Ex 1) $\varphi : \mathbb{R}^2 \longrightarrow \text{Cylinder} = S \subseteq \mathbb{R}^3$

$\varphi(\theta, h) = (\cos\theta, \sin\theta, h)$

$$g_{\mathbb{R}^2} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad g_S = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

standard basis e_θ, e_h $\qquad$ $\varphi_\theta, \varphi_h$ basis

So φ is a <u>local isometry</u>, $\langle e_i, e_j \rangle = \langle d\varphi(e_i), d\varphi(e_j) \rangle$
for $i,j \in \{\theta, h\}$ $\qquad = \langle \varphi_i, \varphi_j \rangle$

<u>Prop</u> Charts! suppose $\exists$ parametrizations $f : U \longrightarrow S$, $\bar{f} : U \longrightarrow \bar{S}$ such that the first fundamental forms g & $\bar{g}$ are equal on this chart, then S & $\bar{S}$ are locally isometric on $f(U)$ & $\bar{f}(U)$

<u>Pf</u> Consider $\varphi = \bar{f} \circ f^{-1} : f(U) \longrightarrow \bar{f}(U)$

$$\begin{array}{ccc} f(U) & & \bar{f}(U) \\ \| & & \| \\ V & \longrightarrow & \bar{V} \\ \cap\!\!| & & \cap\!\!| \\ S & & \bar{S} \end{array}$$

φ is a diffeomorphism since f & $\bar{f}$ are charts

$$d\varphi\left(\frac{\partial f}{\partial u^i}\right) = \frac{\partial \bar{f}}{\partial u^i} \quad (*)$$

$$\Rightarrow \left\langle \frac{\partial \bar{f}}{\partial u^i}, \frac{\partial \bar{f}}{\partial u^j} \right\rangle \overset{\text{assumption}}{=} \left\langle \frac{\partial f}{\partial u^i}, \frac{\partial f}{\partial u^j} \right\rangle \Rightarrow \varphi \text{ is an isometry.}$$

$\| $ by $(*)$

$$\left\langle d\varphi\left(\frac{\partial f}{\partial u^i}\right), d\varphi\left(\frac{\partial f}{\partial u^j}\right) \right\rangle$$

Ex 2) helicoid vs catenoid

$$f(u_1, u_2) = (u_2 \cos u_1, \ u_2 \sin u_1, \ a u_1) \quad (u_1, u_2) \in (0, 2\pi) \times \mathbb{R} = U$$

$$\bar{f}(\bar{u}_1, \bar{u}_2) = (a \cosh(\bar{u}_2) \cos(\bar{u}_1), \ a \cosh(\bar{u}_2) \sin(\bar{u}_1), \ a u_2) \quad \text{same } U$$

$$\begin{bmatrix} \bar{E} & \bar{F} \\ \bar{F} & \bar{G} \end{bmatrix} = \begin{bmatrix} a^2 \cosh^2 \bar{u}_2 & 0 \\ 0 & a^2 \cosh^2 \bar{u}_2 \end{bmatrix} \quad \text{for catenoid}$$

$$u_2 = a \sinh(\bar{u}_2)$$
$$u_1 = \bar{u}_1$$

$$\frac{\partial(u_1, u_2)}{\partial(\bar{u}_1, \bar{u}_2)} = a \cosh(\bar{u}_2) \neq 0$$

Jacobian det of change of variables

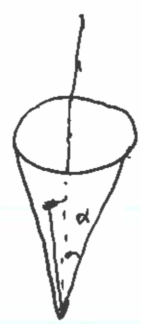
So this change of variables is a local diffeomorphism

In these new coordinates, for the helicoid we have

$$\begin{bmatrix} E & F \\ F & G \end{bmatrix} = \begin{bmatrix} a^2 \cosh^2 \bar{u}_2 & 0 \\ 0 & a^2 \cosh^2 \bar{u}_2 \end{bmatrix} = \begin{bmatrix} \bar{E} & \bar{F} \\ \bar{F} & \bar{G} \end{bmatrix}$$

So by the proposition, they are locally isometric.

3) Cone & plane

C = 

= S

$$\bar{f}(\rho, \theta) = (\rho \cos\theta, \rho \sin\theta, 0)$$

$$(\rho, \theta) \in (0, \infty) \times (0, 2\pi \sin\alpha)$$

parametrizes S

$:(\rho, \theta) = \left(\rho \sin\alpha \cos\left(\frac{\theta}{\sin\alpha}\right), \rho \sin\alpha \sin\left(\frac{\theta}{\sin\alpha}\right), \rho\cos\alpha \right)$

parametrizes the cone

Exercise: Compute the metrics & see these are isometric away ~~from~~ the cone point.

