Let's do some explicit computations in dim 2.

Note first that $R^{s}_{ijk} = -R^{s}_{ikj}$ & Similarly for the expressions in the Codazzi-Mainardi eqn

Thus, suppose $j=1$, $k=2$ wlog

Ex $s=2$, $i=1$

$$\Gamma^{2}_{11,2} - \Gamma^{2}_{12,1} + \Gamma^{1}_{11}\Gamma^{2}_{21} + \Gamma^{2}_{11}\Gamma^{2}_{22} - \Gamma^{1}_{12}\Gamma^{2}_{11} - \Gamma^{2}_{12}\Gamma^{2}_{12}$$

$$= \underbrace{\cancel{(h_{11}h_{21} - h_{12}h_{11})}}_{0}\, g^{12} + (h_{11}h_{22} - h_{12}h_{12})\, g^{22}$$

$$= \frac{\det(II)}{\det(I)} E = KE$$

$S=1$, $i=1$

$$\Gamma^{1}_{11,2} - \Gamma^{1}_{12,1} + \Gamma^{1}_{11}\Gamma^{1}_{21} + \Gamma^{2}_{11}\Gamma^{1}_{22} - \Gamma^{1}_{12}\Gamma^{1}_{11} - \Gamma^{2}_{12}\Gamma^{1}_{12}$$

$$= \overbrace{\cancel{(h_{11}h_{21} - h_{12}h_{11})}}^{0}\, g^{11} + (h_{11}h_{22} - h_{12}h_{12})\, g^{21}$$

$$= -\frac{\det(II)}{\det(I)} F = -KF$$

$i=1 \quad j=1 \quad k=2$ Codazzi-Mainardi

$$h_{11,2} - h_{12,1} = \Gamma^1_{12} h_{11} + \Gamma^2_{12} h_{21} - \Gamma^1_{11} h_{21} - \Gamma^2_{11} h_{22}$$

$$= h^1_{12} h_{11} + \left(\Gamma^2_{12} - \Gamma^1_{11}\right) h_{21} - \Gamma^2_{11} h_{22}$$

Special Case Suppose in the parametrization $f: U \to S \subseteq \mathbb{R}^3$, the coordinate curves are lines of curvature so that $h_{12} = h_{21} = g_{12} = g_{21} \equiv 0$

Using $2\Gamma_{ij,k} = -g_{ij,k} + g_{ki,j} + g_{jk,i}$

$+ \quad \Gamma^k_{ij} = g^{km}\Gamma_{ij,m}$ we get

$$\Gamma^2_{11} = \frac{-1}{2}\frac{g_{11,2}}{g_{22}} = \frac{-1}{2}\frac{E_{u_2}}{G} \qquad \Gamma^1_{12} = \frac{1}{2}\frac{E_{u_2}}{E} = \frac{1}{2}\frac{g_{11,2}}{g_{11}}$$

$$\Gamma^1_{22} = \frac{-1}{2}\frac{g_{22,1}}{g_{11}} = \frac{-1}{2}\frac{G_{u_1}}{E} \qquad \Gamma^2_{12} = \frac{1}{2}\frac{g_{22,1}}{g_{22}} = \frac{1}{2}\frac{G_{u_1}}{G}$$

$$\Gamma^1_{11} = \frac{\Gamma_{11,1}}{g_{11}} = \frac{1}{2}\frac{g_{11,1}}{g_{11}} = \frac{1}{2}\frac{E_{u_1}}{E} \qquad \Gamma^2_{22} = \frac{\Gamma_{22,2}}{g_{22}} = \frac{1}{2}\frac{g_{22,2}}{g_{22}} = \frac{1}{2}\frac{G_{u_2}}{G}$$

$$h_{11,2} = h_{11}\left(\frac{1}{2}\frac{g_{11,2}}{g_{11}}\right) - h_{22}\left(\frac{-1}{2}\frac{g_{11,2}}{g_{22}}\right)$$

$$= \frac{g_{11,2}}{2}\left(\frac{h_{11}}{g_{11}} + \frac{h_{22}}{g_{22}}\right)$$

$$h_{22,1} = \frac{g_{22,1}}{2}\left(\frac{h_{11}}{g_{11}} + \frac{h_{22}}{g_{22}}\right)$$

Ex Surfaces of revolution

$$f(t,\varphi) = \left(r(t)\cos\varphi,\ r(t)\sin\varphi,\ h(t)\right)$$

with $r'^2 + h'^2 = 1$ (revolving arc length curve)

$$I = \begin{pmatrix} 1 & 0 \\ 0 & r^2 \end{pmatrix} \qquad II = \begin{pmatrix} -r''h' + r'h'' & 0 \\ 0 & rh' \end{pmatrix}$$

$$\Gamma^1_{11} = 0 \qquad \Gamma^2_{11} = 0 \qquad \Gamma^2_{12} = \frac{1}{2}\frac{(r^2)'}{r^2} = \frac{rr'}{r^2} = \frac{r'}{r}$$

$$\Gamma^1_{12} = 0$$

$$\Gamma^2_{22} = 0$$

$$\Gamma^1_{22} = \frac{-1}{2}\frac{(r^2)'}{1} = -rr'$$

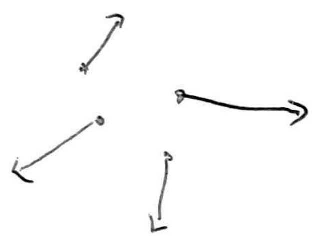
$$\frac{\partial}{\partial\varphi}h_{11} = h_{11,2} = 0 \qquad h_{22,1} = \frac{\partial}{\partial t}h_{22} = \frac{(r^2)'}{2}\left(\frac{-r''h' + r'h''}{1} + \frac{rh'}{r^2}\right)$$

