Pf $X = \xi^i \frac{\partial F}{\partial u^i}$ $Y = \eta^j \frac{\partial F}{\partial u^j}$, since $\frac{\partial F}{\partial u^k}$ form a basis for $T_F U$, it suffices to compute $\left\langle \nabla_X Y, \frac{\partial F}{\partial u^k} \right\rangle$ & see they only depend on g

$$\nabla_X Y = \xi^i \nabla_{\frac{\partial F}{\partial u^i}} \left(\eta^j \frac{\partial F}{\partial u^j} \right) = \xi^i \frac{\partial \eta^j}{\partial u^i} \frac{\partial F}{\partial u^j} + \xi^i \eta^j \nabla_{\frac{\partial F}{\partial u^i}} \frac{\partial F}{\partial u^j}$$

$$\left\langle \nabla_X Y, \frac{\partial F}{\partial u^k} \right\rangle = \xi^i \frac{\partial \eta^j}{\partial u^i} \left\langle \frac{\partial F}{\partial u^j}, \frac{\partial F}{\partial u^k} \right\rangle + \xi^i \eta^j \underbrace{\left\langle \nabla_{\frac{\partial F}{\partial u^i}} \frac{\partial F}{\partial u^j}, \frac{\partial F}{\partial u^k} \right\rangle}_{\Gamma_{ij,k}}$$

$$= \xi^i \frac{\partial \eta^j}{\partial u^i} g_{jk} + \xi^i \eta^j \Gamma_{ij,k}$$

$$\boxed{\Gamma_{ij,k} = \Gamma_{ji,k}}$$

$$\frac{\partial}{\partial u^k} g_{ij} = \frac{\partial}{\partial u^k} \left\langle \frac{\partial F}{\partial u^i}, \frac{\partial F}{\partial u^j} \right\rangle = \left\langle \nabla_{\frac{\partial F}{\partial u^k}} \frac{\partial F}{\partial u^i}, \frac{\partial F}{\partial u^j} \right\rangle + \left\langle \frac{\partial F}{\partial u^i}, \nabla_{\frac{\partial F}{\partial u^k}} \frac{\partial F}{\partial u^j} \right\rangle$$

$$\boxed{\frac{\partial}{\partial u^k} g_{ij} = \Gamma_{ki,j} + \Gamma_{jk,i}}$$

$$\frac{\partial}{\partial u^j} g_{ki} = \Gamma_{jk,i} + \Gamma_{ij,k} \quad \Rightarrow \quad 2\Gamma_{ij,k} = \frac{\partial}{\partial u^i} g_{jk} + \frac{\partial}{\partial u^j} g_{ki} - \frac{\partial}{\partial u^k} g_{ij}$$

$$\frac{\partial}{\partial u^i} g_{jk} = \Gamma_{ij,k} + \Gamma_{ki,j}$$

depends only on g

∎

Christoffel Symbols

Def 1) $\Gamma_{ij,k} := I\left(\nabla_{\frac{\partial F}{\partial u^i}} \frac{\partial F}{\partial u^j}, \frac{\partial F}{\partial u^k}\right)$

$\left(\sum_k\right)$ ← einstein notation

2) $$\nabla_{\frac{\partial F}{\partial u^i}} \frac{\partial F}{\partial u^j} = \Gamma^k_{ij} \frac{\partial F}{\partial u^k}$$

Rmk Recall if $X = \sum \xi^i \frac{\partial F}{\partial u^i}$ $\quad Y = \sum \eta^j \frac{\partial F}{\partial u^j}$

$$I(X,Y) = \sum_{i,j} \xi^i \eta^j g_{ij} = \xi^i \eta^j g_{ij}$$ in einstein notation

Thus, $\Gamma_{ij,k} = \overset{\left(\sum_m\right)}{\Gamma^m_{ij}}\, g_{mk}$, equivalently, $\left(\sum_k\right) g^{mk}\, \Gamma_{ij,k} = \Gamma^m_{ij}$

$$(g^{mk}) = (g_{mk})^{-1}$$

Upshot

$$\Gamma_{ij,k} = -g_{ij,k} + g_{jk,i} + g_{ki,j}$$

$$+ \quad \nabla_X Y = \xi^i\left(\frac{\partial \eta^k}{\partial u^i} + \eta^j \Gamma^k_{ij}\right)\frac{\partial F}{\partial u^k}$$

Cor let $f: U \to V \subseteq S$ a chart, then we have

1) $$\frac{\partial^2 F}{\partial u^i \partial u^j} = \Gamma^k_{ij} \frac{\partial F}{\partial u^k} + h_{ij}\,\nu$$ (Gauss formula)

2) $$\frac{\partial \nu}{\partial u^i} = -h_{ij}\, g^{jk} \frac{\partial F}{\partial u^k} = -h_i^k \frac{\partial F}{\partial u^k}$$ (Weingarten formula)

Pf 1) is just $D_X Y = \nabla_X Y + II(X,Y)\, \nu$

in coordinates + the fact that $D_{\frac{\partial f}{\partial u^j}} \frac{\partial f}{\partial u^i} = \frac{\partial^2 f}{\partial u^i \partial u^j}$

2) is just $D\nu = -L \circ Df$ in coordinates

for $n=2$ we can write this in the

Gauss frame $\frac{\partial f}{\partial u^1}, \frac{\partial f}{\partial u^2}, \nu$ as

$$\frac{\partial}{\partial u^i}\begin{pmatrix} \frac{\partial f}{\partial u^1} \\ \frac{\partial f}{\partial u^2} \\ \nu \end{pmatrix} = \begin{bmatrix} \Gamma^1_{i1} & \Gamma^2_{i1} & h_{i1} \\ \Gamma^1_{i2} & \Gamma^2_{i2} & h_{i2} \\ -h^1_i & -h^2_i & 0 \end{bmatrix}\begin{pmatrix} \frac{\partial f}{\partial u^1} \\ \frac{\partial f}{\partial u^2} \\ \nu \end{pmatrix}$$

Analogue of Frenet eqns.

Def 1) A property of S is intrinsic if it depends only on $(g_{ij}) = g$ the metric / first fundamental form

2) a diffeomorphism ~~[illegible]~~ $\varphi: S \to T$ is an isometry if for all $p \in S$ & vectors $X_1, X_2 \in T_p S$,

$$\langle X_1, X_2 \rangle_p = \langle d\varphi_p(X_1), d\varphi_p(X_2) \rangle_{\varphi(p)}$$

3) S & T are said to be <u>locally isometric</u> <u>at $p \in S$ & $q \in T$</u> if there exist open neighborhoods $p \in U \subseteq S$, $q \in V \subseteq T$ & an isometry

$$\varphi : U \longrightarrow V \qquad \varphi(p) = \varphi(q)$$

4) S & T are <u>locally isometric</u> if there exist local isometries around each point of S & T.

<u>idea</u> A diffeo $\varphi : S \longrightarrow T$ induces an identification

$$TS = \coprod_{p \in S} T_p S \quad \text{with} \quad TT = \coprod_{q \in T} T_q T$$

via $d\varphi_p : T_p S \xrightarrow{\sim} T_{f(p)} T$

thus if T has metric g_T then we obtain a metric $\varphi^* g_T$ the <u>pullback</u> <u>metric on S</u> via

$$(\varphi^* g_T)(X, Y) = g_T(d\varphi(X), d\varphi(Y))$$

if S has a metric g_S, then isometry means $\boxed{\varphi^* g_T = g_S}$

<u>Upshot</u> A property is intrinsic iff it is a (local) isometry invariant

In particular, ∇ the covariant deriv is an isometry invariant.