Upshot: any quantity that can be written in terms of g & Christoffel symbols is intrinsic.

Gauss's theorem egrigium: The Gauss ~~curvature~~ Curvature K is intrinsic.

Recall Gauss' formula: $f : U \longrightarrow S$ chart,

$$\frac{\partial^2 f}{\partial u^i \partial u^j} = \Gamma^k_{ij} \frac{\partial f}{\partial u^k} + h_{ij} \nu \qquad (*)$$

Weingarten formula: $\frac{\partial \nu}{\partial u_i} = -h_{ij} g^{jk} \frac{\partial f}{\partial u^k}$

Note that $\frac{\partial}{\partial u^i}\left(\frac{\partial^2 f}{\partial u^j \partial u^k}\right) = \frac{\partial^3 f}{\partial u^i \partial u^j \partial u^k} = \frac{\partial}{\partial u^i}\left(\frac{\partial^2}{\partial u^k \partial u^j}\right)$

let's apply this to $(*)$:

$$0 = \frac{\partial}{\partial u^k}\left(\frac{\partial^2 f}{\partial u^i \partial u^j}\right) - \frac{\partial}{\partial u^j}\left(\frac{\partial^2 f}{\partial u^i \partial u^k}\right) = \frac{\partial}{\partial u^k}\left(\Gamma^s_{ij} f_s + h_{ij}\nu \right)$$
$$- \frac{\partial}{\partial u^j}\left(\Gamma^s_{ik} f_s + h_{ik}\nu \right)$$

$$= \Gamma^s_{ij,k} f_s - \Gamma^s_{ik,j} f_s + \Gamma^r_{ij} f_{kr} - \Gamma^r_{ik} f_{jr}$$

$$+ (h_{ij,k} - h_{ik,j})\nu + h_{ij}\nu_k - h_{ik}\nu_j$$

Plug in Gauss & Weingarten again

$$= \left(\Gamma^s_{ij,k} - \Gamma^s_{ik,j}\right) f_s + \Gamma^r_{ij}\left(\Gamma^s_{kr} f_s + h_{kr}\nu\right) - \Gamma^r_{ik}\left(\Gamma^s_{jr} f_s + h_{jr}\nu\right)$$

$$+ (h_{ij,k} - h_{ik,j})\nu - h_{ij} h_{km} g^{ms} f_s + h_{ik} h_{jm} g^{ms} f_s = 0$$

Collecting like terms using the basis $\{\{f_s\}_{s=1}^n, \nu\}$:

we must have:

(f_s term)

"Gauss eqn"

$$\boxed{\Gamma^s_{ij,k} - \Gamma^s_{ik,j} + \Gamma^r_{ij}\Gamma^s_{kr} - \Gamma^r_{ik}\Gamma^s_{jr} = (h_{ij} h_{km} - h_{ik} h_{jm}) g^{ms}}$$

for all i, j, k, s

(ν term)

$$\boxed{h_{ij,k} - h_{ik,j} = \Gamma^r_{ik} h_{rj} - \Gamma^r_{ij} h_{kr}}$$

Codazzi - Mainardi eqn

for all i, j, k

Rmk The expression on the LHS of Gauss eqn will be important later as the Riemannian Curvature Tensor,

$$R^s_{ijk} := \Gamma^s_{ij,k} - \Gamma^s_{ik,j} + \Gamma^r_{ij}\Gamma^s_{kr} - \Gamma^r_{ik}\Gamma^s_{jr}$$

Pf of theorem egrigium:

multiply the Gauss eqn by g and consider $i = j = 1$ $k = 2$:

$$\left(h_{11} h_{2m} - h_{12} h_{1m}\right) g^{ms} g_{s2} = h_{11} h_{22} - h_{12} h_{12} = \mathrm{Det}(\mathrm{II})$$

note $(g^{\alpha\beta})$ & $(g_{\alpha\beta})$ are inverse matrices

so $g^{ms} g_{s2} = \delta^m_2 := \begin{cases} 1 & \text{if } m=2 \\ 0 & \text{else} \end{cases}$

On the other side of Gauss eqn, we have an expression depending only on Γ^k_{ij} & g_{ms}, thus $\mathrm{Det}(\mathrm{II})$ is intrinsic, by $K = \dfrac{\det(\mathrm{II})}{\det(\mathrm{I})}$, K is also intrinsic

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Cor: Fold your pizza when you eat it!

Rmk: The Gauss & Codazzi-Mainardi equations are "Integrability constraints" for Γ^k_{ij}, h_{ij} & g_{ms}: They are necessary conditions for a given collection of Christoffel symbols, I^{st} & II^{nd} fundamental form to come from a parametrized surface

$f: U \to S$ (that is to be integrable to such an F)