Parallel transport & geodesics

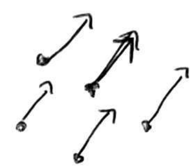
In $\mathbb{R}^n$, we can talk about a vector field Y being "constant" if $D_X Y = 0$ for all $X \in T_p\mathbb{R}^n$

$\iff$ each of the vectors $Y_p \in T_p\mathbb{R}^n$ are parallel (+ same length) for all $p \in \mathbb{R}^n$:

not parallel



Parallel

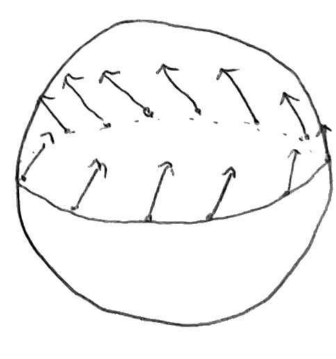


Def 1) A vector field Y tangent to S is called parallel if $\nabla_X Y = 0$ for all $X \in T_pS$ & $p \in S$.

2) let $\gamma : I \to S$ a regular curve. A vector field Y along γ is parallel along γ if $\nabla_{\dot\gamma} Y = 0$

Rmk Note these are both local notions

Ex of a parallel vector field



Prop Suppose X, Y are parallel along $\gamma(t)$, then $\langle X, Y\rangle$ is constant: $\frac{d}{dt}\langle X, Y\rangle_{\gamma(t)} = 0$

Pf $\frac{d}{dt}\langle X, Y\rangle = \nabla_{\dot\gamma}\langle X, Y\rangle = \langle \overset{0}{\cancel{\nabla_{\dot\gamma} X}}, Y\rangle + \langle X, \overset{0}{\cancel{\nabla_{\dot\gamma} Y}}\rangle = 0$

Cor $\|X\|$ is constant for a parallel vector field along γ.

Thm Given i) $\gamma : I \to U$ regular, ii) $f : U \to S \subseteq \mathbb{R}^{n+1}$ a chart of a regular surface iii) $Y_0 \in T_p S$ $p = f(\gamma(t_0))$ for some $t_0 \in I$, then there exists a unique vector field Y parallel along $f \circ \gamma = c$ & with $Y_0 = Y_p$.

Pf Write $Y = \eta^j \frac{\partial f}{\partial u^j}$, $\gamma(t) = (u^1(t), \_\_, u^n(t))$

$\dot\gamma = (\dot u^1(t), \_\_, \dot u^n(t))$ so $\dot c = \dot u^i \frac{\partial f}{\partial u^i}$ $(c = \gamma \circ f)$

$$\nabla_{\dot c} Y = \dot u^i\left(\frac{\partial \eta^k}{\partial u^i} + \eta^j \Gamma_{ij}^k\right)\frac{\partial f}{\partial u^k} = \left(\frac{d\eta^k}{dt} + \dot u^i \eta^j \Gamma_{ij}^k\right)\frac{\partial f}{\partial u^k}$$

So $\nabla_{\dot c} Y = 0 \iff \frac{d\eta^k}{dt} + \dot u^i \eta^j \Gamma_{ij}^k = 0$ for all k

i.e. Y parallel along c

This gives a linear system of diff eqns for $(\eta^k)_{k=1}^{n}$ as a function of t

We have initial conditions $\eta^k(t_0)$ give by the expression $Y_0 = \eta^k(t_0) \left.\frac{\partial F}{\partial u^k}\right|_p \Rightarrow$

$\exists!$ solution to the system, thus $\exists!$ η^k

thus $\exists!$ parallel field $Y = \eta^k \frac{\partial F}{\partial u^k}$ $\boxtimes$

Def Given $Y_0 \in T_pS$ & c a regular curve through p, the parallel transport Y of Y_0 along c is the unique parallel vector field Y along c with $Y_p = Y_0$ given by the theorem.

Rmk The parallel transport along a regular curve is independent of parametrization of c.

$c: I \to S$ a diffeomorphism,

IF $\bar{c} = c \circ \alpha$ for $\alpha: \bar{I} \to I$
$t = \alpha(\bar{t})$

$$\frac{d\bar{c}}{d\bar{t}} = \frac{d\alpha}{d\bar{t}} \frac{dc}{dt} \quad \text{so} \quad \nabla_{\frac{d\bar{c}}{d\bar{t}}} Y = \nabla_{\frac{d\alpha}{d\bar{t}}\frac{dc}{dt}} Y = \frac{d\alpha}{d\bar{t}} \nabla_{\frac{dc}{dt}} Y$$

& $\frac{d\alpha}{d\bar{t}} \neq 0$ for all $\bar{t} \in \bar{I}$ thus

$$\nabla_{\frac{d\bar{c}}{dt}} Y = 0 \iff \nabla_{\frac{dc}{dt}} Y = 0$$