{\partial \eta^j}{\partial u^i} - \eta^i \frac{\partial \xi^j}{\partial u^i} \right) \frac{\partial F}{\partial u^j}$

$X = \sum_i \xi^i \frac{\partial f}{\partial u^i}$ $Y = \sum_j \xi^j \frac{\partial f}{\partial u^j}$

Rmk in Einstein notation, we drop $\sum_i$ & write

$X = \xi^i \frac{\partial F}{\partial u^i}$ top + bottom index = sum over it

then $[X, Y] = \left(\xi^i \frac{\partial \eta^j}{\partial u^i} - \eta^i \frac{\partial \xi^j}{\partial u^i} \right) \frac{\partial F}{\partial u^j}$

Proof

1) $\nabla_{\frac{\partial F}{\partial u^i}} \frac{\partial F}{\partial u^j} = \frac{\partial^2 F}{\partial u^i u^j} - h_{ij}\nu = \frac{\partial^2 F}{\partial u^j \partial u^i} - h_{ji}\nu = \nabla_{\frac{\partial F}{\partial u^j}} \frac{\partial F}{\partial u^i}$

$\Rightarrow \left[\frac{\partial F}{\partial u^i}, \frac{\partial F}{\partial u^j}\right] = 0$

2) $\nabla_X Y = \nabla_{\xi^i \frac{\partial F}{\partial u^i}} \left(\eta^j \frac{\partial F}{\partial u^j}\right) = \xi^i \nabla_{\frac{\partial F}{\partial u^i}} \left(\eta^j \frac{\partial F}{\partial u^j}\right)$

Einstein notation $\sum_i$ & $\sum_j$

$= \xi^i \left(\left(\nabla_{\frac{\partial F}{\partial u^i}} \eta^j\right) \frac{\partial F}{\partial u^j} + \eta^j \nabla_{\frac{\partial F}{\partial u^i}} \frac{\partial F}{\partial u^j} \right)$

$= \xi^i \frac{\partial \eta^j}{\partial u^i} \frac{\partial F}{\partial u^j} + \xi^i \eta^j \nabla_{\frac{\partial F}{\partial u^i}} \frac{\partial F}{\partial u^j}$

$[X, Y] = \nabla_X Y - \nabla_Y X$

$= \xi^i \frac{\partial \eta^j}{\partial u^i} \frac{\partial F}{\partial u^j} - \eta^i \frac{\partial \xi^j}{\partial u^i} \frac{\partial F}{\partial u^j} + \xi^i \eta^j \cancel{\left[\frac{\partial F}{\partial u^i}, \frac{\partial F}{\partial u^j}\right]}_{0}$

$= \left(\xi^i \frac{\partial \eta^j}{\partial u^i} - \eta^i \frac{\partial \xi^j}{\partial u^i} \right) \frac{\partial F}{\partial u^j}$

Thm ∇ depends only on the first fundamental form (metric) $g = (g_{ij})$.

"∇ is intrinsic to S"